COL351: Analysis and Design of Algorithms

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Divide and Conquer

Problem

Median Finding: Given an array of unsorted numbers and an integer k, design an algorithm that finds the k^{th} smallest number in the array. Assume that A contains distinct numbers.

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- Partition the numbers in A into A_L (all numbers $< p$) and A_R (all numbers $> p$)
- If ($|A_L| = k - 1$), then return(p)
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- What is the running time of this algorithm if the pivot is picked arbitrarily? $O(n^2)$

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- Consider groups of 5 elements
- Sort Individual groups
- Consider the median of the medians:
  - Here it is 46.
- Use this as the pivot element.

Divide and Conquer
Median finding

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- How many elements in A are larger than p?
  - Claim 1: There are at least $(3n/10 - 6)$ numbers in A that are larger than p.

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- Consider groups of 5 elements
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- Use this as the pivot element p.

- How many elements in A are larger than p? at least $(3n/10 - 6)$
- How many elements in A are smaller than p?
  - Claim 2: There are at least $(3n/10 - 6)$ numbers in A that are smaller than p.

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- How many elements in A are larger than p? at least $(3n/10 - 6)$
- How many elements in A are smaller than p? at least $(3n/10 - 6)$

Algorithm

`Find-Kth-smallest`(A, k)

- ... *//Base cases*
- Consider groups of 5 numbers, sort each group and create another array B containing the median number from each group
- $p \leftarrow$ `Find-Kth-smallest`$(B, \lfloor \frac{|B|}{2} \rfloor)$
- Partition the array A into A_L and A_R using p as the pivot
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  - $T(n) \leq T(\lceil n/5 \rceil) + T(7n/10 + 6) + O(n)$; $T(1) = O(1)$
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- Suppose we want to find the 12^{th} smallest element in the following array.

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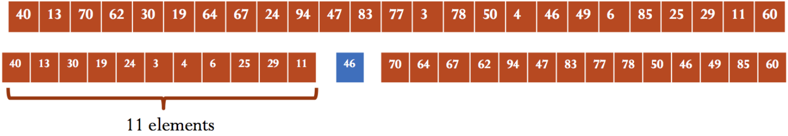

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Given two polynomials:
$A(x) = a_0 + a_1 \cdot x + a_2 \cdot x^2 + ... + a_{n-1} \cdot x^{n-1}$, and
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- We have to obtain the polynomial $C(x) = A(x) \cdot B(x)$
- $C(x)$ may be written as:
  $C(x) = c_0 + c_1 \cdot x + c_2 \cdot x^2 + ... + c_{2n-2} \cdot x^{2n-2}$
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- What is c_i in terms of coefficients of A and B?
  - $c_i = a_i \cdot b_0 + a_{i-1} \cdot b_1 + ... + a_0 \cdot b_i$
- The vector $(c_0, ..., c_{2n-2})$ is called the *convolution* of vectors $(a_0, ..., a_{n-1})$ and $(b_0, ..., b_{n-1})$.

Algorithm

$\texttt{SimpleMultiply}((a_0, ..., a_{n-1}), (b_0, ..., b_{n-1}))$
- For $i = 0$ to $2n - 2$
  - For $j = 0$ to i
    - $c_i \leftarrow c_i + a_j \cdot b_{i-j}$
- return$((c_0, c_1, ..., c_{2n-2}))$

- What is the running time of the above algorithm?

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- What is the running time of the above algorithm? $O(n^2)$
- Is there another way to compute the polynomial $C(x)$?

- Another way to compute the polynomial $C(x)$:
  - Compute $A(s_1), A(s_2), ..., A(s_{2n})$.
  - Compute $B(s_1), B(s_2), ..., B(s_{2n})$.
  - Compute:
    - $C(s_1) = A(s_1) \cdot B(s_1)$
    - $C(s_2) = A(s_2) \cdot B(s_2)$
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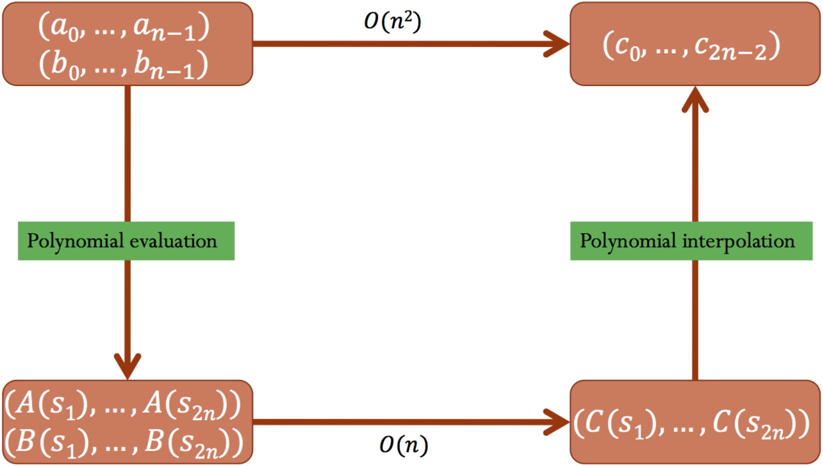
  - Interpolate to obtain the polynomial $C(x)$.
- How fast can you compute $A(s)$ given value of s?
  - $O(n)$ arithmetic operations using *Horner's rule*.
  - $A(s) = a_0 + s \cdot (a_1 + s \cdot (a_2 + ... + s \cdot (a_{n-1}) ...))$


$(a_0, \ldots, a_{n-1})$
$(b_0, \ldots, b_{n-1})$
$O(n^2)$
$(c_0, \ldots, c_{2n-2})$
Polynomial evaluation
Polynomial interpolation
$(A(s_1), \ldots, A(s_{2n}))$
$(B(s_1), \ldots, B(s_{2n}))$
$O(n)$
$(C(s_1), \ldots, C(s_{2n}))$

- Polynomial interpolation: We have $C(s_1), ..., C(s_{2n})$ and we need to compute $(c_0, ..., c_{2n-2})$.

$$\begin{pmatrix} 1 & s_1 & (s_1)^2 & \dots & (s_1)^{2n-1} \\ 1 & s_2 & (s_2)^2 & \dots & (s_2)^{2n-1} \\ 1 & s_3 & (s_3)^2 & \dots & (s_3)^{2n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & s_{2n} & (s_{2n})^2 & \dots & (s_{2n})^{2n-1} \end{pmatrix} \cdot \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_{2n-1} \end{pmatrix} = \begin{pmatrix} C(s_1) \\ C(s_2) \\ C(s_3) \\ \vdots \\ C(s_{2n}) \end{pmatrix}$$

- Is the above square matrix *invertible*?

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$$V = \begin{pmatrix} 1 & s_1 & (s_1)^2 & \dots & (s_1)^{2n-1} \\ 1 & s_2 & (s_2)^2 & \dots & (s_2)^{2n-1} \\ 1 & s_3 & (s_3)^2 & \dots & (s_3)^{2n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & s_{2n} & (s_{2n})^2 & \dots & (s_{2n})^{2n-1} \end{pmatrix}$$

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- So, as long as we use distict values of $s_1, ..., s_{2n}$, we will be able to do polynomial interpolation.

End