Application of VCG Procedures, Cournot Competition, Stackelberg Equilibrium

Scribe: Amit vyas
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1 An Application of VCG Procedure

In last lecture we discussed VCG procedures,which may be used to elicit true preferences of players. In this lecture we begin by discussing an example where VCG procedures may be used

The Broadcasting Problem:

Consider a Content Delivery Service (CDS) over a digital network such as the Internet. The network can be modeled as a graph with n nodes and m edges, where the nodes represent routers or hubs and edges represent the links connecting them. Assume that each link is owned by a network service provider (which we model as a player in the game).

In order to broadcast the content over the entire network, the content delivery service needs to build a spanning tree over the graph and send its content on the spanning tree edges (links). If the content is sent over an edge e, the corresponding player (network service provider) incurs a cost c_e, which it recovers from content delivery system. The socially optimal choice of the tree to broadcast the content will be the minimum cost spanning tree.

The CDS has two problems to solve: one is to select the links to be used for broadcasting content, and second is to decide the payments to the network service providers owning the selected links.

The owner of the links can mis-report their link cost and hence the spanning tree formed may not be socially optimal. So CDS can use VCG Procedure as follows, to get true link costs.

CDS invites bids from, the network service providers for the links they own. Let owner of link e, reports r_e, let G be the given graph with edge e weighted as r_e. CDS calculates T^*, which is a minimum spanning tree of G using the reported link costs r_e. For computing VCG payments, CDS also computes, T^*_e, which is minimum spanning tree of G, considering all edges except e. Let r(T) represent the sum of the reported costs of all edges in a tree T.

To a player owning link e, payment by CDS is given as follows:

$$\text{P(e)} = \begin{cases} 0 & \text{if e} \notin T^* \\ \text{r}(T^*_e) \text{ - } \sum_{d \in T^*, d \neq e} r_d & \text{if e} \in T^* \end{cases}$$

Therefore payoff to player owning edge e is given by

$$\pi\text{(e)} = \begin{cases} 0 & \text{if e} \notin T^* \\ \sum_{d \in T^*_e} r_d \text{ - } (\sum_{d \in T^*, d \neq e} r_d + c_e) & \text{if e} \in T^* \end{cases}$$

It is easy to see that the payoff is maximized when $c_e = r_e$ i.e. the network service providers report their true costs.

2 Cournot Competition

Consider a small household item in the market. There are some producers which produce this item and their are potential consumers which purchase the item depending on the price at which it is sold. Each producer is free to decide the quantity it produces, However the prices will decrease if the production is large. Therefore producer will not produce arbitrarily large amount of the commodity. It will produce an amount that will balance per unit profit and number of units sold (and hence maximize its profit). Assume demand equation is:

$$\text{p+q =M}$$

where p is the per unit price of the item, q is the total quantity produced of the item and M is a constant, Let c be the cost of production of the item and M be much larger than c

We now analyze the equilibrium output for the following cases,

2.1 Single Producer (Monopoly)

There is only a single producer. Let the quantity produced by the producer be q. According to the demand curve, price p = M - q . Therefore, profit π(q) = (M - q - c)q .
The producer will choose the quantity q that maximizes his profits.
Profits for producer are maximized when,

$$\delta\pi(q)/\delta q = 0 \quad i.e.$$
$$M - 2q - c = 0$$

$$\Rightarrow q = (M-c)/2, \quad p = (M+c)/2$$
$$\Rightarrow \pi(q) = ((M-c)/2)^2$$

So, maximum profit for the producer in case of monopoly is $((M-c)/2)^2$
.

2.2 Two Producers (Duopoly)

Now assume that there are two producers, Let quantity produced by the producers be q_1 and q_2 respectively, Therefore, the total quantity produced, q = q_1 + q_2.

According to demand curve, price p = M - q_1 - q_2. Therefore, profit of first producer, $\pi_1(q_1,q_2)$ = (M - q_1 - q_2 - c)q_1 and profit of second producer, $\pi_2(q_1,q_2)$ = (M - q_1 - q_2 - c)q_2

Now both the producers will independently try to maximize their profits, by choosing their quantities q_1 and q_2 respectively. This can be modeled as a two player game, where strategy for each player is the quantity it decides to produce.

Profits for first player are maximized when,

$$\delta\pi_1(q1, q2)/\delta q1 = 0 \quad i.e.$$
$$M - 2q_1 - q_2 - c = 0$$

$$\Rightarrow q_1 = (M - q_2 - c)/2$$

Similarly, profits for second player are maximized when,

$$\delta\pi_2(q1, q2)/\delta q2 = 0 \quad i.e.$$
$$M - 2q_2 - q_1 - c = 0$$

$$\Rightarrow q_2 = (\text{M - } q_1 \text{ - c})/2$$

This gives at Nash equilibrium:

$$\begin{aligned}
q_1^* &= (M - c)/3, \\
q_2^* &= (M - c)/3 \\
Price \;\; p &= (M + 2c)/3 \\
Profit \;\; \pi_1^* = \pi_2^* &= ((M - c)/3)^2 \\
TotalProfit &= (2(M - c)^2)/9 \; < \; ((M - c)/2)^2
\end{aligned}$$

Therefore the price as well as total profit decrease, as another producer is introduced in the market.

2.3 n Producers (Oligopoly)

Similar analysis in case of n producer shows:

$$\begin{aligned}
Total\ quantity\ produced\ q^* &= \sum_{i=1}^{n} q_i^* \\
&= ((M - c)/n + 1)n, \\
Price\ p^* &= (M + nc)/n + 1 \\
Total\ profit &= \sum_{i=1}^{n} \pi_i^* \\
&= N\pi_1^* \\
&= (((M - c)/(n + 1))^2)n
\end{aligned}$$

Clearly as the n increases, total quantity produced increases, price decreases and moves toward c and total profit also decreases.

2.4 Perfect Competition

Extending the analysis to the ideal case where the number of producers are very large i.e. n $\rightarrow \infty$, shows that a total quantity M - c will be produced and sold at price c, leading to a zero profit.

This is an interesting result that shows that in case of perfect competition, consumers get the item at the price equal to its production cost and producers have zero profits.

In real life all producers are not equal. There are some influential producers, who have more knowledge of the market. They are dominant producers, who have the capacity to drive the market prices. Such producers are called as leaders.

Once a leader (say first producer) decides to produce a quantity q_1, others converge to a Nash equilibrium in n-1 person sub-game with price p = m - q - q_1. Therefore the leader chooses a quantity q_1 that maximizes its profit, assuming that the rest of the players converge to the Nash equilibrium of the sub-game with n-1 producers.

Lets consider the two producers case again, with one more assumption that first producer also knows, how second producer will respond if he produces quantity q_1.

Let quantity produced by the producers be q_1 and q_2 respectively. According to demand curve, price p = M - q_1 -q_2. Therefore, Profit of second producer, π_2(q_1,q_2) = (M - q_1 - q_2 - c)q_2

Profits for second player are maximized when,

$$\delta\pi_2(q1, q2)/\delta q2 = 0 \quad i.e.$$
$$M - 2 * q_2 - q_1 - c = 0$$

$$\Rightarrow q_2 = (M - q_1 \text{ - c})/2$$

Profit of the first producer, π_1 = (M - q_1 - q_2 - c)q_1, since the first producer also knows that if he produces quantity q_1, second producer will respond by producing quantity q_2 = (M - q_1 - c)/2, first producer will try to maximize his profits using this additional information available.

Therefore Profit for first producer π_1 = (M - q_1 - (M - q_1 - c)/2 - c)q_1 = ((M - q_1 - c)*q_1)/2
Profits for first producer are maximized when,

$$\delta\pi_1(q1, q2)/\delta q1 = 0 \quad i.e.$$
$$M/2 - q_1 - c/2 = 0$$

$$\Rightarrow q_1^* = (M-c)/2$$
$$Profit\ for\ first\ producer,\ \pi_1^* = (M-c)^2/8$$

Note that here

$$q_1^* = (M-c)/2,$$
$$q_2^* = (M-c)/4,$$
$$P^* = (M+3c)/4,$$

and this is not a Nash equilibrium. First player can seemingly improve its profit by increasing its production. However if it considers the response of the second player after changing his production, it would conclude that the current production schedule is the best for him. This is formally defined as a Stackelberg equilibrium.

Clearly the first producer has more profit here as compared to the case of duopoly, where both the producers had no information about response of the other producers to their production amount. Thus dominant players having more information about the market can improve their profits.

3 Stackelberg Equilibrium

Stackelberg equilibrium prescribes an optimal strategy for the leader if the other players (followers) react by playing their Nash equilibrium strategies in the smaller sub-game, whenever leader announces his strategy, formally

Definition Let S^*_{-i} is the Nash equilibrium of the remaining players if player i chooses to play S^*_i. The strategy profile (S^*_i, S^*_{-i}) is an Stackelberg Equilibrium with player i leading, iff $\forall S_i$, $U_i(S^*_i,\ S^*_{-i}) \geq U_i(S_i,\ S_{-i})$, where, S_{-i} is the Nash equilibrium of the remaining players if player i chooses to play S_i.

Nash equilibrium and Stackelberg Equilibrium may give very different outcomes. Consider the following hypothetical two player game:

Let the strategies of the players be represented as real number x_i between 0 and 1. Let the utility functions for both the players be given as follows:

$$U_1(x_1, x_2) = 2x_2 - x_1$$
$$U_2(x_1, x_2) = 1 - (x_2 - x_1)^2$$

$$1 \geq x_1,\ x_2 \leq 0$$

It is evident that, $x_1 = 0$, $x_2 = 0$ is a unique Nash equilibrium in this game.

Now consider the case where first player has more information. Assume he knows second player's utility function. So he knows that whatever x_1 he chooses, second player will choose $x_2 = x_1$. Therefore he can choose $x_1 = 1$ to maximize his utility and they will reach to a Nash equilibrium with $x_1 = 1$, $x_2 = 1$, where both the players get an utility of 1.

This is a Stackelberg equilibrium with first player as leader. Note that Stackelberg equilibrium with second player leading is $x_2 = 0$, $x_2 = 0$!!