

Lecture-2 (Game Theory, CS905)

2.1 Game Classification :

Games are classified as follows :-

Co-operative Games

Here the players are allowed to form coalitions to share decisions, information and pay-offs.

Non-cooperative games

In such games the players follow the rationality assumption and play to maximize their own payoffs irrespective of other players' payoffs.

There are 2 forms of non-cooperative games.

- a) **Extensive Form** : Here the players make their moves in turns. There are more than one moves in the game and the previous moves of all opponents are known to everyone.
- b) **Normal Form** : Here the game is one shot and the players make simultaneous moves.

2.2 Bimatrix Representation Of 2 Player Games In Normal Form

The two player normal form games with finite number of strategies can be represented as two matrices, one each showing the payoffs of a player for all possible outcomes. Sometimes, these 2 matrices are merged into one in which payoffs of both players are shown for each outcome.

The rows in the payoff matrix represent the strategies of player 1: S_1, S_2, \dots, S_n . The columns represent the strategies of player 2: $S_1, S_2, S_3, \dots, S_m$. Each element of the matrix represents payoffs for a particular outcome (combination of players' strategies) of the game. The Fig. below shown a general payoff matrix with payoffs of both players, denoted as (a_{ij}, b_{ij}) . Here, a_{ij} represents the payoff for player 1 and b_{ij} represents the payoff for player 2 under strategy combination (S_i, S_j) .

P1 \ P2	S_1	S_2	S_3 S_j	S_m
S_1	(a_{11}, b_{11})	(a_{12}, b_{12})
S_2	
S_3		

S_i				(a_{ij}, b_{ij})	
S_{n-1}					
S_n					(a_{nm}, b_{nm})

2.3 Nash Equilibrium

Consider an N player game,

A strategy profile, $(w_1^*, w_2^*, \dots, w_n^*)$ constitutes a Nash Equilibrium, iff

For all i , $w_i : U_i(w_1^*, w_2^*, \dots, w_i^* \dots w_n^*) \geq U_i(w_1^*, w_2^*, \dots, w_i', \dots, w_n^*)$

In a game where pure strategy is used Nash Equilibrium may or may not exist.

In the game of prisoner's dilemma (described in Lecture1), there is only one Nash Equilibrium, (Confess, Confess), i.e. both players will confess.

This can be proved as follows. When, P1 confesses, P2 is better off confessing than denying because he gets 5 years confessing which is less than 10 years if he denies. When, P2 confesses, P1 is better off confessing than denying, because P1 gets 5 years when he confesses but he gets 10 years on denying. The above analysis, shows that (C,C) represents Nash Equilibrium. Similarly, one can show that this is the only Nash Equilibrium in this game.

Prisoner's Dilemma (Payoff Matrix)

P1 / P2	Confess	Deny
Confess	5 , 5	0 , 10
Deny	10 , 0	1 , 1

2.4 Cricket & Movie Game

Consider a husband and wife who have different plans to spend the evening together and have fun. The husband is interested in watching a cricket match while the wife is interested in going out to watch a movie. We can model this as a game with husband as first player (P1) and wife as the second player (P2).

The payoff matrix for this game is given below. The pay-offs for both players in the four outcomes are explained here :-

- a) When P1 and P2 both go for cricket, P1 has payoff of 10 because that option was his preference, while P2 has payoff of 5 because cricket was not her preference.
- b) When P1 goes for cricket while P2 goes for movie, since they are separated from each other, the whole objective of having fun together gets defeated, so each ends up with a payoff of 1.
- c) When P1 goes for movie and P2 goes for cricket, they go for something they don't prefer and also go alone, so each gets a negative payoff of -10.
- d) When P1 goes for movie and P2 goes for movie, then since movie is P2's preference she gets payoff of 10, while P1 gets payoff of 5 because his preference was cricket.

On analyzing this matrix, one can see that there are 2 Nash Equilibrium : (10,5) and (5,10). (10,5) is a Nash Equilibrium because if P2 chooses cricket, then P1 is better off choosing cricket compared to movie and if P1 chooses cricket, then P2 is better off choosing cricket compared to movie. A similar analysis proves that (5,10) is also a Nash Equilibrium

Movie/Cricket Game (Outcome Matrix)

P1 / P2	Cricket	Movie
Cricket	10 , 5	1 , 1
Movie	-10 , -10	5 , 10

2.5 Matching Pennies Game

Consider, the following game of matching pennies. Here, 2 players, P1 & P2, toss one coin each. If the result of toss is same for both players, then P1 wins and gets payoff of 1, while P2 loses and gets payoff of -1. If the result of the toss does not match between 2 players, then P2 wins with payoff of 1, while P1 loses and gets payoff of -1. This is a zero sum game.

In this game it is easy to see that there is no Nash Equilibrium when players use pure strategies. In using pure strategy, the players choose a strategy with probability = 1.

Matching Pennies (Payoff Matrix)

P1 / P2	H	T
H	1, -1	-1, 1
T	-1, 1	1, -1

2.6 Example Of Constant Sum Game :

Consider, the following 2 player game matrix. This matrix has only one Nash Equilibrium, (5,5), located at first row and first column.

Constant Sum Game (Payoff Matrix)

5, 5	10, 0	10, 0
0, 10	5, 5	10, 0
0, 10	0, 10	5, 5

2.7 Cold War

The cold war between US and USSR can also be modelled as a game. Both US and USSR have the option of spending their budget on health or defense and each of them would like to get strategic defense advantage over the other.

If we consider US and USSR as two players P1 and P2 respectively, then both of them have 2 strategies : spend on defense , spend on health. The payoff matrix below shows the four possible outcomes of the game under this strategy set..

When, both players spend on health they get payoff of 10 each. When, one spends on health while the other spends on defense, the one who spends on health gets payoff of -10, because the other one who spends on defence gets strategic defence advantage and hence gets a payoff of 15. When both players spend on defense, they each have payoff of 1 because then the country's health suffers.

Here, it can easily be shown that there is only one Nash Equilibrium (D,D).

Cold War Game (Payoff Matrix)

US / USSR	Health	Defense
Health	10 , 10	-10 , 15
Defense	15 , -10	1 , 1

This game theoretic model seems to explain what happened during the cold war between US and USSR.

2.8 Tragedy Of The Commons

Consider a village with N farmers, that has limited grassland. Each of the N farmers have the option to keep a sheep. Let the utility of milk and wool from the sheep be 1. Let the damage to the environment from one sheep grazing over the grassland be denoted by -5.

Let X_i be a variable that takes values 0 or 1 and it denotes that whether the i th farmer keeps sheep or not.

$X_i = 1$, i^{th} farmer keeps sheep
 $= 0$, i^{th} farmer does not keep sheep

So, the utility of the i^{th} farmer, $U_i(X_i)$, is given by :-

$$U_i(X_i) = X_i - \{ [5 * (X_1 + X_2 + X_3 + \dots X_n)] / N \}$$

since, the total environmental damage is shared by all farmers.

In this game,

$X_i = 1$, is a Nash Equilibrium for all i , if $N \geq 5$

and this Nash Equilibrium is unique if $N > 5$. This can be easily shown because keeping a sheep would add more utility to a farmer from milk/wool than subtract utility from him due to environmental damage. Thus, everyone will end up keeping a sheep and the utility for every farmer will be -4.

In real life this leads to excessive environmental damage. In this case, game theory comes to rescue to by justifying a pollution tax of keeping the sheep, of an amount equal to the damage done to the environment. So, when this pollution tax is added , the utility of the i^{th} farmer becomes

$$U_i(X_1, X_2, \dots X_n) = X_i - 5X_i - \{ [5 * (X_1 + X_2 + X_3 + \dots X_n)] / N \}$$

Now, in this case, the Nash Equilibrium is $X_i = 0$, for all i . This is because any farmer will increase his utility by giving away his sheep. Thus everyone will give away his sheep and each will have utility of 0.

The above example of tragedy of the commons can be used to explain large number of real life situations. Consider, the scenario of industries polluting the environment. Here, the industries can be considered as sheep and the environment can be considered as grassland and similar analysis of tragedy of commons holds true. In another scenario of illegal construction of houses causing infrastructure problems, the houses denote the sheep and the infrastructure denotes the grassland in the tragedy of the commons.

Thus, game theory can be used to design laws and mechanisms to get socially desirable outcomes.

2.9 Traffic Light Modelling

The Traffic light model from game theoretic point of view was described in Lecture 1. Here we re-visit the same model for the purpose of describing the behaviour of the system under the assumptions of increasing and decreasing marginal cost of delay.

The payoff matrix is shown below for reference.

Traffic Light (Outcome Matrix)

P1 / P2	O	D
O	C(d)	C(d+D)
D	0+pf	C(D)+pf

Here assuming that $D \gg d$,

$$C(d+D) = C(D) + d.C'(D),$$

where $C'(D)$ is the first derivative of cost of delay $C(D)$ w.r.t. D . That is it represents the rate of change of cost of delay w.r.t. to delay.

The two cases of marginal cost of delay function are considered below :-

- a) When the marginal cost of delay is a **decreasing function**, then the above game leads to convergence, as shown in Fig.1 below.

In the Fig.1 below, the y axis plots : $d.C'(D)$, x-axis plots : D .

The point (D^*, pf) is marked as an equilibrium point obtained as follows :-
 Consider, point A in the Fig.1. A player at this point has $dC'(D)$ more than $p(f)$, so this player will tend to disobey \Rightarrow the value of D tends to move towards D^* .

Now consider point B. At this point, $pf > dC'(D)$, so a player at this point will tend to obey the traffic light and therefore wait \Rightarrow the congestion delay will tend to reduce and move towards D^* .

Therefore, the congestion in the system tends to move towards D^* , the equilibrium point.

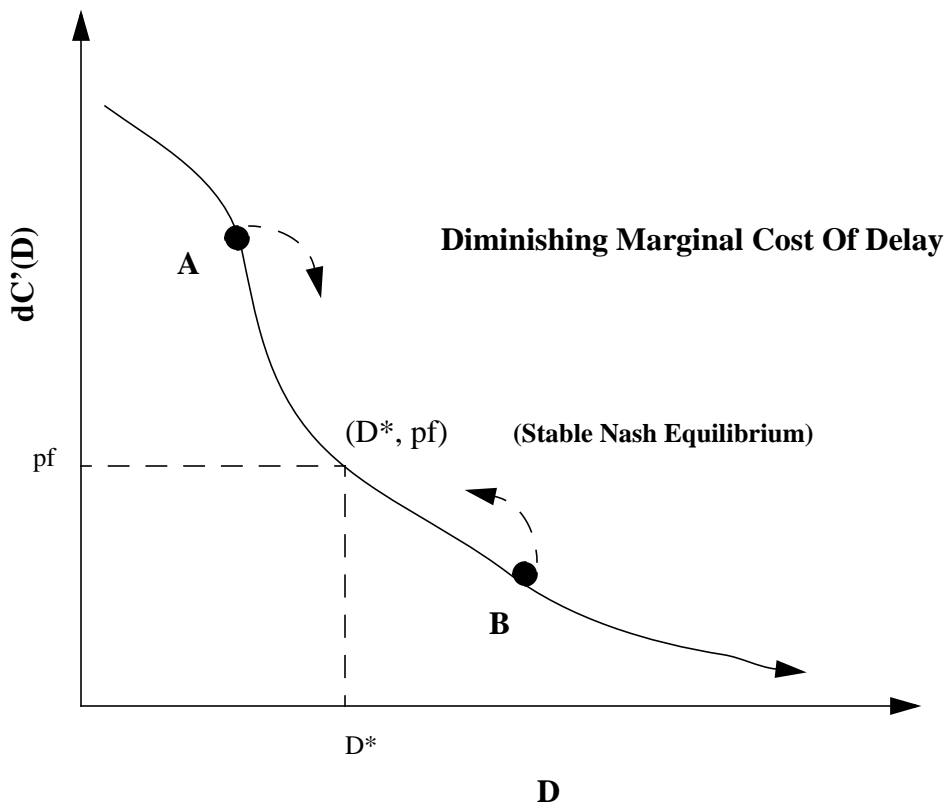


FIG. 1

b) Now, consider the case when the marginal cost of delay is an **increasing function**. then the system tends towards chaos. In the Fig.2 given below, again focus on two points A & B.

At point A, $dC'(D) > pf$, so the player at this point tends to disobey and hence, moves in the direction away from $D^* \Rightarrow$ more traffic congestion.

At point B, $dC'(D) < pf$, so the player at this point tends to obey and hence the congestion drops further.
 Thus, the players at points before and after D^* , tend to move away from D^* and so the equilibrium is not reached.

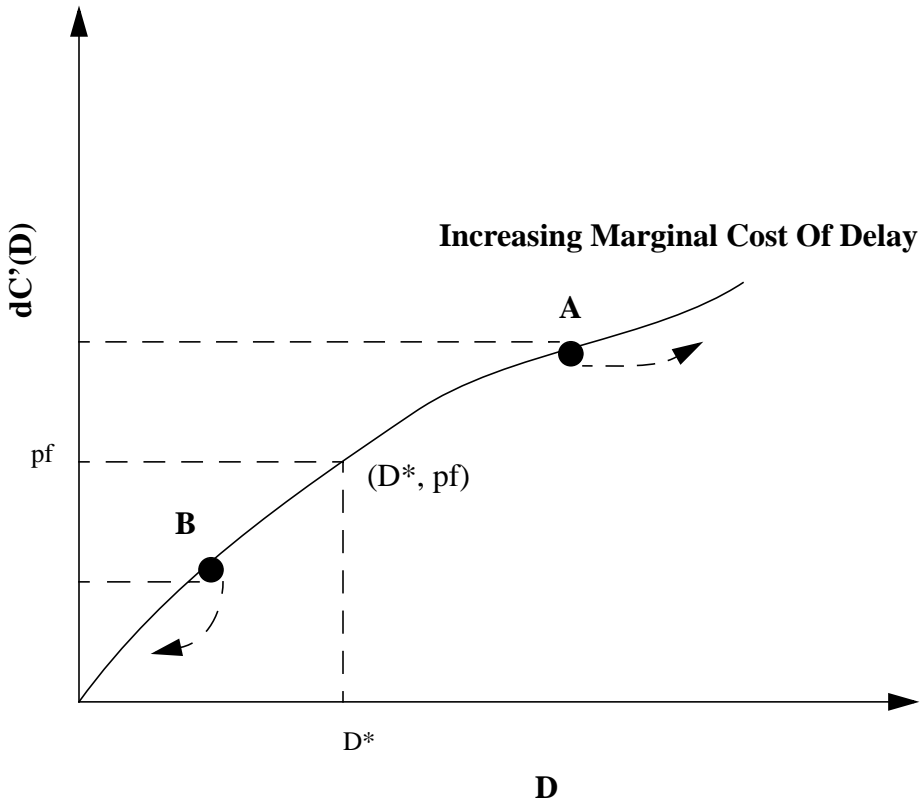


FIG. 2

H.W.

Draw the game matrix for the repeated Prisoner's Dilemma and find the best strategy in the game. Consider 3 repetitions in the game, i.e. $\{C,D\}^3$