

Name: _____

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There are 3 questions for a total of 30 points.

- (10) 1. Let $m = (100 \cdot n \cdot \ln n)$ balls be thrown randomly into n bins. Show that the load of the heaviest and the lightest bins differ by a constant multiplicative factor with probability at least $(1 - 1/n)$.

Solution: Fix a bin i . Let X_j denote the indicator random variable that is 1 if the j^{th} ball is assigned the i^{th} bin and 0 otherwise. We have $\Pr[X_j = 1] = 1/n$. Let $X = \sum_j X_j$. So, X denotes the number of balls in bin i . We have $E[X] = 100 \cdot \ln n$ and from Chernoff bounds we have:

$$\Pr[X \geq 150 \cdot \ln n] \leq \frac{1}{2n^2} \quad \text{and} \quad \Pr[X \leq 50 \cdot \ln n] \leq \frac{1}{2n^2}.$$

The probability that there exists a bin that does not have number of balls in the range $[50 \cdot \ln n, 150 \cdot \ln n]$ is upper bounded by $1/n$. So, with probability at least $(1 - 1/n)$, the load of the heaviest and the lightest bins is bounded by 3.

- (10) 2. Show that there are $O(n^2)$ distinct min-cuts in any graph. Extend your argument to show that there are $O(n^{2\alpha})$ distinct cuts of size at most $\alpha > 1$ times the size of the min-cut.

Solution: Consider the randomized shrinking procedure defined in class. For any fixed min-cut of the graph, the probability that this cut survives is $\Omega(1/n^2)$. The events that two different cuts survive, are disjoint. This gives us that the number of min-cuts in any connected graph is $O(n^2)$.

Let M denote the size of the min cut of the graph. For the second part, consider the randomized procedure that shrinks edges till the number of vertices is 2α . For any fixed cut C of size at most $\alpha \cdot M$, the probability that all edges in C are present in the shrunk graph of size 2α is at least:

$$\begin{aligned} \left(1 - \frac{2\alpha}{n}\right) \cdot \left(1 - \frac{2\alpha}{n-1}\right) \dots \left(1 - \frac{2\alpha}{2\alpha+1}\right) &= \frac{1 \cdot 2 \dots (2\alpha)}{n \cdot (n-1) \dots (n-2\alpha+1)} \\ &= \frac{1}{\binom{n}{2\alpha}} \geq (2\alpha/en)^{2\alpha} \\ &= \Omega(1/n^{2\alpha}) \end{aligned} \tag{1}$$

Let $\mathcal{S}$ be the set of all multi-graphs S containing 2α vertices that can be obtained by shrinking vertices of the original graph. Let p_S denote the probability that a multi-graph S survives after the shrinking process. Given this, we have:

$$\sum_{S \in \mathcal{S}} p_S = 1. \tag{2}$$

Consider a bipartite graph where the vertices on the left are all the cuts of size at most $\alpha \cdot M$ and the vertices on the right are all the multi-graphs with 2α vertices. There is an edge from a vertex u_C on the left to a vertex v_S on the right if all the edges of the cut C represented by u_C , are present (as non-self-loops) in the multi-graph S represented by v_S . We give a weight of p_S to this edge. Consider any vertex u_C on the left. Suppose it has edges to vertices $v_{S_{j_1}}, \dots, v_{S_{j_r}}$, then we know from (1) that $p_{S_{j_1}} + \dots + p_{S_{j_r}} = \Omega(1/n^{2\alpha})$. So the sum of weights of all edges in the bipartite graph is $\Omega(|L|/n^{2\alpha})$, where L is the set of vertices on the left which means that $|L|$ is the number of cuts of size at most

$M \cdot \alpha$ in the graph. Now each vertex on the right has a degree at most $2^{2\alpha}$ (the number of possible partitions of 2α vertices). So we get that the sum of weights of edges is at most $2^{2\alpha} \cdot \sum_{S \in \mathcal{S}} p_S$. This is at most $2^{2\alpha}$ from equation (2). This gives us that $|L| = O(n^{2\alpha} \cdot 2^{2\alpha}) = O(n^{2\alpha})$.

- (10) 3. n pair of users need to communicate using edge-disjoint paths (paths that do not share any edge) on a given network. Each pair $i = 1, \dots, n$ can choose a path from a collection F_i of m paths. Given that any path in F_i shares edges with no more than k paths in F_j for all $i \neq j$, and $8nk \leq m$, show that there is a way to choose n edge-disjoint paths connecting n pairs.

Solution: Consider the randomized procedure that picks one path randomly from each F_i . Let $E_{i,j}$ denote the event that the path between the i^{th} and the j^{th} pair share an edge. We know that $Pr[E_{i,j}] \leq k/m$. For any two events $E_{i,j}$ and $E_{i',j'}$, these events are independent if $i' \notin \{i, j\}$ and $j' \notin \{i, j\}$. So the degree of dependence $d = 2n - 1$. So, we have $e \cdot k/m \cdot 2n \leq 8kn/m \leq 1$. LLL tells us that there is non-zero chance that the randomized procedure finds edge-disjoint paths between all pairs. This gives us that there are edge-disjoint paths between all pairs of vertices in the graph.