Theory of Utility
Scribe notes

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Often the payoff of a game is thought of as equal to the expected value of return from the game. We start with an example which shows why expected value of a game cannot be a measure of our confidence in the payoff from the game.

1 Risk Aversion

In last lectures we discussed the following paradox:

Suppose you are given an option to take Rs. 10 straightaway or Rs. 70 if a coin toss yields head (on tails you get Rs. 0). Then most of you will opt for the second option. Instead of this, if the options are Rs. 10 million straightaway or Rs.70 million on a coin toss, then there is little chance that you will opt for the second option, even though the expected return in the second case is more.

We see a similar situation that occurs in the casino in the next section.

2 St. Petersburg Paradox

There is a casino where a gambler G has to pay an amount of Rs. X to participate. The returns to G is decided by coin tosses. Let T represent tails and H represent heads. The following table relates the outcome of coin tosses to the payoffs.

Rs. 2	H
Rs. 4	TH
Rs. 8	TTH
Rs. 16	$TTTH$
...	...

G gets a return of Rs. 2 if first coin toss gives heads, Rs. 4 if the first coin toss reveals tails and the second reveals heads, and so on as shown in the table above. In this example surely any gambler G will be willing to invest Rs. 2 (i.e $X = 2$) since he is sure to get atleast that much in return. Some gamblers might be willing to invest Rs. 4 but it seems unlikely that anyone will be willing to invest Rs. 1000.

But the expected return of this game is
$(1/2) * 2 + (1/2) * (1/2) * 4 + (1/2) * (1/2) * (1/2) * 8 + \dots$
$= 1 + 1 + 1 + \dots = \infty$.

Even though the expected return from the game is infinity few gamblers will be willing to invest an amount of Rs. 1000 since the probability of winning a good amount of money is very small and the risk involved is very high.

This example shows that the expected return is clearly not what is needed to be maximized for increasing confidence of a player in a game. Then what is to be maximized and why?

We try to model the preferences of a player of a game by a relation in the next section.

3 Preference Relation

Recall that a game consists of:

- Rules,
- Players; and the strategy of each player,
- Outcomes, Ω, and
- Payoffs.

Define a preference relation R on the set of outcomes, Ω, as follows. $R = (\Omega, \geq_R)$. The relation satisfies the following properties:

- *Totality.* $w_1 \in \Omega, w_2 \in \Omega \Rightarrow$ either $w_1 \geq_R w_2$ or $w_2 \geq_R w_1$.
- *Reflexivity.* $w_1 \geq_R w_1$.
- *Transitivity.* $w_1 \geq_R w_2$ & $w_2 \geq_R w_3 \Rightarrow w_1 \geq_R w_3$.

Totality implies that there is an ordering between any two outcomes. For any two outcomes atleast one of them is preferred as much as the other i.e. every two outcomes are comparable.

Reflexivity is due to the fact that an instance of an outcome is preferred as much as any other instance of the same outcome.

Suppose an outcome a is preferred as much as outcome b and b is preferred as much as outcome c. Then a is preferred as much as c. This is Transitivity.

A preference relation is *rational* when it is total, reflexive and transitive. We will be using rational preference relation from now on.

A function $u : \Omega \rightarrow \mathbf{R}$ is said to be a *utility function* representing a preference relation R if

$$u(w_1) \geq (w_2) \Leftrightarrow w_1 \geq_R w_2. \quad (1)$$

If a player is indifferent to two outcomes, then the utlilites of the outcomes are same. The converse is also true.

For example, if the amount paid by the gambler is m and the amount gained by the gambler is x, then the utility function could be $u(m, x) = m - x$.

It is easy to see that if f is a monotonically increasing function and the utility function $u()$ represents R then $f(u)$ also represents R. So $f = au() + b$ also represents R whenever $a > 0$.

Note that we cannot replace the phrase "monotonically increasing" by "monotonically non-decreasing" since the condition in equation $u(w_1) \geq u(w_2) \Leftrightarrow w_1 \geq_R w_2$ above will not be satisfied. For example, when the utility function $u()$ is constant, an outcome w_1 strictly preferred over w_2 ($w_1 \geq_R w_2$ but $w_2 \not\geq_R w_1$) will satisfy $u(w_1) = u(w_2)$ and hence $u(w_2) \geq u(w_1)$. But this contradicts $w_2 \not\geq_R w_1$.

Hence, in particular we cannot have $a = 0$ in the earlier example of f.

Define $w_1 \sim w_2$ if $w_1 \geq_R w_2$ & $w_2 \geq_R w_1$. Clearly $w_1 \sim w_2 \Leftrightarrow u(w_1) = u(w_2)$ for all u representing R.

If $w_1 \sim w_2$ then we say that the player is indifferent to the two outcomes. She prefers each outcome as much as the other.

Lets consider now a game that will formalize our notion of value that we attach to a commodity.

4 Trading Game

There are n traders and m commodities $c_1, \ldots, c_m$. Trader i initially has q_j^i amount of commodity c_j which is represented as a vector $\underline{q}^i$. The mth commodity is a special paper. Traders come to a market and exchange commodities to get an allocation of commodities preferable to their initial endowments.

The set of outcomes is a set of tuples

$$(\underline{q}^1, \underline{q}^2, \ldots, \underline{q}^n) \tag{2}$$

where q_i is the amount of each of the commodity trader i possesses after the trade.

Let the utility function $u_i : \mathbf{R}^m \to \mathbf{R}$ represent the preference relation of trader i.

Let's also assume *free disposal* i.e. the traders can throw away any commodity without incurring any loss to themselves. Equivalently, having more of a commodity cannot decrease the total utility for a trader.

$$\forall i \forall \underline{q}, \forall \underline{q}' \geq 0, \quad u_i(\underline{q}) \leq u_i(\underline{q} + \underline{q}'). \tag{3}$$

Also assume that the mth commodity has no *a priori* utility. That is,

$$\forall i \forall \underline{q}, u_i(\underline{q}) = u_i(\underline{q} + x\underline{e}_m). \tag{4}$$

for all $x \in \mathbf{R}$. $\underline{e}_m$ is the canonical basis vector with 1 at mth position and 0 at all other positions. The mth commodity is also unforgeable and is the only medium of trade.

To start, each trader borrows certain amount of commodity m from a centralised bank and each has to return the same amount to the bank after the trade is over.

Lets look at a particular trader i. Let $u_i(q_1, \ldots, q_m)$ be her utility function. Suppose she gets c_1 at a rate r_1 and is able to sell c_2 at rate r_2. Then she would get δ_1 amount of c_1 by spending $\delta_1 r_1$ amount of c_m and sell δ_2 amount of c_2 to get $\delta_2 r_2$ amount of c_m back. If there is no change in her possession of c_m (this is also called conservation of paper),

$$\delta_1 r_1 = \delta_2 r_2. \tag{5}$$

A trade involving her is possible only if she does not lose in utility. That is,

$$u_i(q_1, \ldots, q_m) \leq u_i(q_1 + \delta_1, q_2 - \delta_2, \ldots, q_m) \tag{6}$$

given $q_2 > 0$.

Assuming u_i are continuous and differentiable, and the amount of trade involved is small, the right hand side is same as

$$\frac{\partial u_i(\underline{q})}{\partial q_1}\delta_1 - \frac{\partial u_i(\underline{q})}{\partial q_2}\delta_2 + u_i(\underline{q}). \tag{7}$$

Then the condition for trade is

$$\frac{\partial u_i(\underline{q})}{\partial q_1}\delta_1 > \frac{\partial u_i(\underline{q})}{\partial q_2}\delta_2, \tag{8}$$

which is same as

$$\frac{\partial u_i(\underline{q})}{\partial q_1}\frac{1}{r_1} > \frac{\partial u_i(\underline{q})}{\partial q_2}\frac{1}{r_2} \tag{9}$$

using the equation 5 above due to conservation of paper and the fact that $\delta_1, \delta_2 > 0$.

If the above inequality is in the opposite direction then the trade is possible in the opposite direction (assuming the trader i has a non-zero amount of the commodity).

The condition for no trade is no change in utility for small trades i.e.

$$\frac{\partial u_i(\underline{q})/\partial q_1}{\partial u_i(\underline{q})/\partial q_2} = r_1/r_2 \qquad (q_1 > 0, q_2 > 0). \tag{10}$$

In general for all commodities

$$\forall 1 \leq j \leq m,\ \forall 1 \leq k \leq m, \qquad \frac{\partial u_i(\underline{q})/\partial q_j}{\partial u_i(\underline{q})/\partial q_k} = r_j/r_k. \tag{11}$$

When $q_1 = 0$ and $q_2 > 0$, trade can happen only in the direction in which trader i *increases* the amount of q_1. Hence in this case equation 9 (derived assuming the same direction of trade) is the condition of equilibrium.

$$\frac{\partial u_i(\underline{q})/\partial q_1}{\partial u_i(\underline{q})/\partial q_2} > r_1/r_2 \qquad (q_1 = 0, q_2 > 0). \tag{12}$$

Here r_n is the relative price of commodity c_n. If the similar ratios of all traders are the same, there is no change of utility by any trade and this is a state of equilibrium.

But if the above ratios are not same for two traders, that is, if

$$\frac{\partial u_i(\underline{q})/\partial q_1}{\partial u_i(\underline{q})/\partial q_2} = r_{12} \tag{13}$$

and

$$\frac{\partial u_{i'}(\underline{q})/\partial q_1}{\partial u_{i'}(\underline{q})/\partial q_2} = r_{12}^{'} \tag{14}$$

with $r_{12} \neq r_{12}^{'}$, trade can occur. For example let $r_{12} > r_{12}^{'}$. Then trade can occur in the following manner:

Let $r_{12}^{'} < \delta_1/\delta_2 < r_{12}$. Trader i can sell an amount δ_1 of commodity c_1 to trader $i^{'}$ and buy an amount δ_2 of c_2 from the same trader. From the point of view of trader i, her change in utility is given by,

$$\tfrac{-\partial u_i(\underline{q})}{\partial q_1}\delta_1 + \tfrac{\partial u_i(\underline{q})}{\partial q_2}\delta_2$$

$$= \delta_2(-\tfrac{\partial u_i(\underline{q})}{\partial q_1}\delta_1/\delta_2 + \tfrac{\partial u_i(\underline{q})}{\partial q_2})$$

$$> \delta_2(-\tfrac{\partial u_i(\underline{q})}{\partial q_1}r_{12} + \tfrac{\partial u_i(\underline{q})}{\partial q_2})$$

$$= 0.$$

So

$$\frac{-\partial u_i(\underline{q})}{\partial q_1}\delta_1 + \frac{\partial u_i(\underline{q})}{\partial q_2}\delta_2 > 0,$$

and from the point of view of trader $i^{'}$, change in utility after this trade is,

$$\frac{\partial u_{i'}(\underline{q})}{\partial q_1}\delta_1 - \frac{\partial u_{i'}(\underline{q})}{\partial q_2}\delta_2 > 0,$$

in a similar manner as above.

Hence trade can occur and both traders end up increasing their utilities.

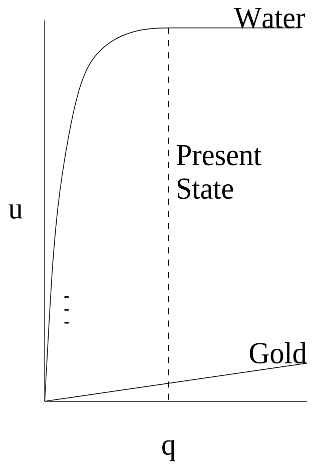
It is appreciable that with so few assumptions (free disposal, continuity and differentiability of utility functions, and a rational preference relation) we were able to show that the prices will be proportional to the ratio of change of utility and the change of quantity.

5 Gold vs. Water

The above arguments reinforces the notion that value of a commodity is proportional to the ratio of change of utility and the change in quantity that forces that change in utility i.e.

$$\text{value} = \frac{\partial u}{\partial q},$$

where u is the utility function of a commodity and q is the quantity of the commodity.



This explains the high cost and low utility of gold compared with low cost and high utility of water, as shown in the figure. Since the marginal utility of water is much smaller than that of gold in the present state, the slope of the graph for gold is much more than for water.