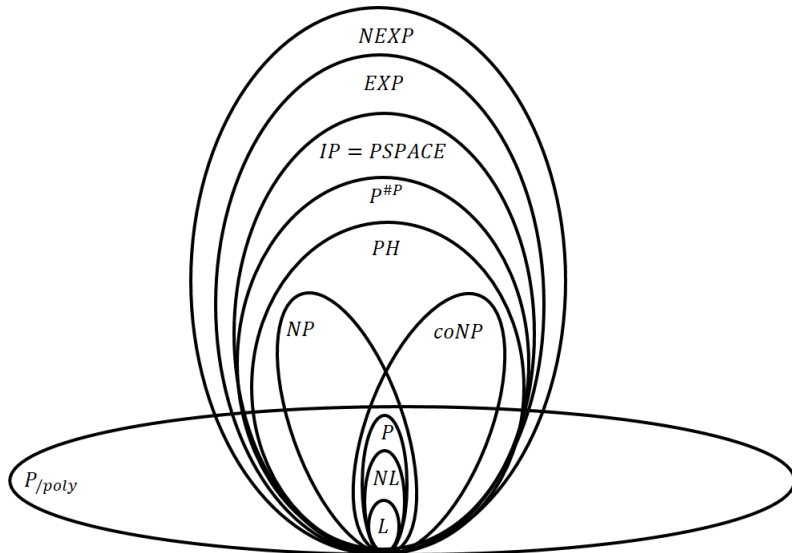


CSL853: Complexity Theory

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Complexity Diagram

Our current view of complexity



PCP Theorem and Hardness of Approximation

PCP Theorem and Hardness of Approximation

Hardness of approximation

- One way to deal with NP-hard optimization problems is to design approximation algorithms.
- We will work with the following NP-hard optimization problems.
 - MAX-3SAT: Given a Boolean formula in 3 CNF form, output an assignment that makes the maximum number of clauses true.
 - MIN-VC: Given a graph G , find a vertex cover of G with minimum cardinality.
 - MAX-IS: Given a graph G , find an independent set of G with maximum cardinality.
 - MIN-SC: Given a set S and subsets $S_1, \dots, S_m$, find the set cover with minimum cardinality.
 - MAX-CUT: Given a graph G , find a partition of the vertices such that the number of edges across the partition is maximized.

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- There is a $7/8$ -approximation algorithm for MAX-3SAT.

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 - MAX-CUT: Given a graph G , find a partition of the vertices such that the number of edges across the partition is maximized.
- There is a $7/8$ -approximation algorithm for MAX-3SAT.
- Is there an efficient algorithm with better approximation factor than $7/8$?

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 - MAX-CUT: Given a graph G , find a partition of the vertices such that the number of edges across the partition is maximized.
- There is a 2-approximation algorithm for MIN-VC.
- Is there an efficient algorithm with better approximation factor than 2?

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 - MAX-CUT: Given a graph G , find a partition of the vertices such that the number of edges across the partition is maximized.
- Claim 1: C is a vertex cover of any graph $G = (V, E)$ if and only if $V \setminus C$ is an independent set of G .
- Does the above claim mean that there is a constant factor approximation algorithm for MAX-IS?
- What is the best approximation algorithm that you can think of for MAX-IS?

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 - MAX-CUT: Given a graph G , find a partition of the vertices such that the number of edges across the partition is maximized.
- There is an $O(\log n)$ -approximation algorithm for MIN-SC.
- Is there an efficient algorithm with better approximation factor than $O(\log n)$?

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 - MAX-CUT: Given a graph G , find a partition of the vertices such that the number of edges across the partition is maximized.
- There is a 2-approximation algorithm for MAX-CUT.
- Is there an efficient algorithm with better approximation factor than 2?

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 - MAX-CUT: Given a graph G , find a partition of the vertices such that the number of edges across the partition is maximized.
- How do we show that there cannot exist an efficient algorithm for a problem that gives approximation factor better than some quantity α ?

PCP Theorem and Hardness of Approximation

Hardness of approximation

- How do we show that there cannot exist an efficient algorithm for a problem that gives approximation factor better than some quantity α ?

An idea

Suppose there exists a efficiently computable function f that takes as input Boolean formulas and outputs 3-CNF formulas such that the following holds:

$$\psi \in \text{SAT} \Rightarrow f(\psi) \in \text{SAT}, \text{ and}$$

$$\psi \notin \text{SAT} \Rightarrow < \alpha\text{-fraction of clauses in } f(\psi) \text{ are satisfiable}$$

Then there cannot exist an α -approximation algorithm for MAX-3SAT, unless $P = NP$.

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- We may further extend such *inapproximability* to other optimization problems using **gap preserving reductions**.
 - For instance, we can argue that if such a function f exists, then there cannot exist an α -approximation algorithm for MAX-IS, unless $P = NP$.

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Then there cannot exist an α -approximation algorithm for MAX-3SAT, unless $P = NP$.

- Does such a function f exist?

End