The 52 playing cards in a deck are arranged randomly, face up, to form an array with 4 rows and 13 columns. Show that it is always possible to pick one card from each column so that the 13 cards you get are all different in value i.e. ignoring the suit you get all the cards ace, king, queen, jack, 2, 3, 4, 5, 6, 7, 8, 9, 10. (10)

Create a bipartite graph $G=(U,V,E)$. For every column we have a vertex in U & for every value a vertex in V.

Hence $|U|=|V|=13$.

The edge $(u,v)\in E$ if column u contains the card with value v. Since every column has 4 cards & every value, there are 4 cards of that value, the degree of every vertex is 4. Hence the graph is 4-regular.

The graph G has a perfect matching. For contradiction assume that it does not. Hence there is a Hall set, a set $S\subseteq U$ s.t. $|N(S)|<|S|$. Now there are $4|S|$ edges incident to vertices in S & all these edges are also incident to vertices in $N(S)$.

So $4|S|\geq 4N(S)$ & hence the contradiction.

The edges of the perfect matching give the card to pick from each column.

Marking scheme: 4 marks for constructing the bipartite graph or an equivalent flow network

+4 marks for ~~claiming that this graph has a perfect matching~~ arguing that this is a 4-regular graph.

Common mistakes:

1) building a flow-network & saying (without proof) that it will have a flow of 13 units.

2) going column by column to pick a card. No such approach is likely to work without backtracking. Proving that by backtracking you will succeed is even more difficult.