COL351: Analysis and Design of Algorithms

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Applications of Network Flow

- Suppose there are four teams in IPL with their current number of wins:
  - Daredevils: 10
  - Sunrisers: 10
  - Lions: 10
  - Supergiants: 8
- There are 7 more games to be played. These are as follows:
  - Supergiants plays all other 3 teams.
  - Daredevils Vs Sunrisers, Sunrisers Vs Lions, Daredevils Vs Lions, Sunrisers Vs Daredevils

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  - Supergiants plays all other 3 teams.
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- A team is said to be eliminated if it cannot end with maximum number of wins.
- Can we say that Supergiants have been eliminated give the current scenario?

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  - Supergiants plays all other 3 teams.
  - 4 games between Daredevils and Sunrisers.
- Can we say that Supergiants have been eliminated give the current scenario?

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There are n teams. Each team i has a current number of wins denoted by $w(i)$.There are $G(i,j)$ games yet to be played between team i and j. Design an algorithm to determine whether a given team x has been eliminated.

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- Consider the following flow network

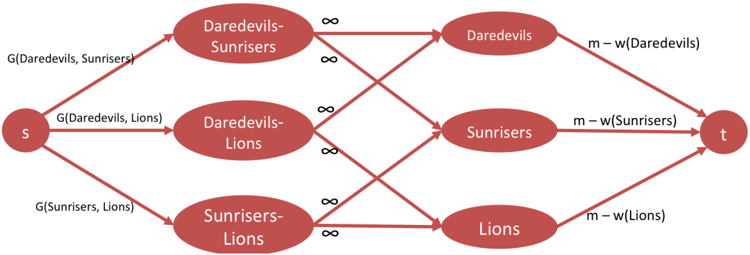

Figure: Team x can end with at most m wins, i.e., $m = w(x) + \sum_j G(x,j)$

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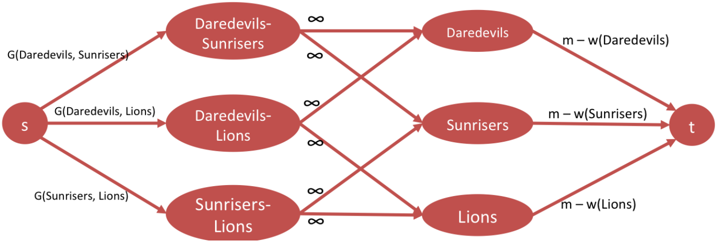
- Claim 1: Team x has been eliminated **iff** the maximum flow in the network is $< g^*$, where $g^* = \sum_{i,j \text{ s.t. } x \notin \{i,j\}} G(i,j)$.



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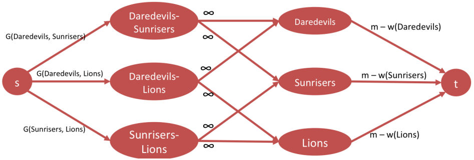
- Comment: If we can somehow find a subset T of teams (not including x) such that $\sum_{i \in T} w(i) + \sum_{i<j \text{ and } i,j \in T} G(i,j) > m \cdot |T|$. Then we have a witness to the fact that x has been eliminated.



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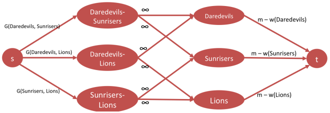
- Can we find such a subset T?



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Proof.

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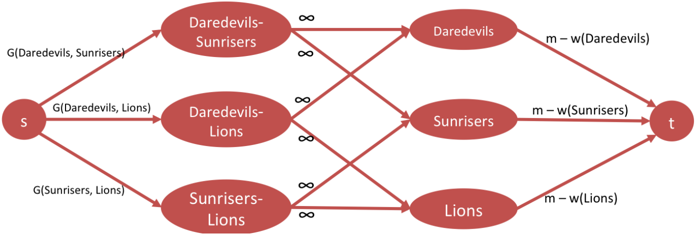
$\square$



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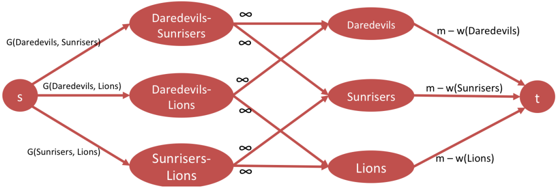
Proof of Claim 1

- Claim 1.1: If x has been eliminated, then the max flow in the network is $< g^*$.
- Claim 1.2: If the max flow is $< g^*$, then team x has been eliminated.

Proof of Claim 1.2

- Consider any s-t min-cut (A, B) in the graph.
- Claim 1.2.1: If v_{ij} is in A, then both v_i and v_j are in A.



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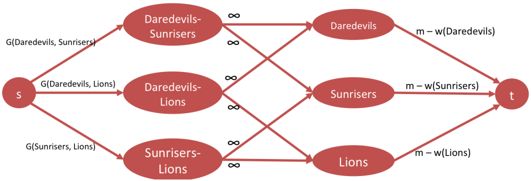

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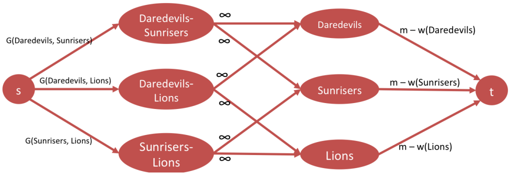

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- Let T be the set of teams such that $i \in T$ **iff** $v_i \in A$. Then we have:

$$
\begin{aligned}
& C(A,B) = \sum_{i \in T} (m - w(i)) + \sum_{\{i,j\} \not\subset T} G(i,j) < g^* \\
\Rightarrow \quad & m \cdot |T| - \sum_{i \in T} w(i) + (g^* - \sum_{\{i,j\} \subset T} G(i,j)) < g^* \\
\Rightarrow \quad & \sum_{i \in T} w(i) + \sum_{\{i,j\} \subset T} G(i,j) > m \cdot |T| \quad \square
\end{aligned}
$$

$\square$

End