Repeated Game with Alternating Offers

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1 Why Study Bargaining?

> Cats seem to go on the principle that it never does any harm to ask for what you want.
>
> *Krutch Joseph Wood*

Story : Once upon a time there was a piece of bread and two cats in close neighbourhood. There was contention for the piece of bread. Obviously there was a judge around, a monkey in this case and the matter was presented to him. He generously(!) agreed to resolve the matter by dividing the piece into two equal parts. But Alas! the parts were not equal. The cat with the smaller piece complained. So the monkey took a bite from the bigger piece only to find that it became smaller than the other. Now the other cat complained. Monkey was only too happy to continue offering his services and with each passing iteration ended up with a fuller stomach. Soon the bread piece reduced to an invisible size and the cats returned home being hungry equals!

Moral of the story : Knowledge of Game Theory and Nash Bargaining solutions can help you get a good share of bread.

2 An Example

Suppose there is an interaction between a customer and a seller for the sale of an object. The customer is willing to pay upto Rs. 50 while the object costs Rs 10 to the seller. The customer wants to minimize the payment while the seller wants to maximize it (she won't accept a loss). The surplus of Rs. 40 is to be divided between the two. This is done through a bargain process. This can be seen as a two player Extensive Form Game.

At each iteration one player proposes a payment value. If it is not acceptable to the other player she rejects it and proposes a new value in the next iteration. With each iteration, the contenders lose precious time and thus their gain from this bargain is discounted by a factor.

3 A Model for Bargains

We have two players, who have to share an amount of 1 unit between themselves. An outcome is specified by (X, t) where player 1 gets X and player 2 gets $1 - X$. If the bargain ends after t iterations, then the utility functions for the two players are:

$$u_1(X,t) = X.\delta_1^t$$
$$u_2(X,t) = (1-X).\delta_2^t$$

Also, the bargain breaks down after N iterations and both get no utility, i.e, $u_i(X, N) = 0$ for $i = 1, 2$.

4 Alternating offers game

Let us formally define the game. The first player proposes a value for X. Now, player 2 has two choices (either accept the offer or decline). If she accepts then the game ends there but, if she declines then she proposes another value for X. Now, player 1 either accepts or declines and the same thing repeats till the other player accepts. The strategy of a player is the values of X that she is ready to accept, and the values to offer in the next round in case she rejects.

In the case when $N = 1$ it can be easily seen that the Nash Equilibrium is $X = 1$. That is, player 1 proposes $X = 1$. Since only one iteration is allowed player 2 has no choice because this is the last iteration. The utilities are 1, 0 respectively. Note that by changing their strategies , none of the two players can increase her utility. Hence $X = 1$ is a Nash Equilibrium.

If $N = 2$, i.e., two iterations are allowed, the game tree is shown below. The subgame perfect equilibrium offer amounts are also shown.

If player 2 is offered a share of less than δ_2 she can easily reject it and herself propose a share of 1 for herself (which will give her a utility of $1.\delta_2^1 = \delta_2$). This will have to be accepted by Player 1 because this would be the last iteration. In this case the utilities are 0, δ_2 respectively. On the other hand they can get utilites of $1 - \delta_2$, δ_2 respectively if player 1 proposes $X = 1 - \delta_2$ to player 2 who accepts it (she has no reason to reject it since she cannot do better by rejecting it). This strategy pair actually forms a subgame perfect Nash Equilibrium because none can deviate from it to increase her utility.

Now, let us consider the case for a general N. We will see that just like the case for N=2 the players are better off by finishing the game in the first step itself rather than carrying on bargaining.

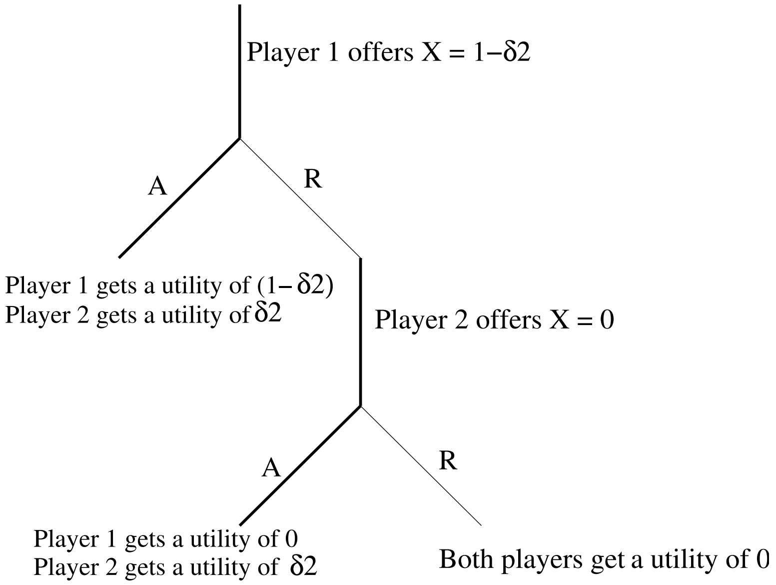

Figure 1: N=2 game tree

Since, the last player, say player i can always get a utility of δ_i^{N-1} by rejecting the previous offer and in the Nth turn offer the other player 0. Since, this would be the last iteration so the other player cannot do better by not accepting. So, the second last player (call her player j) must give the last player atleast δ_i in the $N-1$th turn. This would offer player i a utility of δ_i^{N-1} which she would have no reason to reject. Going 1 step further since player j can ensure a utility of atleast $(1-\delta_i)\delta_j^{N-2}$ so player i must offer her atleast this much. That is player i must offer $(1-\delta_i)\delta_j$.

Let us denote by $X_1(k)$ the amount that player 1 can ensure for herself in the first turn itself if the game runs for $N = k$ turns. That is she can ensure a utility of atleast $X_1(k)$.

Let us try to find a recurrence relation for the function $X_1(k)$. Let us consider the case where $N = k+2$. The amount that player 1 can ensure is $X_1(k+2)$. Suppose that the bargain goes on for 2 steps. Now only k more steps are allowed. The situation now is similar to a new game with $N = k$. Thus player 1 can ensure a amount of $X_1(k)$. Thus for the original game, after 2 steps player 1 can ensure a utility of atleast $X_1(k)\delta_1^2$. Thus player 2 must offer player 1 atleast $X_1(k)\delta_1$ in the second turn. This would

give player 2 a utility of $(1 - X_1(k)\delta_1)\delta_2$. Hence in the first step, player 1 must offer player 2 a utility of atleast $(1 - X_1(k)\delta_1)\delta_2$ if she wants her to accept. To do this she must offer her $(1 - X_1(k)\delta_1)\delta_2$. With this she will get $(1 - (1 - X_1(k)\delta_1)\delta_2)$ in the first turn.
So,

$$\begin{aligned} X_1(k+2) &= 1 - (1 - X_1(k)\delta_1)\delta_2), \\ &= 1 - \delta_2 + X_1(k)\delta_1\delta_2. \end{aligned}$$

If $N = 2M$ this gives

$$X_1(N) = (1-\delta_2) + \delta_1\delta_2(1-\delta_2) + (\delta_1\delta_2)^2(1-\delta_2)\ldots + (\delta_1\delta_2)^{M-1}(1-\delta_2),$$

$$\Rightarrow X_1(N) = (1-\delta_2)\sum_{i=0}^{M-1}(\delta_1\delta_2)^i,$$

$$\Rightarrow X_1(N) = \frac{(1-\delta_2)(1-(\delta_1\delta_2)^M)}{1-\delta_1\delta_2}.$$

If $N \to \infty$ then player 1 will get

$$X_1(\infty) = \frac{1-\delta_2}{1-\delta_1\delta_2}.$$

Let $\delta_1 = 1 - \beta_1$, $\delta_2 = 1 - \beta_2$.
Now if β_1 and β_2 are very small then ;

$$X_1(\infty) = \frac{\beta_2}{\beta_1 + \beta_2}.$$

So, we can see that the patient player (that is one who has a smaller β_i) is the one who gets the bigger piece of the total.