

# CSL 857: Randomized Algorithms

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# Administrative stuff

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- **Lectures:**
  - *Location:* CSE Seminar room.
  - *Time:* TBD.

# Warm-up

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# Linearity of Expectations

- Linearity of expectations: For any random variables  $\mathbf{X}_1, \mathbf{X}_2$  and constants  $\mathbf{c}_1, \mathbf{c}_2$  we have

$$\mathbf{E}[\mathbf{c}_1 \cdot \mathbf{X}_1 + \mathbf{c}_2 \cdot \mathbf{X}_2] = \mathbf{c}_1 \cdot \mathbf{E}[\mathbf{X}_1] + \mathbf{c}_2 \cdot \mathbf{E}[\mathbf{X}_2]$$

- Hat-Check Problem:  $\mathbf{n}$  men go to a party and their hats get mixed up. They randomly pick up a hat. What is the expected number of men who get their own hats?

# Linearity of Expectations

- Coupon collector problem: There are  $n$  different type of coupons. You get a random coupon every day. What is the expected number of days in which you should collect at least one coupon of each type?

# Linearity of Expectations

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- Define  $i^{\text{th}}$  epoch to be the sequence of days starting the day after the  $(i-1)^{\text{th}}$  new coupon was collected and ending on the day the  $i^{\text{th}}$  coupon was collected.
- Define  $X_i$  to be a random variable denoting the number of days in the  $i^{\text{th}}$  epoch. Note that  $X_1=1$ .
- We are interested in knowing expected value of
$$\mathbf{X} = \mathbf{X}_1 + \mathbf{X}_2 + \dots + \mathbf{X}_n$$

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- What is the value of  $E[\mathbf{X}_i]$ ?

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- We are interested in knowing expected value of

$$\mathbf{X} = \mathbf{X}_1 + \mathbf{X}_2 + \dots + \mathbf{X}_n$$

- What is the value of  $E[\mathbf{X}_i]$ ?

- $E[\mathbf{X}_i] = n / (n - i + 1)$ .

- $$\begin{aligned} E[\mathbf{X}] &= E[\mathbf{X}_1] + E[\mathbf{X}_2] + \dots + E[\mathbf{X}_n] \\ &= n \cdot (1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}) \\ &= n \cdot H_n \\ &= O(n \cdot \ln n) \end{aligned}$$

# Deviation from Expectation

- Markov's Inequality: Let  $\mathbf{X}$  be a non-negative random variable and  $\mathbf{a} > \mathbf{0}$ , then

$$\Pr[\mathbf{X} \geq \mathbf{a}] \leq \mathbf{E}[\mathbf{X}] / \mathbf{a}.$$

- Corollary: Let  $\mathbf{X}$  be a non-negative random variable and  $\mathbf{c} \geq 1$ , then

$$\Pr[\mathbf{X} \geq \mathbf{c} \cdot \mathbf{E}[\mathbf{X}]] \leq 1 / \mathbf{c}.$$

# Deviation from Expectation

- Hat-check problem: What is the probability that at least **10** people out of **n** get their own hats?
  - $E[X] = 1$ , so from Markov we get  $\Pr[X \geq 10] \leq 0.1$ .
- Note:
  - $\Pr[\text{everyone gets their own hats}] = 1/n!$
  - On the other hand from Markov, we get that
$$\Pr[X \geq n] \leq 1/n.$$

# Deviation from Expectation

- Chebychev's Inequality: Let  $\mathbf{X}$  be a random variable and  $\mathbf{a} > \mathbf{0}$ , then

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- Define the random variable  $\mathbf{Y} = (\mathbf{X} - \mathbf{E}[\mathbf{X}])^2$ .
- Lemma 1:  $\mathbf{E}[\mathbf{Y}] = \mathbf{E}[\mathbf{X}^2] - \mathbf{E}[\mathbf{X}]^2 = \mathbf{Var}[\mathbf{X}]$

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- Proof of Chebychev's inequality:

- $$\begin{aligned} \Pr[|\mathbf{X} - \mathbf{E}[\mathbf{X}]| \geq \mathbf{a}] &= \Pr[\mathbf{Y} \geq \mathbf{a}^2] \\ &\leq \mathbf{E}[\mathbf{Y}] / \mathbf{a}^2 && \text{(by Markov)} \\ &\leq \mathbf{Var}[\mathbf{X}] / \mathbf{a}^2 && \text{(by Lemma 1)} \end{aligned}$$

# Deviation from Expectation

- Birthday Problem: You uniformly sample  $q$  items with replacement from a collection of  $n$  items. What is the probability that two items are the same?
- Birthday Problem(popular version): There are  $q$  people in a room. What is the value of  $q$  such that the probability of two people having the same birthday is at least  $\frac{1}{2}$ . Each person's birthday is assumed to be a random day in the year.

End

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