Von Neumann And Morgenstern Utility Function

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1 Risk Aversion

In last lecture we discussed the St. Petersburg Paradox, where although the expected return from the game is infinity, only a few gamblers are willing to invest large amount of money. Now we take a simplified version of the above paradox.

Consider a player being given two options : Take Rs.10 for sure or Rs.70 with probability 0.5. In this case, the player is likely to go for the second option.

Now consider the case when Rs 10 and Rs.70 were replaced by Rs. 10 million and Rs.70 million respectively. In this case, the user is likely to go for the first option. Why is it so? Why people may not want to maximize their expected gain?

Let m_0 be the initial amount of money with the user. Let $u(m)$ be a utility function representing the user's preference relation. The expected utility in each of the options in the cases discussed above is summarized below :

	Option 1	**Option 2**
Case A	$u(m_0 + 10)$	$1/2[u(m_0) + u(m_0 + 70)]$
Case B	$u(m_0 + 10^8)$	$1/2[u(m_0) + u(m_0 + 7 * 10^8)]$

Assume that the player has initially Rs. 10,000. i.e. $m_0 = 10^4$.
Consider the case when $u(m) = m$. Now the expected utilities are given in the following table :

	Option 1	**Option 2**
Case A	$10^4 + 10$	$10^4 + 35$
Case B	$10^4 + 10^8$	$10^4 + 35 * 10^7)$

Here we see that the *expected utility* is more for option 2 in both the cases. Therefore players should prefer option 2 in both the cases

In earlier lecture we saw that if a utility function $u()$ represents a preference relation R, then f(u()) also represents R, where f() is a monotonically increasing function. Therefore $u(m) = m^{1/3}$ should represent the same preference relation as $u(m) = m$. If we assume $u(m) = m^{1/3}$, then the expected utilities are given in the following table :

	Option 1	**Option 2**
Case A	$10^{4/3}[1 + \frac{1}{3000}]$	$10^{4/3}[1 + \frac{3.5}{3000}]$
Case B	$(10^4 + 10^8)^{1/3}$	$1/2[10^{4/3} + (10^4 + 7 * 10^8)^{1/3}]$

Here we see that that *expected utility* is more for option 2 in case A and is more for option 1 in case B as one would generally expect in real life. Notice that $u(m) = m^{1/3}$ does not represent the same preference relation as $u(m) = m$. This can be explained using Von Neumann and Morgenstern utility function.

2 Von Neumann and Morgenstern Postulates

Let R be a preference relation over a set of deterministic outcomes Ω and $u() : \Omega \rightarrow \Re$ be a utility function representing R. If we have probabilistic outcomes, then the preference relation R should be defined over probabilistic outcomes $\Omega' = \Re^{|\Omega|}$. Now the utility function has to be *redefined* so as to represent this new preference relation. Therefore, now $u() : \Omega' \rightarrow \Re$.

Notation : Let $(w_1(p_1), w_2(p_2), \ldots, w_m(p_m)) \in \Omega'$, where w_i is i^{th} outcome in Ω and p_i is the probability of occurrence of w_i s.t. $\sum_{i=1}^{m} p_i = 1$.

Von Neumannn and Morgenstern gave a set of rationality postulates, which define preference relation in probabilistic outcomes.

2.1 First Postulate

Consider any two outcomes (namely w_1 and w_2) from the set Ω, with outcome w_1 being preferred over outcome w_2. Let $p, q \in [0, 1]$ and $q > p$. Consider two outcomes $(w_1(p), w_2(1-p))$ and $(w_1(q), w_2(1-q))$ in Ω'. A rational player should choose second outcome over the first. Formally :

$\forall w_1, w_2 \in \Omega,\ \forall p, q \in [0, 1]$ s.t. $w_1 \succ w_2$ and $q > p$,
$\Rightarrow \quad (w_1(p), w_2(1-p)) \prec (w_1(q), w_2(1-q))$

2.2 Second Postulate

If outcome w_1 is preferred over outcome w_2 and outcome w_2 is preferred over outcome w_3, then there exists a probability p, such that the player is indifferent to the outcome $(w_1(p), w_2(1-p))$ and w_3. Formally :

If $w_1 \succ w_2 \succ w_3$, then $\exists p \in [0, 1]$ s.t. $(w_1(p), w_2(1-p)) \sim w_3$

Corollary : For w_1, w_2, w_3 defined as above, there exists a unique $p \in [0, 1]$ s.t. $(w_1(p), w_2(1-p)) \sim w_3$.
Proof : Suppose there exist $p_1, p_2 \in [0, 1]$ and $p_1 > p_2$ (without loss of generality) s.t.

$(w_1(p_1), w_2(1-p_1)) \sim w_3 \quad$ and $\quad (w_1(p_2), w_2(1-p_2)) \sim w_3$

From first postulate, we know that:

$(w_1(p_1), w_2(1-p_1)) \succ (w_1(p_2), w_2(1-p_2))$

$\Rightarrow w_3 \succ w_3$ which is a contradiction.

2.3 Third Postulate

According to this postulate, if the player is indifferent between two outcomes say w_1 and w_2, and in another outcome say w_3, w_1 happens with probability p, then the player remains indifferent if w_1 in w_3 is replaced by w_2. To state this formally, label outcomes $w_1, w_2, \ldots, w_m$ in Ω s.t. $w_1 \succ w_2 \succ \ldots w_{m-1} \succ w_m$. Consider an outcome O in Ω' given as :

$(w_1(p_1), w_2(p_2), \ldots, w_m(p_m))$

Let u_k be the probability s.t. $(a_1(u_k), a_m(1-u_k)) \sim a_k$.

If $O' = (w_1(p_1+p_ku_k), w_2(p_2), \ldots, w_{k-1}(p_{k-1}), w_k(0), w_{k+1}(p_{k+1}), \ldots, w_m(p_m + (1-u_k)p_k))$, then $O \sim O'$

3 Von Neumann And Morgenstern Utility Function

Von Neumann and Morgenstern utility function U is defined over $\Omega' = \Re^{|\Omega|}$. Let $w_1 \in \Omega$ is the *most preferred* outcome ($\forall w \in \Omega,\ w_1 \succ w$) and $w_m \in \Omega$ is the *least preferred* outcome ($\forall w \in \Omega,\ w \succ w_m$) . For each outcome $w \in \Omega$, define

$$u(w) = p \qquad s.t. \qquad w \sim (w_1(p), w_m(1-p)) \tag{1}$$

From postulate 2, such a p always exists and is unique.

Von Neumann and Morgenstern utility function for $w \in \Omega'$ is the expected value of the utility function u() as defined by equation 2. i.e.

$$U(w_1(p_1), w_2(p_2), \ldots, w_m(p_m)) = p_1u(w_1) + p_2u(w_2) + \ldots + p_mu(w_m) \tag{2}$$

To check that the Von Neumann and Morgenstern utility function represents the preference relationship on Ω', consider two outcomes O and O' in Ω' .i.e.

$O = (w_1(p_1), w_2(p_2), \ldots, w_m(p_m))$ and

$O' = (w_1(q_1), w_2(q_2), \ldots, w_m(q_m))$

By repeated application of postulate 3, we get :

$O \sim (w_1(\sum_{i=1}^{m} p_i u(w_i)), w_m(1 - \sum_{i=1}^{m} p_i u(w_i)))$ and

$O' \sim (w_1(\sum_{i=1}^{m} q_i u(w_i)), w_m(1 - \sum_{i=1}^{m} q_i u(w_i)))$

From postulate 1, it is clear that given two outcomes O and O' in Ω' (as defined above), a rational player will prefer the one, which corresponds to higher expected value of the utility function, u().

Therefore, $U : \Omega' \rightarrow \Re$ represents a preference relationship on Ω'.