Nash Bargaining Solutions

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1 Bargaining Solution

In a transaction when the seller and the buyer value a product differently, a surplus is created. A bargaining solution is then a way in which both agree to divide the surplus.
Example. Consider a house made by a builder A. It costed him Rs.10 Lacs. A potential buyer is interested in the house and values it at Rs.20 Lacs. This transaction can generate a surplus of Rs.10 Lacs. These people now need to trade at a price. Buyer knows that the cost is less than 20 Lacs and the seller knows that the value is greater than 10 Lacs. The two of them need to agree at a price. Both try to maximize their surplus. Buyer would want to buy it for 10 Lacs, while the seller would like to sell it for 20 Lacs. They bargain on the price, and either trade or dismiss. Trade would result in the generation of surplus, whereas no surplus is created in case of no-trade. Bargaining Solution provides an acceptable way to divide the surplus among the two parties.
Formally, a Bargaining Solution is defined as

$$F : (X, d) \rightarrow S$$

where $X \subseteq R^2$ and $S, d \in R^2$
In the above example, $price \in [10, 20]$, bargaining set is simply $x + y \leq 10, x \geq 0, y \geq 0$. A point (x, y) in the bargaining set represents the case, when seller gets a surplus of x, and buyer gets a surplus of y, i.e. seller gets $10 + x$ and the buyer pays $20 - y$.
Assumption Bargainging Set X is convex and bounded

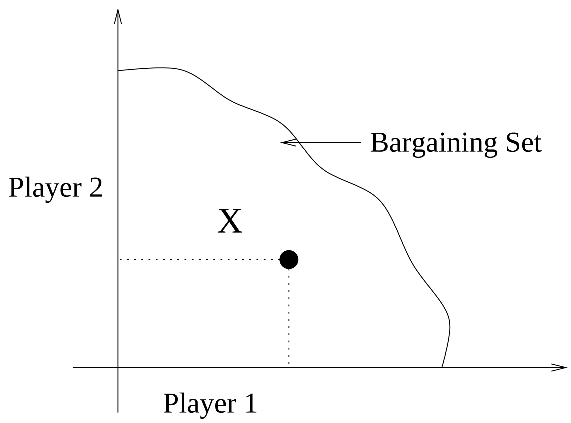

Figure 1: Bargaining Set

2 Pareto Optimality

A Pareto Optimal solution is one in which none of the players can increase their payoff without decreasing the payoff of atleast one of the other players.
A solution ω is Pareto optimal iff
$\forall \omega' \in \Omega$, $\exists i, s.t.$
$u_i(\omega') < u_i(\omega)$, or $\forall i$ $u_i(w') = u_i(w)$
where $u_i(w)$ is the utility function for player i at outcome w.
All points on the boundary of the Bargaining Set are Pareto Optimal solutions. In a bargaining situation, players would like to settle at a pareto optimal outcome, because if they settle at an outcome which is not pareto optimal, then there exists another outcome where atleast one player is better off without hurting the interest of the other players. Pareto optimal solutions are not unique in most of the cases.
Example. In the earlier example, $x + y = 10$ is a pareto optimal frontier.

3 Properties of a Bargaining Solution

Nash gave four axioms that any bargaining solution should satisfy.

- Invariant to affine transformations.
- Pareto optimality.

- Independence from Irrelevant Alternatives.
- Symmetry

1. **Invariant to affine transformations**
   An affine transformation $\tau_{Ab} : R^2 \to R^2$ is defined by a matric A, and a vector b of the following form.
   $A = \begin{bmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{bmatrix}$ $b = \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}$
   Now the transformation can be defined as
   $$\tau_{Ab}(x) = Ax + b$$
   A bargaining solution is invariant to an affine transformation iff
   $\forall A, b$, if $F(X, d) = S$
   then $F(\tau_{Ab}(X), \tau_{Ab}(d)) = \tau_{Ab}(S)$

2. **Pareto Optimality**
   $F(X, d)$ should be a Pareto optimal solution.
   Any bargaining solution should be better off than the disagreement point.

3. **Independent from Irrelevant Alternatives**
   If S is the Nash bargaining solution for a bargaining set X then for any subset Y of X containing S, S continues to be the Nash Bargaining Solution. This axiom of Nash is slightly controversial unlike the previous two axioms, since more alternatives give you better bargaining power. This can be intuitively justified.
   Let us say that the set Y has a NBS S' and S be another NBS of X (refer figure 2). Now $S' \in Y, S \in Y$ and $S' \in X, S \in X$. In both the bargaining sets X and Y, both the options S, S' are available to the players. They should be expected to settle to the same outcomes. The presence of irrelevant alternatives in X should not influence the bargaining solution.
   If
   $$F(X, d) = S$$
   and
   $$Y \subset X$$
   $$S \in Y, d \in Y,$$
   $$\Rightarrow F(Y, d) = S$$

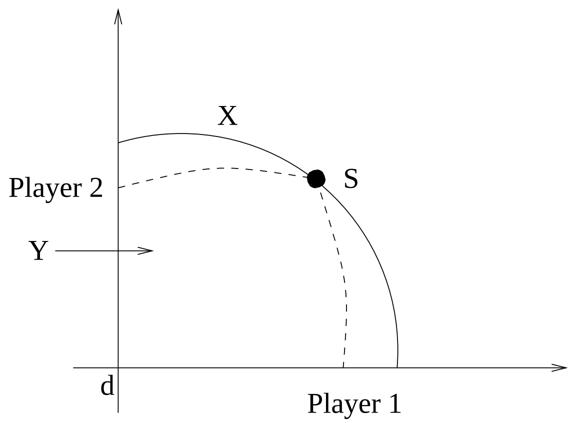

Figure 2: Independence from Irrelevant Alternatives

4. **Symmetry**
   The principle of symmetry says that symmetric utility functions should ensure symmetric payoffs. Payoff should not discriminate between the identities of the players. It should only depend on their payoff functions. Put simply, symmetry implies the bargaining solution for region $X = x + y \leq 1, x \geq 0, y \geq 0, d = (0, 0)$, should be $(1/2, 1/2)$ as shown in figure 3. If both players have the same utility functions, then symmetry demands that both get equal payoffs.

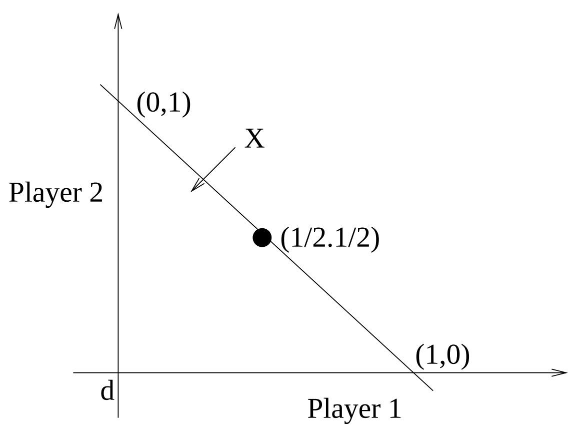

Figure 3: Symmetry

Nash characterized the NBS and proved that there is a unique solution satisfying the axioms given by Nash.

Theorem: *If a tangent is drawn to the curve defining the boundary of the bargaining set at s - the Nash bargaining solution, it intersects the lines parallel to the axes and passing through the disagreement point (d) at points r and t. Then* $s = (r + t)/2$

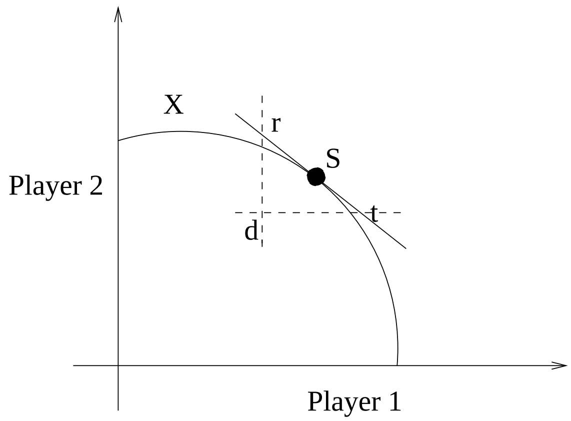

Figure 4: The bargaining solution s = (r+t)/2

Proof: Let $d = (d_1, d_2)$ where d_1 and d_2 are the utilities of the two players in the event of disagreement. The bargaining problem is shown in Figure 4. Let S be a pareto optimal point of X such that it is the midpoint of the line joining the points r and t. We will prove that S is a NBS of (X, d).
Lets define an Affine Function τ_{Ab} where

$$A = \begin{bmatrix} \frac{1}{t_1-d_1} & 0 \\ 0 & \frac{1}{t_2-d_2} \end{bmatrix} \text{ and } b = \begin{bmatrix} \frac{-d_1}{t_1-d_1} \\ \frac{-d_2}{r_2-d_2} \end{bmatrix}$$

It is easy to see that

$$\tau_{Ab}(d) \rightarrow (0, 0)$$
$$\tau_{Ab}(r) \rightarrow (0, 1)$$
$$\tau_{Ab}(t) \rightarrow (1, 0)$$
$$\Rightarrow \tau_{Ab}(s) = (\frac{1}{2}, \frac{1}{2})$$

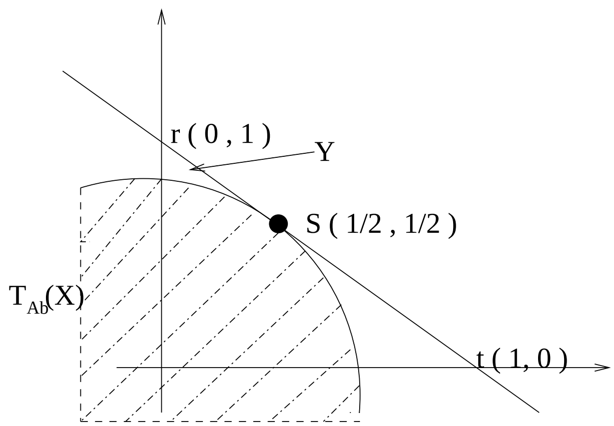

Figure 5:

Let $Y = x_1 + x_2 \leq 1, x_1 \geq 0, x_2 \geq 0$. Note that $\tau_{Ab}(s)$ is a NBS for $(Y, 0)$. Also,
$\tau_{Ab}(X) \subset Y$ [Since X is convex]
$(0,0) \in \tau_{Ab}(X), \tau_{Ab}(s) \in \tau_{Ab}(X)$
$\Rightarrow \tau_{Ab}(s)$ is a NBS for $(\tau_{Ab}(X), \tau_{Ab}(d))$ [Independence from irrelevant alternatives]
$\Rightarrow s$ is a NBS for (X, d)

$$\Rightarrow s = \frac{r+t}{2}$$

Generalised Nash Bargaining Solution
If the players were asymetric in their bargaining strengths, then NBS can be generalized by dropping the symmetry axiom. In this case the NBS satisfies $s = \alpha r + \beta t$ where α and β are bargaining powers of the two players, and $\alpha + \beta = 1$

4 NBS is a solution to the Alternating Offers Game

Define the alternating offers game as an extensive form game (done in last lecture). In this game, two players bargain to settle on a price. First of all player 1 makes an offer to player 2. Player 2 can either accept or reject. If player 2 accepts the deal takes place, otherwise he incurs a discount on his

utility and makes an offer to the player 1. The game goes on like this until someone accepts the offer. Let $u_1(x)$ and $u_2(x)$, $x \in (0,1)$ be the utility functions of the two players. If agreement settles in time t at x^*, their payoff will be $((\delta_1)^t u_1(x^*), (\delta_2)^t u_2(x^*)$
The Subgame Perfect Equilibrium for this game is defined by x^*, y^* s.t.

$$\delta_1 u_1(x^*) = u_1(y^*)$$
$$\delta_2 u_2(y^*) = u_2(x^*)$$

Player 1 offers x^* and accepts any offer that is atleast y^*. Similarly player 2 offers y^* and accepts anything that is atleast x^*. If $\delta_1 = \delta_2 = \delta$, then this is a symetric game.

Theorem :*Nash Bargaining Solution is same as the solution to the symmetric alternating offers game in the limit $\delta \to 1$*
Define Nash product

$$g(x) = (x_1 - d_1)(x_2 - d_2)$$

To prove the theorem we use the following lemma.
Lemma. NBS S of (X, d) is the unique solution $S \in X$ that maximizes $g(x)$. The proof of this lemma is simple and can also be found in the prescribed text book. Let x^*,y^* correspond to the solutions of the alternating offers

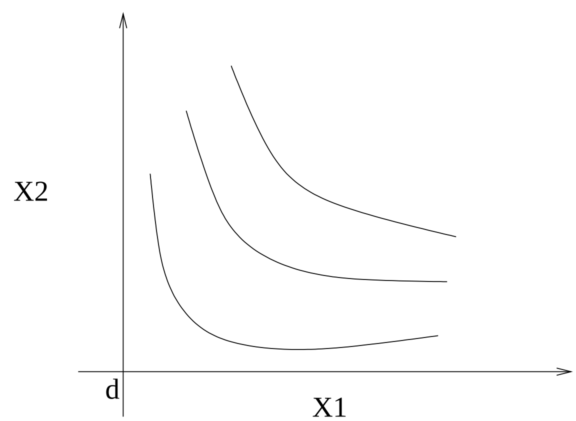

Figure 6: NBS maximizes $g(x) = (x_1 - d_1) * (x_2 - d_2)$

game. Now

$$\delta u_1(x^*) = u_1(y^*),$$

$$\delta u_2(y^*) = u_2(x^*).$$

Now,

$$\begin{aligned} g(x^*) &= u_1(x^*)u_2(x^*) \\ &= u_1(x^*).\delta u_2(y^*) \\ &= u_1(y^*)u_2(y^*) \\ &= g(y^*) \end{aligned}$$

In the limiting case, when δ is close to 1, $x^* = y^*$
x^* maximizes $g(x)$ and $x^* \in X$
$\Rightarrow x^*$ is Nash Bargaining Solution for (X, d)

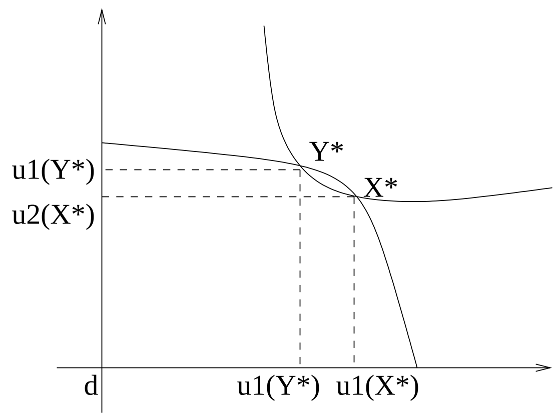

Figure 7: Correspondence with Repeated Game Bargain

In the figure, the curve facing outward is the curve for $g(x) = k$, where k is a constant. The farther we shift the curve from the origin, more value it attains. Hence in the limiting case, the value of $g(x)$ is maximum when it barely touches the convex curve, i.e. $x^* = y^*$.

4.1 HomeWork

Say Rs.1 Lac was to be divided between two players with the following utilities.

$$u_1(m) = m^\alpha,$$

$$u_2(m) = m^\beta$$

The bargaining set X is given by $(x_1, x_2) : x_1 + x_2 \leq 1, x_1 \geq 0, x_2 \geq 0$.
If $x \in [0, 1]$, the utilities are given by

$$u_1 = x^\alpha,$$

$$u_2 = (1 - x)^\beta.$$

What is symmetric NBS for this game and how does it depend on α, β?
What can you conclude about the outcome of bargaining between a risk neutral and a risk averse player?