CSL 857: Randomized Algorithms (CSE, IIT Delhi, Semester II, 2011-12) Major Exam (Take-home)

Name: _____

Entry number: _____

There are 3 questions for a total of 30 points.

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(10) 1. Show that the cover time for any regular graph is $O(n^2 \log n)$.

(10) 2. Let G be a d-regular, connected, non-bipartite graph. Let M denote the adjacency matrix of G and let $Q = (I + M/d)/2$. Note that Q is the transition matrix of the lazy random walk on G. For any non-empty subset of vertices $S \subseteq V$ and $\bar{S} = V \setminus S$, let $E(S, \bar{S})$ denote the set of edges that is crosses the cut $(S, \bar{S})$. Let $\phi_S = \frac{1}{2d} \cdot \frac{|E(S,\bar{S})|}{\min(|S|,|\bar{S}|)}$ and let $\phi = \min_S \phi_S$. The following interesting fact relates the eigenvalue gap of Q with ϕ defined previously (ϕ denotes something known as the *conductance* of the graph).

Fact: $\lambda_2 \leq 1 - \phi^2/2$ (where λ_2 is the second eigen value of Q.)

We know from the discussion in class that if the eigenvalue gap $(1 - \lambda_2)$ is large, then the Markov chain is rapidly mixing. From the above fact, it will be sufficient to give a lower bound on ϕ in order to argue that the mixing time for Q is small. Show the following:

(a) Show that $\tau(\epsilon) = O\left(\frac{\ln(2n/\epsilon)}{\phi^2}\right)$.

Here is a simple idea to obtain a lower bound on ϕ. Define a method that gives a path between any pair (x, y) of vertices. Let us call such paths standard paths (w.r.t. the defined method). For any edge e in the graph, the congestion of e, $C(e)$, is defined to be the number of standard paths that include e. Suppose by some means we are able to prove the following statement:

$$\forall e, C(e) \leq c$$

Then we have $\forall S, |S| \cdot |\bar{S}| \leq |E(S, \bar{S})| \cdot c$. This gives us $\frac{|E(S,\bar{S})|}{\min(|S|,|\bar{S}|)} \geq \frac{n}{2c}$ and this implies that $\phi \geq \frac{n}{4cd}$.

Let us use this idea to bound the mixing time of the lazy random walk on a hypercube that we discussed in class:

> Pick an index $i \in \{1, 2, ..., n\}$ randomly. With probability $1/2$ each make the value of the i^{th} bit 0 or 1.

(b) Using the bit-fixing path as the standard path between vertices, obtain a bound on congestion and use it to get a bound on ϕ and then $\tau(\epsilon)$.

(10) 3. Let $G = (X, Y, E)$ be a bipartite graph such that $|X| = |Y| = n$ and there exists a perfect matching in G. Let M_n denote the set of matchings of size n and let M_{n-1} denote the set of matchings of size $(n-1)$. Consider the following Markov Chain C defined on the state space $\Omega = M_n \cup M_{n-1}$:

> Let X_t denote the current state (note X_t is a matching). Flip a fair coin. If the coin turns heads $X_{t+1} = X_t$. Else, pick an edge $e = (u, v)$ uniformly at random from E and do the following:
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> - If $X_t \in M_n$ and $e \in X_t$, then $X_{t+1} = X_t - \{e\}$.
> - If $X_t \in M_{n-1}$ and both u and v are not part of the matching in X_t, then $X_{t+1} = X_t \cup \{e\}$.
> - If $X_t \in M_{n-1}$ and exactly one of u or v is part of the matching in X_t, then $X_{t+1} = X_t - \{e'\} \cup \{e\}$. Here e' is the edge that is part of the matching X_t with one end point either u or v.
> - If all the above cases do not apply, $X_{t+1} = X_t$

(a) Show that the above markov chain is irreducible. What is the stationary distribution of this Markov Chain?