

CSL 857: Randomized Algorithms

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Martingales

Sequence of random variables satisfying certain conditions that arise in applications such as random walks and gambling problems

Martingales: Introduction

- Definition(Martingale): A sequence of random variables $Z_0, Z_1, \dots$ is a martingale with respect to the sequence $X_0, X_1, \dots$ if for all $n \geq 0$, the following conditions hold:
 1. Z_n is a function of $X_0, X_1, \dots, X_n$;
 2. $E[|Z_n|] < \infty$;
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- Example: A gambler plays a fair game. Let X_i denote the amount the gambler wins in the i^{th} game. Let Z_i denote the total amount won by the gambler after the i^{th} game.
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 - $\forall i, E[X_i] = 0$;
 - $E[Z_{i+1} | X_1, \dots, X_i] = Z_i + E[X_{i+1}] = Z_i$.

Martingales: Useful results

There are 3 main results that we will look at with respect to Martingales:

1. Martingale stopping theorem;
2. Wald's equation;
3. Azuma-Hoeffding inequality.

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- Problem: Consider the gambling problem. The gambler decides that he will quit after playing k games. What is the expected winnings?
 - Expected winnings = 0.
- Claim: If the sequence $Z_0, \dots, Z_n$ is a martingale w.r.t. $X_0, \dots, X_n$, then $E[Z_n] = E[Z_0]$.

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- Proof:
 - $Z_i = E[Z_{i+1} | X_0, \dots, X_i]$.
 - $E[Z_i] = E[E[Z_{i+1} | X_0, \dots, X_i]] = ?$

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- Proof:
 - $Z_i = E[Z_{i+1} | X_0, \dots, X_i]$.
 - So, $E[Z_i] = E[E[Z_{i+1} | X_0, \dots, X_i]] = E[Z_{i+1}]$.
 - So, $E[Z_n] = E[Z_0] = E[X_0] = 0$.

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- Suppose the time T at which the gambler quits depends on the outcome of the games:
 - The gambler stops if his winnings is more than 1000.
 - The gambler stops after the first time he wins 5 games in a row.
- Definition(Stopping time): A non-negative random variable T is a stopping time for the sequence $\{Z_n, n \geq 0\}$ if the event $T = n$ depends only on the random variables $Z_0, \dots, Z_n$.

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- Examples:
 1. T is the first time the gambler has won more than Rs. 1000.
 2. T is the first time the gambler has won 5 games in a row.
- Non-examples:
 1. T is the last time the gambler has won 5 games in a row.

Martingales: Stopping Theorem

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- Theorem(Martingale Stopping Theorem): If $Z_0, Z_1 \dots$ is a martingale with respect to $X_1, X_2, \dots$, then $E[Z_T] = E[Z_0]$ whenever one of the following holds:
 1. The Z_i 's are bounded, so there is a constant C such that, for all i , $|Z_i| \leq C$;
 2. T is bounded;
 3. $E[T] < \infty$, and there is a constant C such that $E[|Z_{i+1} - Z_i| | X_1, \dots, X_i] < C$.

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- Problem: Consider the gambling problem. Suppose the gambler gets \$1 for a win and loses \$1 for a loss. Let T be the first time the gambler has either won $\$l_2$ or lost $\$l_1$. Assuming that the gambler quits after T games, what is the probability that the player wins $\$l_2$ before losing $\$l_1$?

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- Problem 1: Consider the gambling problem. Suppose the gambler gets \$1 for a win and loses \$1 for a loss. Let T be the first time the gambler has either won $\$l_2$ or lost $\$l_1$. Assuming that the gambler quits after T games, what is the probability that the player wins $\$l_2$ before losing $\$l_1$?
- Solution: Z_i 's are bounded, we may apply the Stopping Theorem. Let q be the probability that the gambler wins $\$l_2$. So, $E[Z_T] = 0 = q \cdot l_2 - (1 - q) \cdot l_1 \Rightarrow \frac{l_1}{l_1 + l_2}$.

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- Corollary(Wald's equation): Let $X_1, X_2, \dots$ be non-negative, independent, identically distributed random variables with distribution X . Let T be a stopping time for this sequence. If T and X have bounded expectation, then
$$E\left[\sum_{i=1}^T X_i\right] = E[T] \cdot E[X].$$

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- Proof:

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- Problem: You roll a fair dice. Let X be the outcome. After this, you roll the dice X times independently take the sum of the outcomes. What is the expected value of the sum?

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- Problem: You roll a fair dice. Let X be the outcome. After this, you roll the dice X times independently take the sum of the outcomes. What is the expected value of the sum?
- Solution: Let $Y_1, \dots, Y_X$ be the random variables denoting the outcomes of the rolls in the second round. Note that X is the stopping time. From Wald's equation, we get that:

$$E\left[\sum_{i=1}^X Y_i\right] = E[X] \cdot E[Y_1] = \left(\frac{7}{2}\right) \cdot \left(\frac{7}{2}\right) = \frac{49}{4}.$$

Azuma-Hoeffding Inequality

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- Theorem(Azuma-Hoeffding Inequality): Let $X_0, \dots, X_n$ be a martingale such that $|X_k - X_{k-1}| \leq c_k$. Then for all $t \geq 0$ and any $\lambda > 0$,

$$\Pr[|X_t - X_0| \geq \lambda] \leq 2 \cdot e^{-\frac{\lambda^2}{2 \sum_{k=1}^t (c_k)^2}}$$

- Corollary: Let $X_0, \dots, X_n$ be a martingale such that $|X_k - X_{k-1}| \leq c$. Then for all $t \geq 1$ and any $\lambda > 0$,

$$\Pr[|X_t - X_0| \geq \lambda c \sqrt{t}] \leq 2 \cdot e^{-\frac{\lambda^2}{2}}$$

End
