

# CSL 857: Randomized Algorithms

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# Lovász Local Lemma (LLL)

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# LLL: Motivation

- Suppose there are independent events  $E_1, \dots, E_m$  such that the probability of any event  $E_i < 1$ . Then the following holds true:

$$\Pr[\wedge_{i=1}^m \neg E_i] = ?$$

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$$\Pr[\bigwedge_{i=1}^m \neg E_i] = \prod_{i=1}^m (1 - \Pr[E_i]) > 0.$$

- Can we say a similar statement even when the events are not completely independent but have some independence in some limited sense.

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- Warm-up exercise: Given a 3-CNF Boolean formula with each variable having a frequency  $\leq 3$ , show that this formula is Satisfiable.

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- Main idea: Let us randomly assign 0/1 values to the variables and see whether there is a non-zero probability that such a boolean formula is true for a random assignment.
- Let  $E_i$  denote the event that the  $i^{th}$  clause is not satisfied.
- Claim 1:  $\forall i, \Pr[E_i] = ?$

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- First attempt using union bound:  
$$\Pr[\bigwedge_{i=1}^m \neg E_i] = 1 - \Pr[\bigvee_{i=1}^m E_i] \geq 1 - \sum_{i=1}^m \Pr[E_i] = 1 - m \cdot 2^{-k}.$$

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  - This attempt fails if  $m > 2^k$ .
- Degree of dependence: Each event is independent of all but at most  $2^k/e$  events.
  - Question: Does this help?

# LLL: Statement

- Lovász Local Lemma (LLL): Let  $G = (V, E)$  denote the dependency graph for events  $E_1, \dots, E_m$ . Suppose there exists  $x_i \in [0, 1]$  for all  $0 \leq i \leq m$  such that

$$\forall i, \Pr[E_i] \leq x_i \cdot \prod_{(i,j) \in E} (1 - x_j),$$

then

$$\Pr[\bigwedge_{i=1}^m \neg E_i] \geq \prod_{i=1}^m (1 - x_i).$$

- Dependency graph: A dependency graph for dependent events  $E_1, \dots, E_m$  is defined as follows: there is a node  $i$  for each event  $E_i$  and there is an edge  $(i, j)$  for every pair of dependent events  $(E_i, E_j)$ .
  - Note that there may be an edge between two independent events.

# LLL: Proof

- Claim: Let  $S \subset \{1, \dots, m\}$  and  $i \notin S$ , then

$$\Pr[E_i | \bigwedge_{j \in S} \neg E_j] \leq x_i.$$

- Proof of LLL (using above claim):

$$\begin{aligned} \Pr[\bigwedge_{i=1}^m \neg E_i] &= \\ (1 - \Pr[E_1 | \bigwedge_{i=2}^m \neg E_i]) \cdot \\ (1 - \Pr[E_2 | \bigwedge_{i=3}^m \neg E_i]) \dots (1 - \Pr[E_m]) \\ &\geq \prod_{i=1}^m (1 - x_i). \end{aligned}$$

# LLL: A useful corollary

- LLL Corollary: Let  $0 < p < 1$ . Let  $E_1, \dots, E_m$  be some events in some probability space such that  $\forall i, \Pr[E_i] \leq p$ . If each event is mutually independent of all other events except for at most  $d$  events, and if  $e \cdot p \cdot (d + 1) \leq 1$ , then  $\Pr[\bigwedge_{i=1}^m \neg E_i] > 0$ .

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- Problem: Consider boolean formulas in  $k$ -CNF (i.e., each clause has exactly  $k$  literals), and that the frequency of each variable is  $\leq \frac{2^k}{ek}$ . Show that any such formula is satisfiable.
- Let  $E_i$  denote the event that clause  $i$  is not satisfied (for random assignment). The degree of dependence is at most  $\frac{2^k}{e} - (k - 1)$  and we have  $e \cdot 2^{-k} \cdot \left( \frac{2^k}{e} - (k - 1) \right) \leq 1$  for  $k \geq 3$ . So,  $\Pr[\bigwedge_{i=1}^m \neg E_i] > 0$ .

End

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