

CSL853: Complexity Theory

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PCP Theorem and Hardness of Approximation

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Two Prover one round proof systems (2P1R)

Theorem (Weak hardness for Label Cover)

There exists a universal constant $\epsilon > 0$ and Σ_0 of constant size such that $\text{GapLabelCover}_{1,1-\epsilon}(\Sigma_0)$ is NP-hard.

Theorem (2P1R proof system characterisation of NP)

There is a constant $\epsilon > 0$ such that $\text{NP} = 2\text{P1R}_{1,1-\epsilon}(\log n)$.

Theorem (Ran Raz's Parallel repetition theorem (informal version))

Let the soundness error probability of a two prover one round proof system be $\delta < 1$. Then the error probability on k parallel repetitions is δ^{dk} , where d is a constant that depends only on the length of the answers of the original proof system.

Theorem (Strong hardness for Label Cover)

*For **all** constant $\epsilon > 0$ there exists a constant $|\Sigma|$ such that $\text{GapLabelCover}_{1,\epsilon}(\Sigma)$ is NP-hard.*

Theorem

The problem $\text{GapLabelCover}_{1, \frac{1}{(\log n)^3}}$ is not in P unless $\text{NP} \subseteq \text{DTIME}(n^{O(\log \log n)})$.

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Set cover

Theorem

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Definition (Set Cover)

Given a universe U with $|U| = n$, and sets $S_1, \dots, S_m \subset U$ such that $\cup_i S_i = U$, find a set of indices $I \subseteq [m]$ with minimum cardinality such that $\cup_{i \in I} S_i = U$.

Theorem

There exists $c > 0$, such that no polynomial time $c \log n$ approximation algorithm exists for MIN-SC unless $\text{NP} \subseteq \text{DTIME}(n^{O(\log \log n)})$.

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Definition (Set System)

An (m, l) -set system consists of a universe B and collections of subsets $C_1, \dots, C_m$ such that if the union of any collection of l sets in $\{C_1, \dots, C_m, \overline{C_1}, \dots, \overline{C_m}\}$ is B , then the collection must contain both C_i and $\overline{C_i}$ for some i .

Theorem

There exists an (m, l) set system with universe B such that $|B| = O(2^{2l} m^2)$. Moreover, such a set system can be constructed in $2^{O(l)} m^{O(1)}$ -time.

PCP Theorem and Hardness of Approximation

Set cover

- Given an instance of the GapLabelCover problem $(G = (V_1, V_2, E), \Sigma, \Pi)$ where $|\Sigma| = m$, we use an (m, l) -set system with universe B to construct the following instance of Set Cover.
 - The universe $U = E \times B$.
 - For any vertex $v \in V_2$ and $x \in \Sigma$ let $S_{v,x} = \cup_{e \text{ contains } v} \{e\} \times C_x$.
 - For any vertex $u \in V_1$ and $x \in \Sigma$ let $S_{u,x} = \cup_{e \text{ contains } u} \{e\} \times \overline{C_{\Pi_e(x)}}$.
 - The following subsets of U form the instance of the set cover problem:
 $\{S_{w,x} : w \in V_1 \cup V_2, x \in \Sigma\}$.

Lemma

If the Label Cover instance has a labeling that satisfies all the edges, then the Set Cover instance produced has a set cover of size $|V_1| + |V_2|$.

Lemma

If the Label Cover instance has no labeling that satisfies more than $\frac{2}{l^2}$ fraction of edges, then the Set Cover instance has no set cover of size $\leq \frac{l}{8}(|V_1| + |V_2|)$.

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Proof.

- Given an instance of the GapLabelCover problem $(G = (V_1, V_2, E), \Sigma, \Pi)$ where $|\Sigma| = m$, we use an (m, l) -set system with universe B to construct the following instance of Set Cover.
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- If all edges are satisfied, then there is a set cover of size $(|V_1| + |V_2|)$ and if at most $\frac{1}{\log^3 n} \leq \frac{2}{\log^2 n}$ edges are satisfiable, then there is a set cover of size at least $\frac{\log n}{8}(|V_1| + |V_2|)$.



PCP Theorem and Hardness of Approximation

Set cover

Definition (MAX-CUT)

Given a graph $G = (V, E)$ and a weight function $w : E \rightarrow \mathbb{R}^+$ such that for each edge (u, v) has a weight w_{uv} . Find a cut $S \subset V$ that maximises

$$\sum_{u \in S, v \in \bar{S}, (u,v) \in E} w_{uv}.$$

Theorem

There exists an efficient approximation algorithm with approximation ratio $1/2$.

- Claim 1: The following quadratic program is equivalent to the MAX-CUT problem:

$$\begin{array}{ll} \text{Maximize} & \sum_{(i,j) \in E} w_{ij} \cdot \frac{(1 - x_i x_j)}{2} \\ \text{Subject to} & x_i \in \{+1, -1\} \quad \forall i \in V \end{array}$$

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- Consider the following **semi-definite program (SDP)** that is a relaxed version of the quadratic program.

$$\begin{aligned} \text{Maximize} \quad & \sum_{(i,j) \in E} w_{ij} \frac{(1 - \langle u_i, u_j \rangle)}{2} \\ \text{Subject to} \quad & \langle u_i, u_i \rangle = 1 \quad \forall i \in V \end{aligned}$$

- Claim 2: Let O_{quad} denote the optimal solution to the quadratic program and O_{sdp} denote the optimal solution to the SDP. Then $O_{sdp} \geq O_{quad}$.

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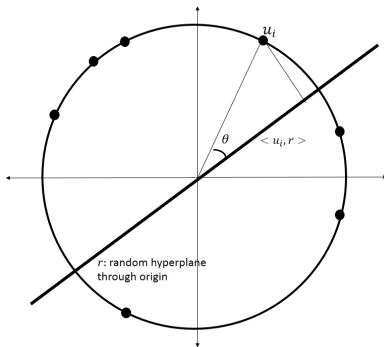
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- How do we use the vectors u_i to obtain integer solution to the MAX-CUT problem?
 - Note that these vectors are located on a unit size *ball* centred at the origin.

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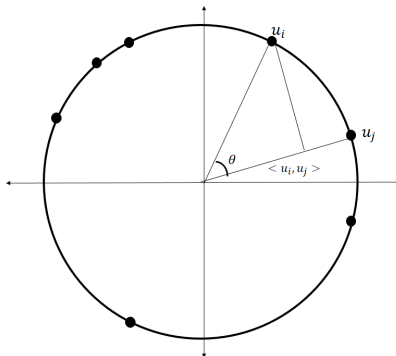


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- Consider the cut $S = \{i : \langle r, u_i \rangle \geq 0\}$ and $\bar{S} = \{i : \langle r, u_i \rangle < 0\}$.
- Consider any edge $(i, j) \in E$. What is the probability that this edge is a cut-edge? $\frac{\theta}{\pi} = \frac{\cos^{-1} \langle u_i, u_j \rangle}{\pi}$.
- The expected value W of the cut can be calculated in the following manner:

$$\begin{aligned} W &= \sum_{(i,j) \in E} w_{ij} \cdot \Pr[(i,j) \text{ is in the cut}] \\ &= \sum_{(i,j) \in E} w_{ij} \cdot \frac{\cos^{-1} \langle u_i, u_j \rangle}{\pi} \\ &\geq \alpha \cdot \sum_{(i,j) \in E} w_{ij} \cdot \frac{(1 - \langle u_i, u_j \rangle)}{2} \quad (\text{where } \alpha = \min_{-1 \leq \rho \leq 1} \frac{\cos^{-1} \rho / \pi}{(1 - \rho)/2}) \\ &= \alpha \cdot O_{sdp} \geq \alpha \cdot OPT \end{aligned}$$

- Claim: $\alpha = 0.87856$.
- So, we get a randomised algorithm that gives α approximation to the MAX-CUT problem.

End