Alternating Offers with General Utilities
Lecture 14

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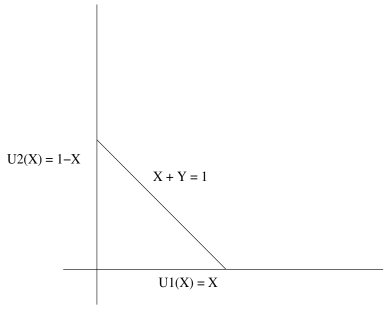
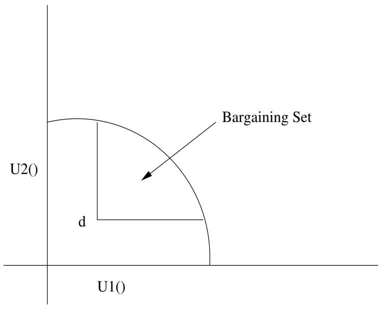
In the last lecture, we discussed *Bargain Solution* for the two-player case where the utilities are linear. The game terminates in disagreement after N rounds and both the player get a zero payoff. In case the utilities are linear, the set of feasible outcomes (also called bargaining set) looks as shown below.

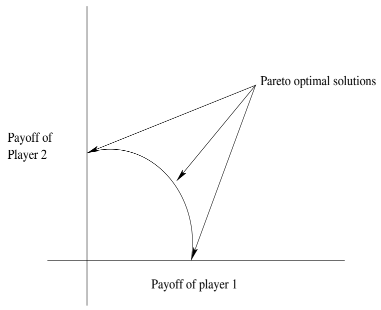


However, if the utilities are non-linear, the set of feasible outcomes can take any form. One of the possibilities is shown below.



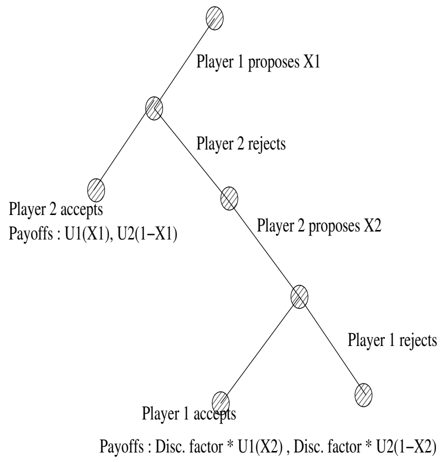
Here, d represents the disagreement point, and $U_1(d)$ and $U_2(d)$ are the utilities of the two players in the event of disagreement.
If the utilities of the players are consistent with Van Neumann Morganstern utility postulates, then the bargaining set will always be convex. To see this, consider two outcomes w_1 and w_2, with utilities given by $U_1()$ and $U_2()$. Now consider an outcome $w(p) : w_1$ with probability p and w_2 with prob $1-p$. The utility for player 1 on w(p) is $U_1(w(p)) = pU_1(w1) + (1-p)U_1(w2)$. Similarly, $U_2(w(p)) = pU_2(w1) + (1-p)U_2(w2)$. So, $(U_1(w(p)), U_2(w(p)))$ is a point in the line joining $(U_1(w1), U_2(w1))$ and $(U_1(w2), U_2(w2))$, and so, $w(p)$ is in the bargaining set. The above analysis implies that any convex combination of two outcomes in the bargaining set is also in the bargaining set. Therefore, we will assume that the Bargaining set is a convex region.

Pareto Optimality



Define region R $= (u_1, u_2) : u_1 \geq U_1(d), u_2 \geq U_2(d)$. An outcome w is pareto optimal iff $\forall w' \in R$, either $\exists i : u_i(w') < u_i(w)$ or $\forall i \; u_i(w') = u_i(w)$. Note that if the bargaining set is convex, the pareto optimal solutions will lie on the boundary of the bargaining set curve.

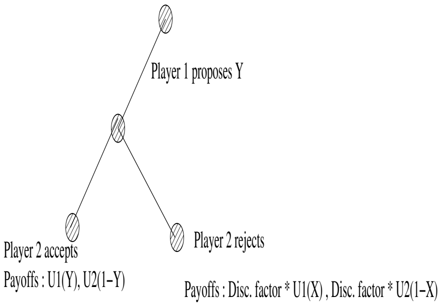
Subgame Perfect Equilibrium in Alternating Offers Game



Consider the alternating offers game as discussed in the last lecture. Assume that the utilities of the two players are not linear in the share x they receive. Let the utilities of players be $U_1()$ and $U_2()$ and δ_1 and δ_2 be their discount factors. Assume that player 1 gets a share of x and player 2 gets $1 - x$. Unlike the last lecture, we assume that the game can continue to arbitrary number of steps.
Let us recall the definition of Subgame Perfect Equilibrium. A strategy profile $s^\star$ is a Subgame Perfect Equilibrium of a game if it is a Nash equilibrium of every subgame of the game.

Claim 1 *There exists a value $x^\star$ such that, in the SPE (Subgame Perfect Equilibrium) of this game, player 1 proposes $x^\star$ at the first stage and player 2 accepts. That is in the scenario discussed above, the best solution can be achieved in the first turn itself.*

Proof 1 *The decision tree of the subgame perfect equilibrium can not be infinite, because if it were to be so, both the players will get a zero return in the end (as δ_1 and δ_2 are both less than 1 and $lim_{n->\inf}\delta_1^n -> 0$ and $lim_{n->\inf}\delta_2^n -> 0$). If the first player were to offer anything more than zero (in his first chance), player 2 being rational would have accepted it, giving strictly better payoffs to both the players (thereby reducing the decision tree to just 1 level).*



Now, let's assume that the subgame perfect equilibrium was attained after the player 1's first offer was rejected by player 2. Let the game terminate in n steps with an offer of x*.The resultant payoffs for the two player will be* $\delta_1^n U_1(x)$ *and* $\delta_2^n U_2(1-x)$*, respectively. Now, since* δ_1^n *and* δ_2^n *are both less than 1 and* u_1 *and* u_2 *are both monotonically non-decreasing functions, there exist a* y *such that,*

$$\begin{array}{lcl} U_1\ (y) & \geq & \delta_1^n U_1\ (x) \\ U_2\ (1-y) & \geq & \delta_2^n U_2\ (1-x) \end{array}$$

If player 1 will offer $1-y$ *to player 2, the player 2 being rational won't reject it, as she knows that if she rejects she will get* $\delta_2^n U_2\ (1-x)$*, which is less than* $U_2\ (1-y)$*. And since, for player 1,* $U_1\ (y) > \delta_1^n U_1\ (x)$*, he will have no hesitation in offering this to player 2. This will result in termination of decision tree at the first level itself, which contradicts the assumption taken above that subgame perfect equilibrium was attained after the rejection of first proposal of player 1 by player 2. Hence, it is proved that any subgame perfect equilibrium can be achieved in the first turn itself.*

Claim 2 *In a SPE, the final outcome will be either of* $x^\star$*,* $y^\star$ *(depending on whether player 1 proposes first or player 2) satisfying*

$$\begin{array}{lcl} U_1\ (y^\star) & = & \delta_1 U_1\ (x^\star) \\ U_2\ (1-x^\star) & = & \delta_2 U_2\ (1-y^\star) \end{array}$$

Proof 2 *Let's assume that the final outcome in a subgame perfect equilibrium where player 2 makes the first offer is* $y^\star$*. So, in the game where the first offer is to be given by player 1, if he were to give* x *such that* $U_2(1-x) \geq \delta U_2(1-y^\star)$*, player 2 being rational will accept. However, if player 1 offers an x:* $x > x^\star$

(where $x^\star$ is a critical value for which ,$U_2\ (1 - x^\star) = \delta_2 U_2\ (1 - y^\star)$), player 2 will reject the offer, as she knows that since in the SPE where she makes the first offer, the outcome is $y^\star$, she can always expect to get a payoff of $\delta U_2(1 - y^\star)$. On the other hand, if player 1 were to offer an x: $x < x^\star$, he knows that he can get a better payoff by offering $x^\star$ as x ¡$x^\star$ -¿ $U_1(x) < U_1(x^\star)$. This implies that player 1 being rational, will offer $x^\star$ which will be accepted by player 2. Now, consider the SPE where player 2 is to make his first offer. We have already assumed it's outcome to be $y^\star$. However, for player 1 to accept this offer, $U_1(y^\star) \geq \delta U_1(x^\star)$. Also, among all outcomes accepted by player 1, player 2 will offer the least possible, as $\forall y > y'$ (where y' is such that $U_1(y') = \delta U_1(x^\star)$, $1 - y < 1 - y'$ and so, $U_2(1 - y) < U_2(1 - y')$. So, $y^\star = y'$. Therefore, the final outcome of the SPE will either be $x^\star$ or $y^\star$, depending on whether player 1 proposes first or player 2 and $x^\star, y^\star$ satisfies

$$
\begin{aligned}
U_1\ (y^\star) &= \delta_1 U_1\ (x^\star) \\
U_2\ (1 - x^\star) &= \delta_2 U_2\ (1 - y^\star)
\end{aligned}
$$