COL351: Analysis and Design of Algorithms

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Course Overview

- Basic graph algorithms
- Algorithm Design Techniques:
  - Greedy Algorithms
  - Divide and Conquer
  - Dynamic Programming
  - Network Flow
- Computational Intractability

Computational Intractability

Definition (Efficient Algorithms)

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- There are lots of problems arising in various fields for which this question is unresolved.
- Question 2: Are these problems related in some manner?
- Question 3: If someone discovers an efficient algorithm to one of these difficult problems, then does that mean that there are efficient algorithms for other problems? If so, how do we obtain such an algorithm.

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- Polynomial-time reduction:
  - Consider two problems X and Y.
  - Suppose there is a *black box* that solves arbitrary instances of problem X.
  - Suppose any arbitrary instance of problem Y can be solved using a polynomial number of standard computational steps and a polynomial number of calls to the black box that solves instance of problem X.
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- Claim 1: BIPARTITE-MATCHING $\leq_p$ MAX-FLOW.

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- Claim 2: Suppose $Y \leq_p X$. If X can be solved in polynomial time, then Y can be solved in polynomial time.
- Claim 3: Suppose $Y \leq_p X$. If Y cannot be solved in polynomial time, then X cannot be solved in polynomial time.

Definition (Independent Set)

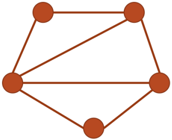
Given a graph $G = (V, E)$, a subset $I \subseteq V$ of vertices is called an independent set of G iff there are no edges between any pair of vertices in I.

Problem

INDEPENDENT-SET: Given a graph $G = (V, E)$ and an integer k, check if there is an independent set of size at least k in G.

Problem

MAXIMUM-INDEPENDENT-SET: Given a graph $G = (V, E)$, output the size of independent set of G of maximum cardinality.



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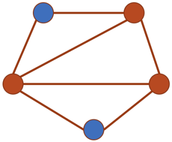

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Definition (Vertex Cover)

Given a graph $G = (V, E)$, a subset $S \subseteq V$ of vertices is called a vertex cover of G iff for any edge (u, v) in the graph at least one of u, v is in S.

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VERTEX-COVER: Given a graph $G = (V, E)$ and an integer k, check if there is a vertex cover of size at most k in G.

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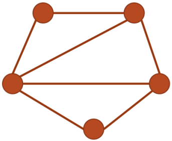
MINIMUM-VERTEX-COVER: Given a graph $G = (V, E)$, output the size of vertex cover of G of minimum cardinality.



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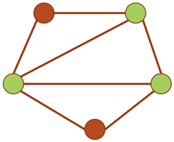

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- Claim 3: MINIMUM-VERTEX-COVER $\leq_p$ VERTEX-COVER.

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- Claim 3: MINIMUM-VERTEX-COVER $\leq_p$ VERTEX-COVER.
- Claim 4: VERTEX-COVER $\leq_p$ MINIMUM-VERTEX-COVER.

- Claim 5: INDEPENDENT-SET $\leq_p$ VERTEX-COVER.

Proof of Claim 5

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- Claim 5.2: Let S be a vertex cover of G, then $V - S$ is an independent set of G.
- Claim 5.3: G has an independent set of size at least k if and only if G has a vertex cover of size at most $n - k$.

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- Claim 5.3: G has an independent set of size at least k if and only if G has a vertex cover of size at most $n - k$.
- Given an instance (G, k) of the independent set problem, create an instance $(G, n - k)$ of the vertex cover problem, make a single query to the block box for solving the vertex cover problem and return the answer that is returned by the black box. $\square$

- Claim 6: MINIMUM-VERTEX-COVER $\leq_p$ MAXIMUM-INDEPENDENT-SET.

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- Claim 6.1: G has an independent set of size k if and only if G has a vertex cover of size $n - k$.
- Make a single call to the black box for the maximum independent problem with input G. If the black box returns k, then return $n - k$. $\square$

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Another proof of Claim 6

- MINIMUM-VERTEX-COVER $\leq_p$ VERTEX-COVER
- VERTEX-COVER $\leq_p$ INDEPENDENT-SET
- INDEPENDENT-SET $\leq_p$ MAXIMUM-INDEPENDENT-SET $\square$

Theorem

If $X \leq_p Y$ and $Y \leq_p Z$, then $X \leq_p Z$.

Problem

DEG-3-INDEPENDENT-SET: Given a graph $G = (V, E)$ of bounded degree 3 *(i.e., all vertices have degree* ≤ 3*)* and an integer k, check if there is an independent set of size at least k in G.

- Claim 1: INDEPENDENT-SET $\leq_p$ DEG-3-INDEPENDENT-SET

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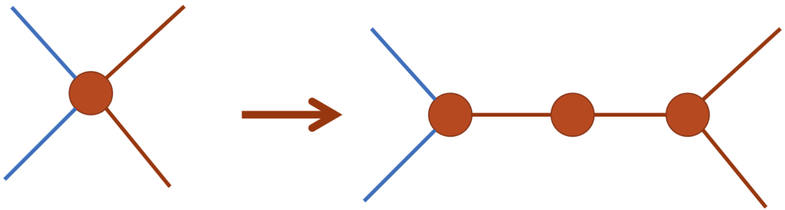
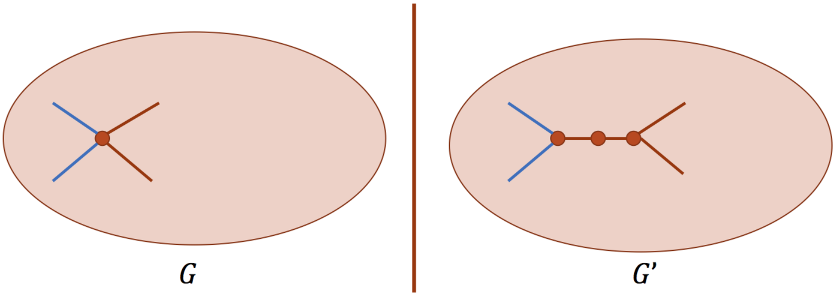
- Claim 1: INDEPENDENT-SET $\leq_p$ DEG-3-INDEPENDENT-SET
  - Idea: "Split" all vertices.



- Claim 1: INDEPENDENT-SET $\leq_p$ DEG-3-INDEPENDENT-SET

Proof of Claim 1

- Consider graph G' constructed by "splitting" a vertex of G.
- Claim 1.1: G has an independent set of size at least k if and only if G' has an independent set of size at least $(k+1)$.



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