COL351: Analysis and Design of Algorithms

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Computational Intractability: NP, NP-complete, NP-hard

Definition (NP)

A problem X is said to be in NP iff there is an efficient certifier for X.

Definition (NP-complete)

A problem is said to be NP-complete iff the following two properties hold:

- $X \in \text{NP}$
- For all $Y \in \text{NP}$, $Y \leq_p X$

Theorem (Cook-Levin Theorem)

3-SAT is NP*-complete.*

Definition (NP-hard)

A problem X is said to be NP-hard iff the following property holds:

- $\cancel{X \in \text{NP}}$
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Theorem (Cook-Levin Theorem)

3-*SAT is* NP-*complete.*

- Claim 1: INDEPENDENT-SET, VERTEX-COVER, SET-COVER are also NP-complete.

Proof of Claim 1

- These problems are in NP.
- 3-SAT $\leq_p$ INDEPENDENT-SET $\leq_p$ VERTEX-COVER $\leq_p$ SET-COVER

Problem

TSP: Given a complete, weighted, directed graph G and an integer k, determine if there is a tour in the graph of total length at most k.

- Claim 1: TSP $\in$ NP
  - Proof sketch: A tour of length at most k is a certificate.

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- Claim 1: TSP $\in$ NP
  - Proof sketch: A tour of length at most k is a certificate.
- Claim 2: 3-SAT $\leq_p$ TSP

Proof of Claim 2

- Claim 2.1: 3-SAT $\leq_p$ HAMILTONIAN-CYCLE
- Claim 2.2: HAMILTONIAN-CYCLE $\leq_p$ TSP

Problem

HAMILTONIAN-CYCLE: Given an unweighted, directed graph, determine if there is a Hamiltonian cycle in the graph.

- Hamiltonian cycle: A cycle that visits each vertex exactly once.

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- Claim 2.2: HAMILTONIAN-CYCLE $\leq_p$ TSP

Proof of Claim 2.2

- Given an unweighted, directed graph G, construct the following complete, directed, weighted graph G':
  - For each edge (u, v) in G, give the weight of 1 to edge (u, v) in G'
  - For each pair (u, v) such that there is no edge from u to v in G, add an edge (u, v) with weight 2 in G'
- Claim 2.2.1: G has a Hamiltonian cycle if and only if G' has a tour of length at most n

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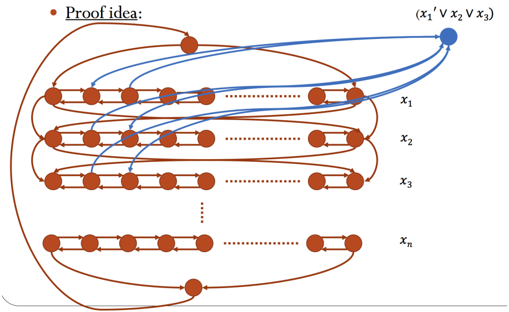
Proof of Claim 2.1

- Given an instance of the 3-SAT problem (a formula Ω with n variables and m clauses), we need to create a directed graph G such that Ω is satisfiable if and only if G has a Hamiltonian cycle.



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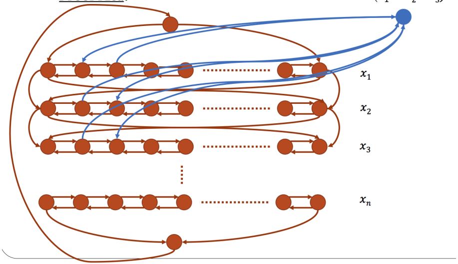
- Claim 2.1.1: If the 3-SAT formula is satisfiable, then there is a Hamiltonian cycle in the constructed graph.

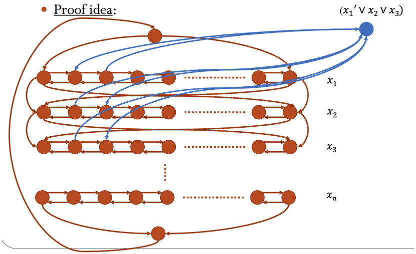
- Proof idea:



- Claim 2.1: 3-SAT $\leq_p$ HAMILTONIAN-CYCLE

Proof of Claim 2.1

- Given an instance of the 3-SAT problem (a formula Ω with n variables and m clauses), we need to create a directed graph G such that Ω is satisfiable if and only if G has a Hamiltonian cycle.
- Claim 2.1.1: If the 3-SAT formula is satisfiable, then there is a Hamiltonian cycle in the constructed graph.
- Claim 2.1.2: If the constructed graph has a Hamiltonian cycle, then the 3-SAT formula has a satisfying assignment.



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Proof of Claim 1

- Claim 1.1: HAMILTONIAN-PATH $\in$ NP

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Proof of Claim 1

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  - A Hamiltonian path acts as a certificate.

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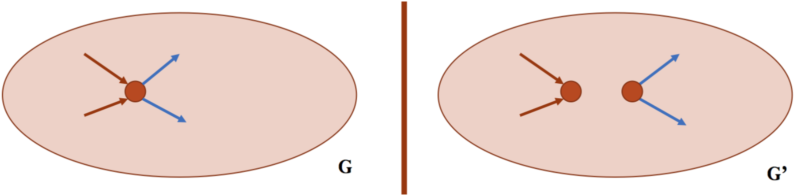
Proof of Claim 1

- Claim 1.1: HAMILTONIAN-PATH $\in$ NP
  - A Hamiltonian path acts as a certificate.
- Claim 1.2: HAMILTONIAN-PATH is NP-hard.
  - Claim 1.2.1: HAMILTONIAN-CYCLE $\leq_p$ HAMILTONIAN-PATH

- Claim 1.2.1: HAMILTONIAN-CYCLE $\leq_p$ HAMILTONIAN-PATH

Proof of Claim 1.2.1

- Consider the graph G' constructed from graph G.
- There is a Hamiltonian cycle in G if and only there is a Hamiltonian path in G'.



Definition (k-colorable)

A graph is said to be k-colorable is it is possible to assign one of k colors to each node such that for every edge (u, v), u and v are assigned different colors.

Problem

$\underline{k\text{-COLORING}}$: Given a graph G, determine if G is k-colorable.

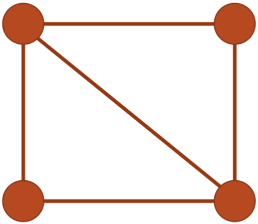


Figure: Is this graph 2-colorable?

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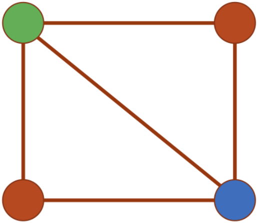

Figure: Is this graph 2-colorable? Yes

Definition (k-colorable)

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Problem

$\underline{k\text{-COLORING}}$: Given a graph G, determine if G is k-colorable.

Problem

$\underline{\text{2-COLORING}}$: Given a graph G, determine if G is 2-colorable.

- How hard is the 2-COLORING problem?

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Problem

k-COLORING: Given a graph G, determine if G is k-colorable.

Problem

2-COLORING: Given a graph G, determine if G is 2-colorable.

- How hard is the 2-COLORING problem?
  - 2-COLORING $\in$ P since G is 2-colorable if and only if G is bipartite and we know an efficient algorithm for checking if a given graph is bipartite.

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Problem

<u>k-COLORING</u>: Given a graph G, determine if G is k-colorable.

Problem

<u>3-COLORING</u>: Given a graph G, determine if G is 3-colorable.

- How hard is the 3-COLORING problem?

Definition (k-colorable)

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Problem

k-COLORING: Given a graph G, determine if G is k-colorable.

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3-COLORING: Given a graph G, determine if G is 3-colorable.

- How hard is the 3-COLORING problem?
- Claim 1: 3-COLORING is NP-complete.

Problem

3-COLORING: Given a graph G, determine if G is 3-colorable.

- Claim 1: 3-COLORING is NP-complete.

Proof of Claim 1

- Claim 1.1: 3-COLORING is in NP
  - A short certificate is a 3-coloring of the graph.
- Claim 1.2: 3-SAT $\leq_p$ 3-COLORING

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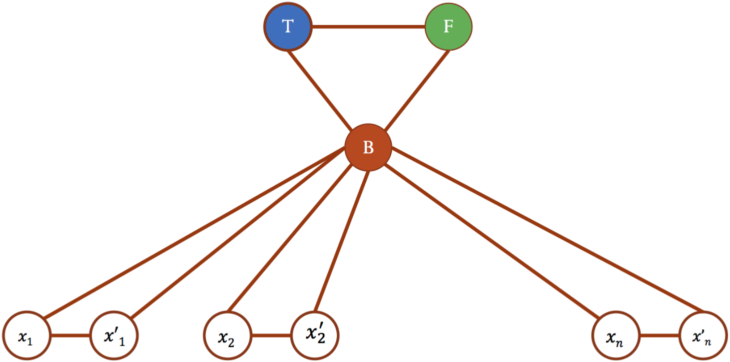
Proof ideas for Claim 1.2

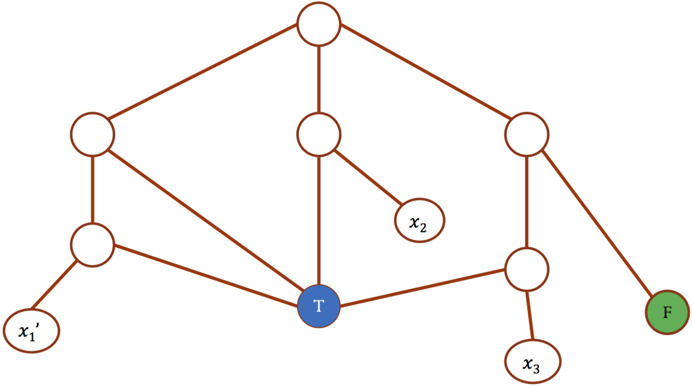
- Consider the following gadget. There is a bijection between colors and truth values.



- Claim 1.2: 3-SAT $\leq_p$ 3-COLORING

Proof ideas for Claim 1.2

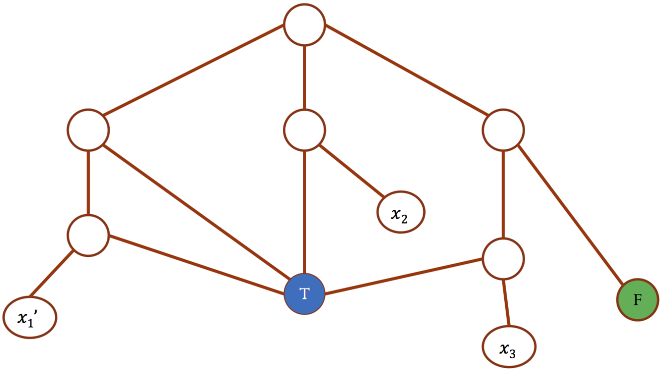
- How we encode a clause, say $(\bar{x}_1 \vee x_2 \vee x_3)$.



- Claim 1.2: 3-SAT $\leq_p$ 3-COLORING

Proof ideas for Claim 1.2

- Claim 1.2.1: There is no 3 coloring of the graph below with nodes $\bar{x}_1, x_2$, and x_3 assigned F color.



- Claim 1.2: 3-SAT $\leq_p$ 3-COLORING

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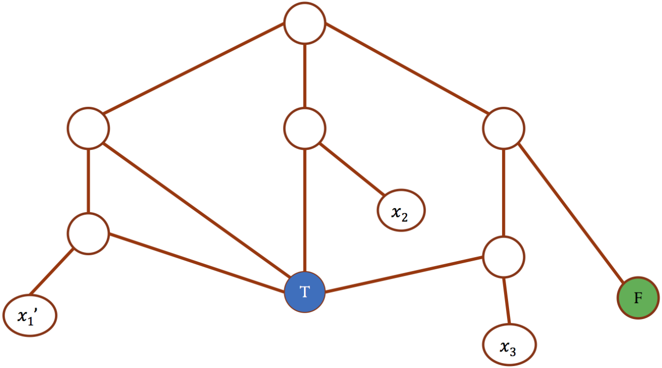
- Claim 1.2.2: There is a 3 coloring of the graph below with at least one of the nodes $\bar{x}_1, x_2$, and x_3 assigned T color.



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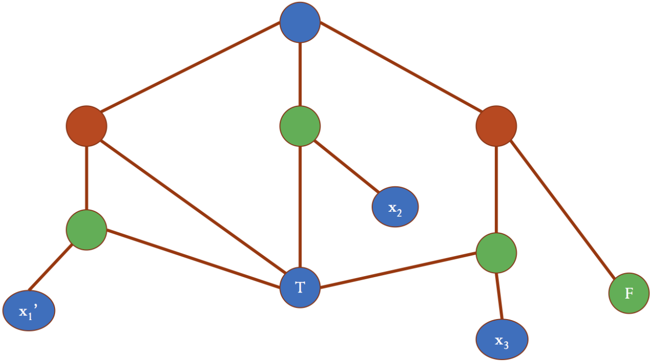
  - $\bar{x}_1 : T, x_2 : T, x_3 : T$



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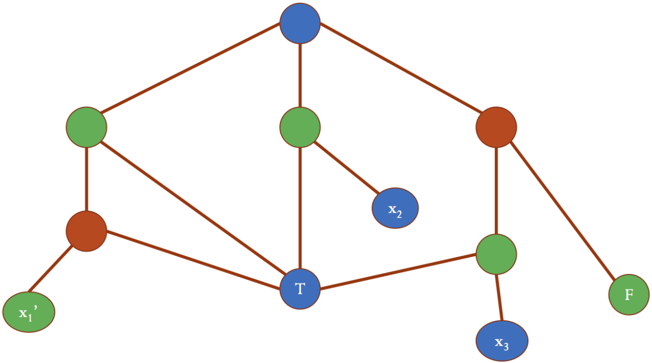

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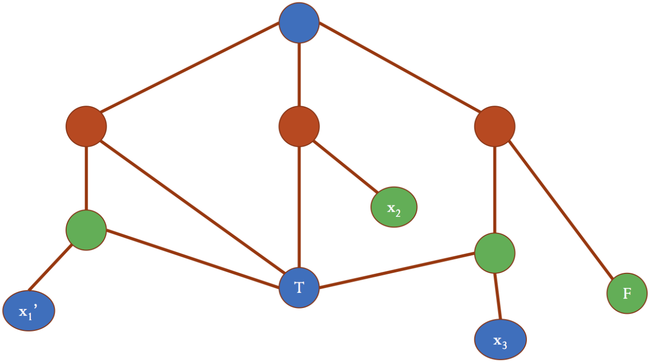

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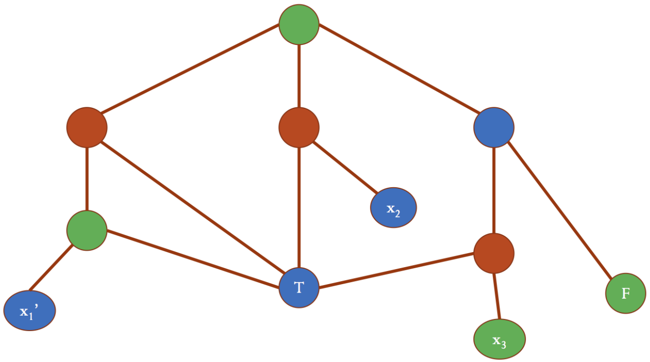

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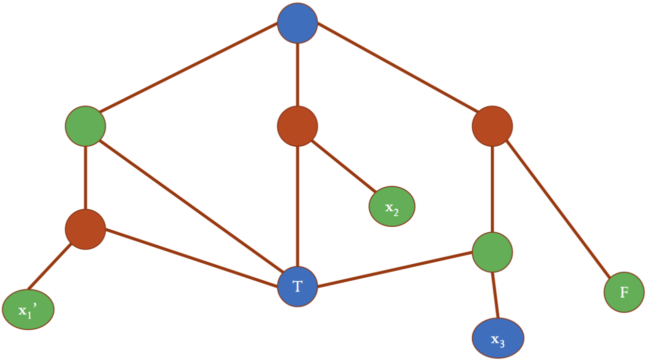

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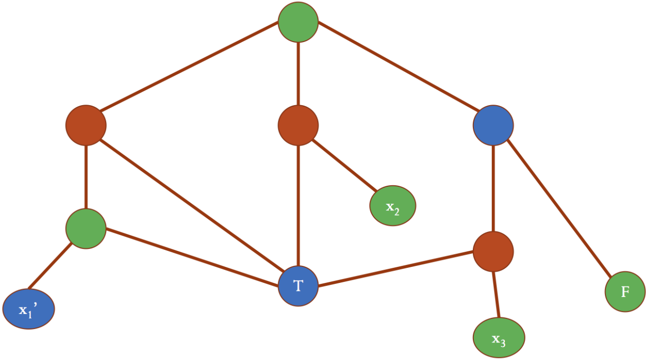

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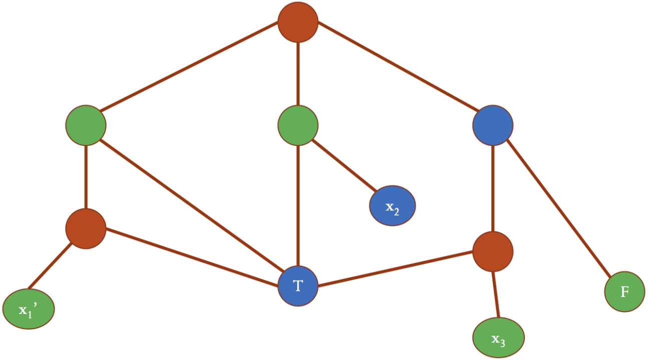

- Claim 1.2: 3-SAT $\leq_p$ 3-COLORING

Proof ideas for Claim 1.2

- Claim 1.2.3: The given formula is satisfiable if and only if the constructed graph has a 3 coloring.

End