COL351: Analysis and Design of Algorithms

Ragesh Jaiswal, CSE, IITD

Applications of Network Flow

Definition (Perfect Matching)

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- Claim 3: If there is a perfect matching, then for all subsets $A \subseteq X$, $|A| \leq |N(A)|$.
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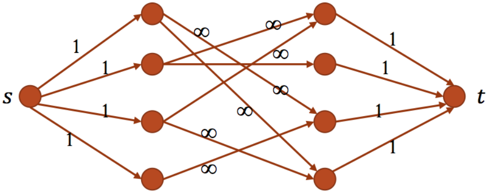
Proof of Claim 4

- Consider the flow network in figure below constructed using the bipartite graph.
- Claim 4.1: The max-flow in the network is equal to the maximum matching in G.



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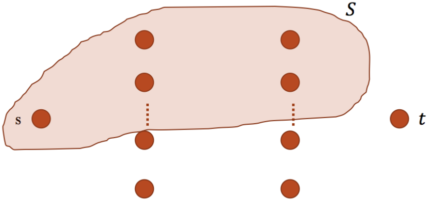

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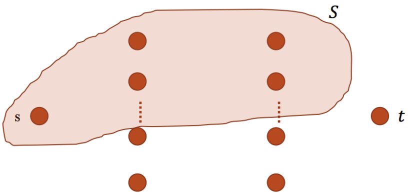

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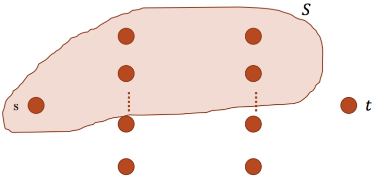

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- This is a constructive proof since we can find a subset A' such that $|A'| > |N(A')|$.
- Such an A' may be interpreted as a *certificate* of the fact that there is no perfect matching in G.

- Suppose there are four teams in IPL with their current number of wins:
  - Daredevils: 10
  - Sunrisers: 10
  - Lions: 10
  - Supergiants: 8
- There are 7 more games to be played. These are as follows:
  - Supergiants plays all other 3 teams.
  - Daredevils Vs Sunrisers, Sunrisers Vs Lions, Daredevils Vs Lions, Sunrisers Vs Daredevils

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- A team is said to be eliminated if it cannot end with maximum number of wins.
- Can we say that Supergiants have been eliminated give the current scenario?

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  - 4 games between Daredevils and Sunrisers.
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Problem

There are n teams. Each team i has a current number of wins denoted by $w(i)$.There are $G(i,j)$ games yet to be played between team i and j. Design an algorithm to determine whether a given team x has been eliminated.

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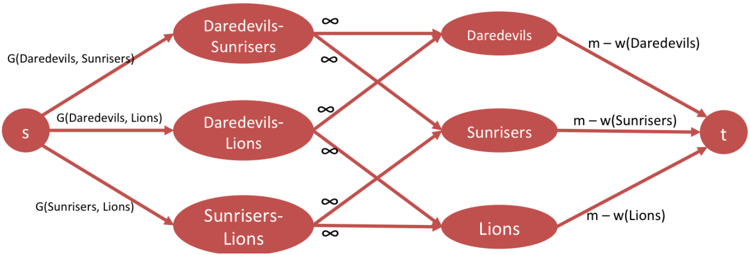

Figure: Team x can end with at most m wins, i.e., $m = w(x) + \sum_j G(x,j)$

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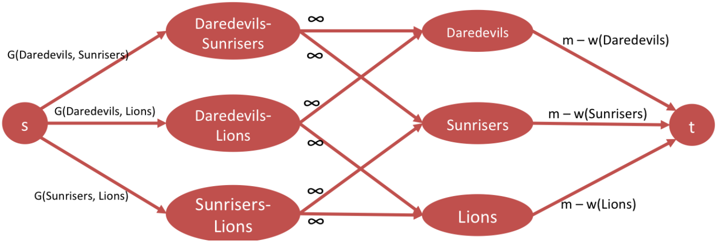

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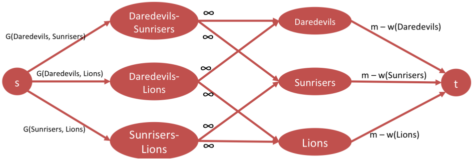

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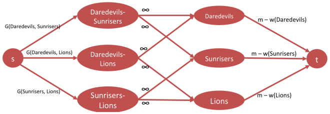
- Can we find such a subset T?



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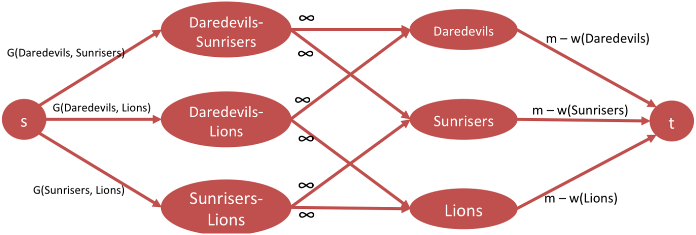
Proof.

- Claim 1.1: If x has been eliminated, then the max flow in the network is $< g^*$.

$\square$



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