CSL758 Advanced Algorithms

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Linear Programming and **Duality**

1. Motivation

Many real world examples can be cast into the framework of linear programming problems. For instance, the fact that the value of maximum-flow of a graph is equal to the value of minimum cut is a result of duality. In the last class we have seen that how we convert the maximum weight bipartite matching to its dual problem of finding minimum weight vertex cover.

2. Introduction

A linear programming problem is a set of linear inequalities and an objective function. Each linear inequality takes the form:

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n \{\leq, =, \geq\}\ b.$$

The objective function takes the form:

$$c_1x_1 + c_2x_2 + \ldots + c_nx_n\,.$$

The goal is to find an assignment of values to $x_1, x_2, ..., x_n \in$ ¡ such that the objective function is maximized over all assignments and obey each of inequalities.

3. Definitions

We will represent the variables by $x = (x_1, x_2, \ldots, x_n) \in$ ¡ . We have following definitions.

Definition 1 *$x \in R^n$ is feasible for an LP if it satisfies all the constraints.*

Definition 2 *$x \in R^n$ is optimal if it is feasible and optimizes the objective function over feasible x.*

Definition 3 *An LP is feasible if there exists a feasible vector x with it.*

Definition 4 *An LP is unbounded if there is a feasible vector x with arbitrary good objective value.*

Lemma 1 *Every LP is either infeasible, has an optimal solution, or is unbounded.*

The proof follows by compactness of R^n and because polytopes are closed set.

Definition 5 *Every LP is in canonical form if it has the form:*

$$\begin{array}{ll} \textit{Maximize} & c^T x \\ \textit{Subject to} & Ax \leq b \\ & x \geq 0 \end{array}$$

where $c \in R^n$ is the objective function, and each row of A is a constraint.

Rules for converting an arbitrary LP to its canonical form

- If it is a minimization problem, negate c and maximize.
- If a constraint is $\geq$, negate it and convert it to $\leq$
- If a constraint is =, add two constraints with $\leq$ and $\geq$

Below is the example of canonical form:

Maximize $4x_1 + x_2 + 5x_3 + 3x_4$

Subject to $x_1 - x_2 - x_3 + 3x_4 \leq 1$ ……(1)

$5x_1 + x_2 + 3x_3 + 8x_4 \leq 55$ ………(2)

$-x_1 + 2x_2 + 3x_3 + 5x_4 \leq 3$ ………(3)

$x_1, x_2, x_3, x_4 \geq 0$

Definition 6 *An LP is in standard form if it has the form:*

$$\begin{array}{ll} \textit{maximize} & c^T x \\ \textit{subject to} & Ax = b \\ & x \geq 0 \end{array}$$

To convert an LP to standard form we can add a slack variable to an inequality. That is, if the inequality is: $a^T x \leq b$ we can convert this to $a^T x + s = b$. To ensure that $x \geq 0$ we can replace it two variables(x^+, x^-) with $x^+ - x^-$, which can take any value even when $x^+, x^- \geq 0$.

Crude Upper Bound

So how do we found crude bounds on the value of the maximizing function above?

Multiply (2) by 2 we get

$$10x_1 + 2x_2 + 6x_3 + 16x_4 \leq 110 \qquad \ldots\ldots\ldots\ldots(4)$$

Note that (4) dominates objective function and hence we can be sure that :

$$4x_1 + x_2 + 5x_3 + 3x_4 \leq 110$$

But this is a very crude upper bound. We can do better.
Add (2) and (3) to give the below inequality-

$$4x_1 + 3x_2 + 6x_3 + 13x_4 \leq 58 \qquad \ldots\ldots\ldots\ldots(5)$$

Clear (5) also dominates objective function and hence we have found a better upper bound

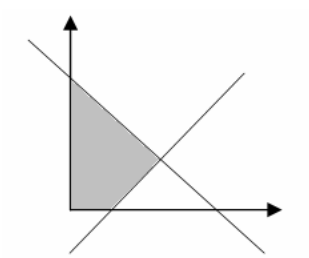
$$4x_1 + x_2 + 5x_3 + 3x_4 \leq 58$$

4. Geometry

Consider following LP:

$$\max x_1$$
$$2x_1 + 3x_2 \leq 5$$
$$3x_1 - 4x_2 \geq 2$$

We can describe it graphically as



Shaded region is feasible region.

- Any equality of the form $a^T x = b$ is a hyperplane.
- Any inequality of the form $a^T x \leq b$ is a halfspace.
- Intersection of halfspaces is a polyhedron ($Ax \leq b$)
- A bounded polyhedron is a polytope and it is convex.
- Maximum values of objective function occur at the extremes or the corners of polytope.

So the Linear programming problem is analogous to the problem of finding a point in polytope that is furthest in the direction specified by objective function

5. Duality

Every LP has its dual LP. If we have following primal LP:

$$Max \ \sum_i c_i x_i \ \ s.t. \ \ \sum_i A_{ij} x_i \leq b_j \ \ \forall j = 1 \ , \ \ldots \ , \ m \ and \ x_i \geq 0$$

We will have its dual as

$$Min \ \sum_j b_j y_j \ \ s.t. \ \ \sum_j A_{ij} y_j \geq c_i \ \ \forall \ i = 1, \ \ldots \ , \ n \ and \ y_j \geq 0$$

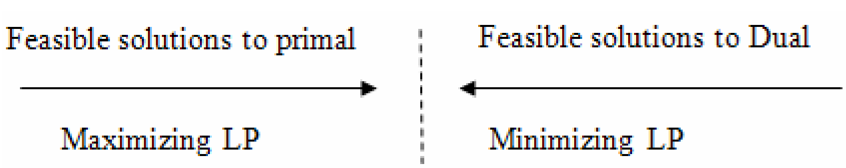
Thus we have a variable y_j for every constraint in the primal LP. The objective function is linear combination th b_j multiplied by the y_j. To get the constraints of the new LP, we multiply each of the constraints of the primal LP by the multiplier y_j, then the coefficients of every x_i must sum up to no more than c_i

Theorem 1: Weak Duality Theorem

If $\mathbf{x}$ *is a feasible solution to Primal and* $\mathbf{y}$ *is a feasible solution to Dual then the value of* $\mathbf{c}^T\mathbf{x}$ *is smaller than or equal to the value of* $\mathbf{b}^T\mathbf{y}$.

Proof:

$c^T x \leq (y^T A)x = y^T(Ax) \leq y^T b$ -----4.1



Primal is unbounded, then Dual is infeasible.
The Dual of Dual program is the Primal.
If Dual is unbounded then Primal is infeasible.

Theorem 2: Strong Duality Theorem

If Primal has an optimal solution $\mathbf{x^*}$ *then Dual has optimal solution* $\mathbf{y^*}$ *and* $\mathbf{c}^T\mathbf{x^*} = \mathbf{b}^T\mathbf{y^*}$.

Theorem 3: Complimentary Slackness Theorem

If optimal solutions for primal and dual are equal then we have

$c^T x = (y^T A)x$ or ----4.2

$$\sum_i (\sum_j A_{ij} y_j) x_i = \sum_i c_i x_i$$

$y^T(Ax) = y^T b$ or ----4.3

$$\sum_j b_j y_j = \sum_j (\sum_i A_{ij} x_i) y_j$$

Each term on left side of equation 4.2 must be equal to the corresponding term on the right hand side of equation 4.2 and each term on the left hand side of the equation 4.3 must be equal to corresponding term on the right hand side of equation 4.3. If in eq 4.2 $b_i \neq \sum_i A_{ij} x_i$ we have yj = 0. If ci $\neq \sum_j A_{ij} y_j$ in equation 4.2 we have $x_i = 0$.

Therefore we have

1. $\forall$ *j: either yj = 0 or bj =* $\sum_i A_{ij} x_i$.
2. $\forall$ *i: either xi = 0 or ci =* $\sum_j A_{ij} y_j$.

Condition 1 is called dual complementary slackness theorem(DCS) and condition 2 is called primal complimentary slackness theorem(PCS). If *bj* $= \sum_i A_{ij} x_i$ we say that j^{th} constraint in primal is tight. If *ci* $= \sum_j A_{ij} y_j$ we say that ith constraint in dual is tight.

If (x,y*) are primal and dual optimal solutions respectively if and only if they satisfy DCS and PCS.*