

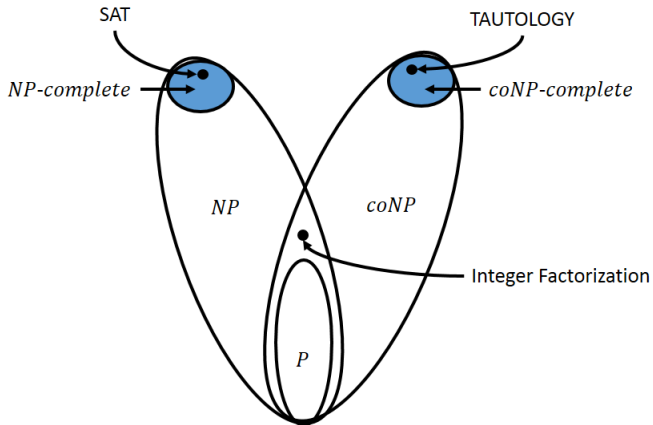
CSL853: Complexity Theory

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Complexity Diagram

Complexity Diagram

Our current view of complexity classes



Complexity class EXP and NEXP

Definition (EXP and NEXP)

$\text{EXP} = \cup_{c \geq 0} \text{DTIME}(2^{n^c})$.

$\text{NEXP} = \cup_{c \geq 0} \text{NTIME}(2^{n^c})$.

- What is the relationship between classes P, NP, EXP, and NEXP?

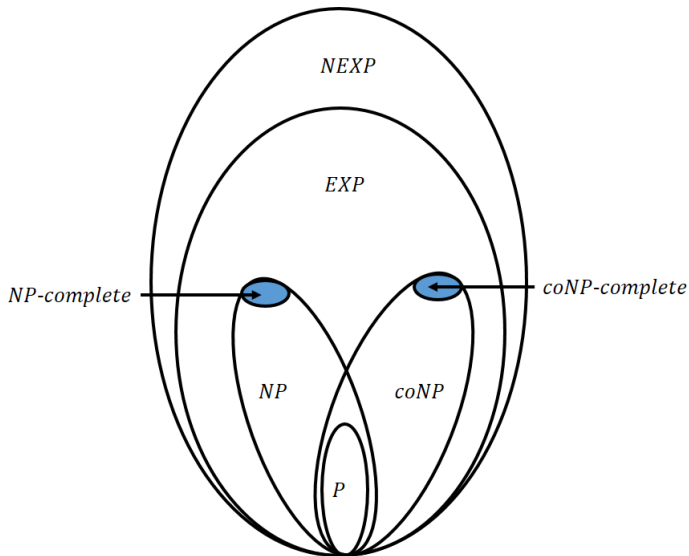
Theorem (Padding argument)

If $\text{EXP} \neq \text{NEXP}$, then $\text{P} \neq \text{NP}$.

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Our current view of complexity classes



What if $P = NP$?

Diagonalization: A tool for separating Complexity Classes

Diagonalization

The Time Hierarchy

Theorem (Time Hierarchy Theorem, Hartmanis and Stearns, 1965)

If f and g are time-constructible functions satisfying $f(n) \log f(n) = o(g(n))$, then

$$\text{DTIME}(f(n)) \subsetneq \text{DTIME}(g(n)).$$

- We will prove the following simpler lemma.

Lemma

$$\text{DTIME}(n) \subsetneq \text{DTIME}(n^{1.5}).$$

Diagonalization

The Time Hierarchy

Lemma

$$\text{DTIME}(n) \subsetneq \text{DTIME}(n^{1.5}).$$

Proof.

- Consider the following TM D : On any input x , run the universal TM U to simulate M_x on x for $|x|^{1.4}$ steps. If the simulation ends before $|x|^{1.4}$ steps and produces output b , then D outputs $(1 - b)$, otherwise D outputs 0.

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- Let L_D denote the language decided by D .
- Claim 1: $L_D \in \text{DTIME}(n^{1.5})$.
- Claim 2: $L_D \notin \text{DTIME}(n)$.

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 - Assume otherwise. Let M be the machine that decides L_D in n -time.

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 - Assume otherwise. Let M be the machine that decides L_D in n -time.
 - U simulates M in $(c \cdot n \log n)$ -time.

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 - For a large enough n_0 , for all $n \geq n_0$, we have $n^{1.5} \geq cn \log n$.

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 - Let x be the encoding of M such that $|x| \geq n_0$.
 - But then $D(x) \neq M(x)$ which is a contradiction. □

Diagonalization

The Time Hierarchy

Theorem (Time Hierarchy Theorem, Hartmanis and Stearns, 1965)

If f and g are time-constructible functions satisfying $f(n) \log f(n) = o(g(n))$, then

$$\text{DTIME}(f(n)) \subsetneq \text{DTIME}(g(n)).$$

- Some interesting corollaries:
 - $\text{DTIME}(n) \subsetneq \text{DTIME}(n \log^2 n) \subsetneq \text{DTIME}(n^2)$.
 - $P \subsetneq \tilde{P} \subsetneq \text{EXP}$
 - Here $\tilde{P} = \bigcup_{c \geq 0} \text{DTIME}(2^{\log^c n})$ (referred as “quasi-polynomial time”).

Diagonalization

The Time Hierarchy

Theorem (Nondeterministic Time Hierarchy Theorem, Cook, 1972)

If f and g are time-constructible functions satisfying $f(n+1) = o(g(n))$, then $\text{NTIME}(f(n)) \subsetneq \text{NTIME}(g(n))$.

- Again, we will show that following simpler lemma.

Lemma

$\text{NTIME}(n) \subsetneq \text{NTIME}(n^{1.5})$.

Diagonalization

The Time Hierarchy

Lemma

$$\text{NTIME}(n) \subsetneq \text{NTIME}(n^{1.5}).$$

Proof.

- For every $i \in \mathbb{N}$, let J_i be the interval $(l_i, u_i]$, where $u_i = 2^{l_i^{1.2}}$, $l_1 = 2$, and $l_{i+1} = u_i$.
- Define NDTM D on input x as follows:
 - If $x \neq 1^n$ for some $n \in \mathbb{N}$, then reject.
 - Let $n \in J_i$.
 - (a) If $n < u_i$, then simulate the non-deterministic universal TM on M_i and 1^{n+1} for $n^{1.1}$ steps and output the result of the simulation.
 - (b) If $n = u_i$, then *deterministically simulate* M_i on $1^{l_{i+1}}$, and accept iff the simulation rejects in $(l_i + 1)^{1.1}$ time.

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$\text{NTIME}(n) \subsetneq \text{NTIME}(n^{1.5})$.

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- Claim 1: $L_D \in \text{NTIME}(n^{1.5})$.
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- Claim 1: $L_D \in \text{NTIME}(n^{1.5})$.
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 - Assume otherwise. Let M be the NDTM that decides L_D in n -time.
 - Let k be sufficiently large such that $M_k = M$ and all simulations of D can be done in $n^{1.1}$ time for any inputs 1^n such that $n \in J_k$.

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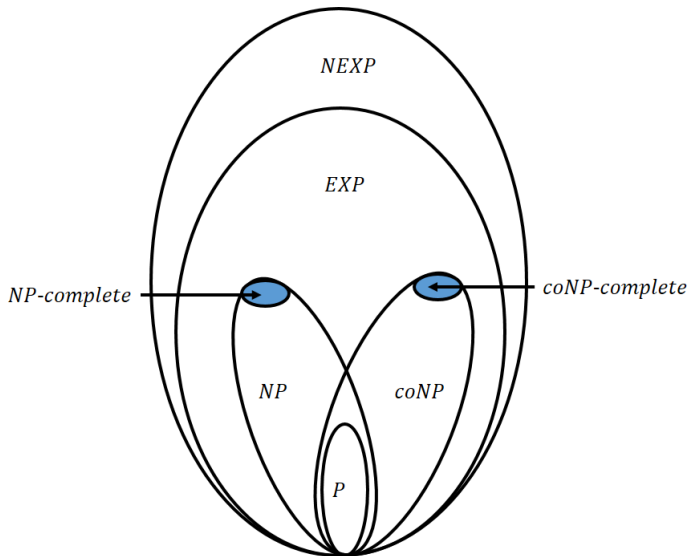
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 - Let k be sufficiently large such that $M_k = M$ and all simulations of D can be done in $n^{1.1}$ time for any inputs 1^n such that $n \in J_k$.
 - Then,
 $D(1^{l_k+1}) = M(1^{l_k+1})$, $D(1^{l_k+2}) = M(1^{l_k+2})$, ..., $D(1^{u_k}) = M(1^{u_k})$ and
 $D(1^{l_k+1}) = M(1^{l_k+2})$, $D(1^{l_k+2}) = M(1^{l_k+3})$, ..., $D(1^{u_k-1}) = M(1^{u_k})$.
 - But $D(1^{u_k}) \neq M(l_k + 1)$ which gives a contradiction. $\square$

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Our current view of complexity classes



Diagonalization: NP-Intermediate Problems

Theorem (NP-Intermediate Languages, Ladner, 1975)

Suppose that $P \neq NP$. Then there exists a language $L \in NP \setminus P$ that is not NP-complete.

Proof (based on unpublished proof by Impagliazzo).

- For any $H : \mathbb{N} \rightarrow \mathbb{N}$, define
$$\text{SAT}_H = \{\psi 01^{n^{H(n)}} \mid \psi \in \text{SAT and } n = |\psi|\}.$$
- Define $H : \mathbb{N} \rightarrow \mathbb{N}$ in the following manner:
 - $H(n)$ is the smallest number $i < \log \log n$ such that for every $x \in \{0,1\}^*$ with $|x| \leq \log n$, M_i outputs $\text{SAT}_H(x)$ within $i \cdot |x|^i$ steps. If there is no such number i , then $H(n) = \log \log n$.

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- Claim 1: $\text{SAT}_H \in P$ if and only if $H(n) = O(1)$. Moreover, if $\text{SAT}_H \notin P$, then $H(n)$ tends to infinity with n .

End