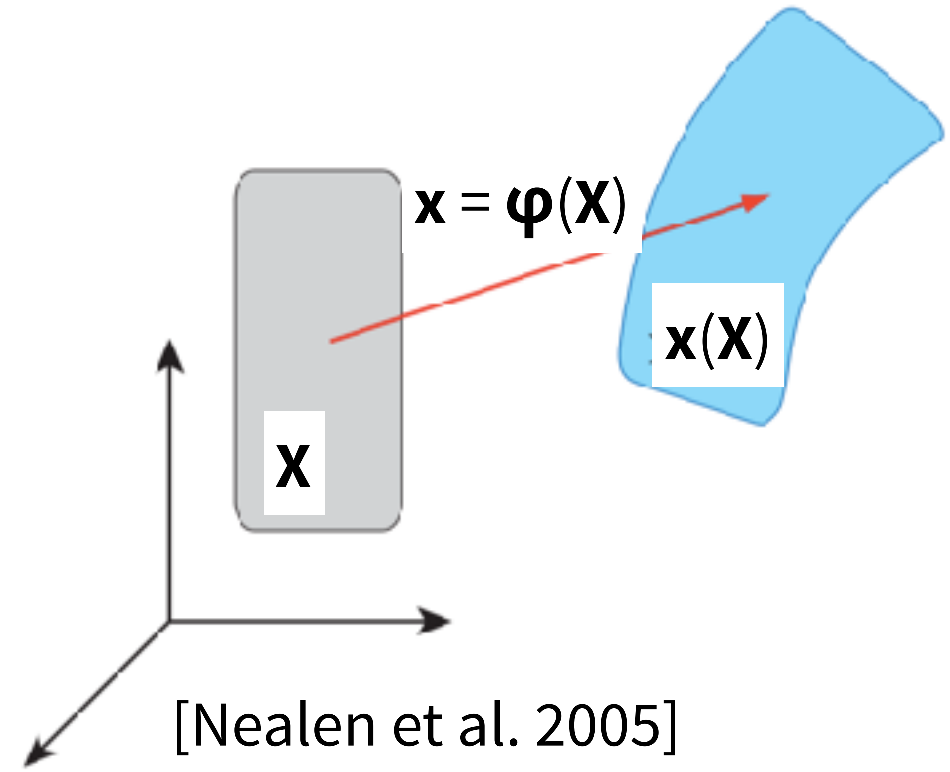
COL865: Special Topics in Computer Applications

Physics-Based Animation

16 — Elasticity

Elastic solids



[Nealen et al. 2005]

Elastic body = reference shape
+ nonrigid transformation

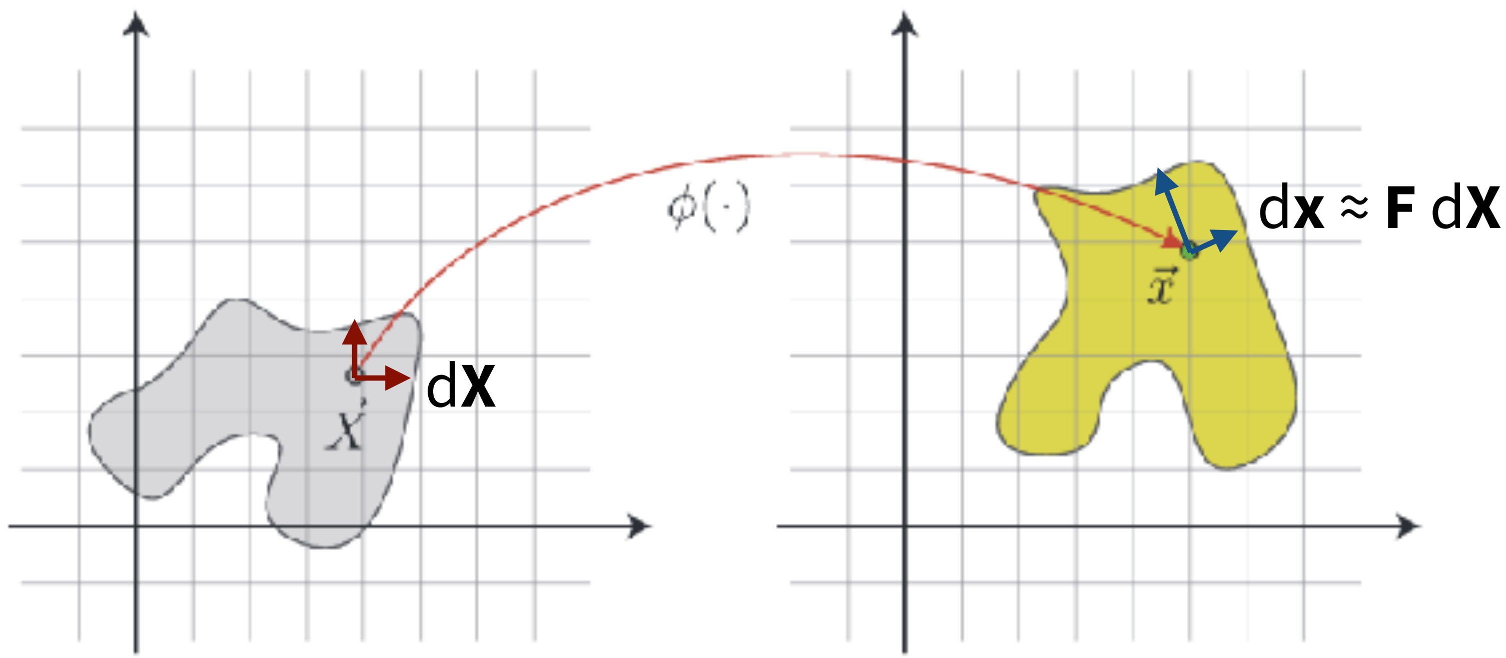
$$\mathbf{x} = \boldsymbol{\varphi}(\mathbf{X})$$
$$\mathbf{v} = \mathrm{d}\boldsymbol{\varphi}(\mathbf{X})/\mathrm{d}t$$

Lagrangian representation: track material points at fixed reference coordinates **X**, varying spatial coordinates **x**

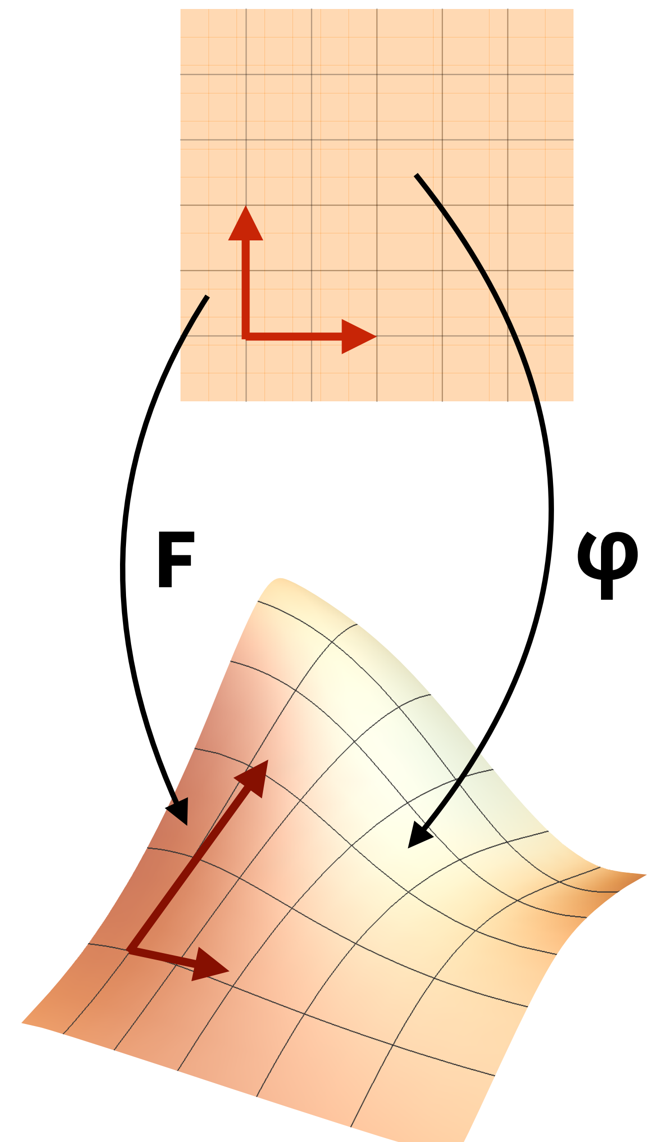
Reading

- Sifakis & Barbic, *FEM Simulation of 3D Deformable Solids*, Part 1: "The classical FEM method and discretization methodology", Ch. 2 and 3
- ***Theoretical background***: Bonet & Wood, *Nonlinear Continuum Mechanics for Finite Element Analysis*

Kinematics



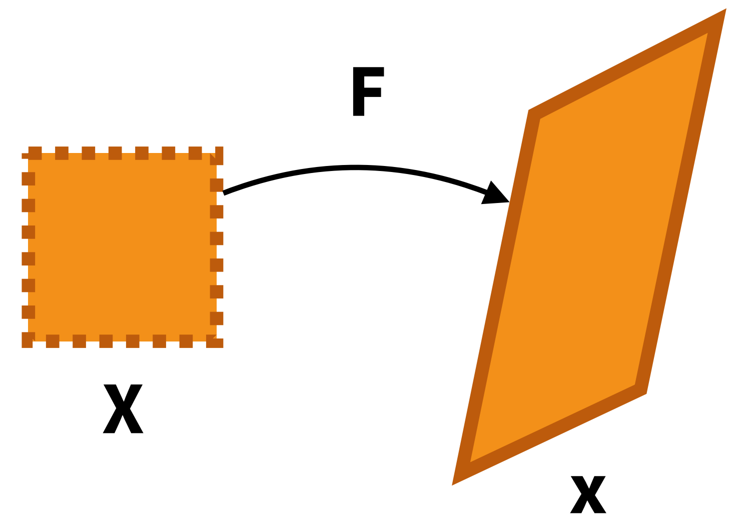
[Sifakis & Barbic]



Deformation function $\boldsymbol{\varphi} : \mathbb{R}^3 \to \mathbb{R}^3$

Deformation gradient $\mathbf{F} = d\boldsymbol{\varphi}/d\mathbf{X}$

Strain



Change in length:

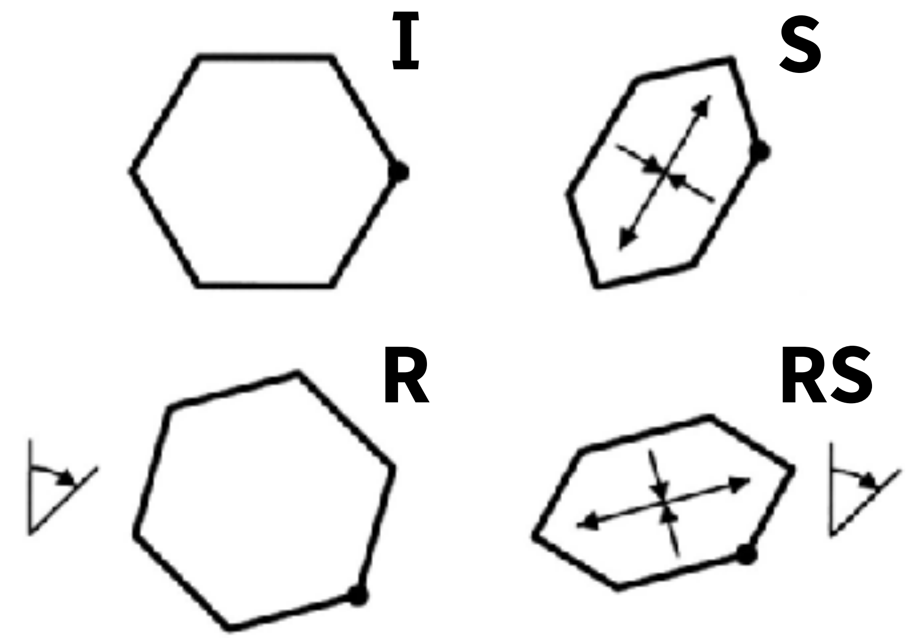
$$\|d\mathbf{x}\| - \|d\mathbf{X}\| = ?$$

$$\tfrac{1}{2}\left(\|d\mathbf{x}\|^2 - \|d\mathbf{X}\|^2\right) = \tfrac{1}{2}\, d\mathbf{X}^T \left(\mathbf{F}^T\mathbf{F} - \mathbf{I}\right) d\mathbf{X}$$

Green strain tensor: $\mathbf{E} = \tfrac{1}{2}\left(\mathbf{F}^T\mathbf{F} - \mathbf{I}\right)$

- What do its diagonal and off-diagonal entries represent?
- If $\boldsymbol{\varphi} = \mathbf{R}\,\mathbf{X} + \mathbf{t}$, what is $\mathbf{F}$? What is $\mathbf{E}$?

Strain



[Shoemake & Duff 1992]

Polar decomposition:

$$\mathbf{F} = \mathbf{R}\,\mathbf{S}$$

- **S** symmetric: stretching in material space
- **R** orthogonal: rotation into world space

$$\mathbf{E} = \tfrac{1}{2}\,(\mathbf{S}^2 - \mathbf{I})$$

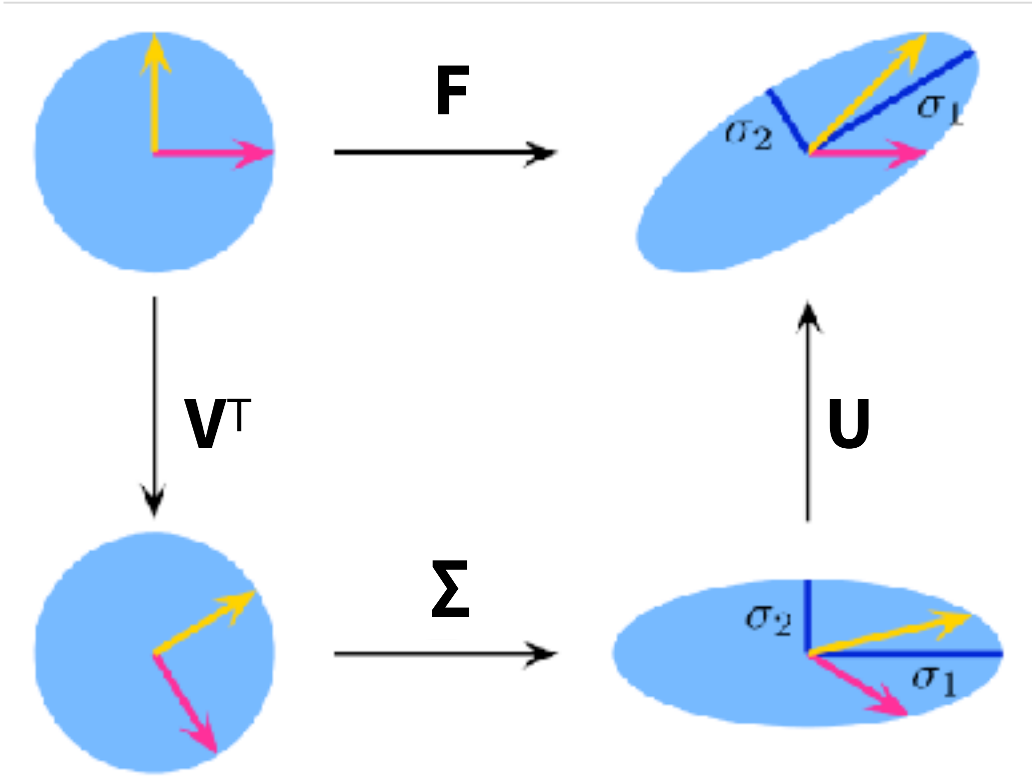
Principal strains

Singular value decomposition:

$$\mathbf{F} = \mathbf{U}\,\boldsymbol{\Sigma}\,\mathbf{V}^T$$

- **V** orthogonal: principal directions in material space
- **Σ** diagonal: principal strains
- **U** orthogonal: principal directions in world space

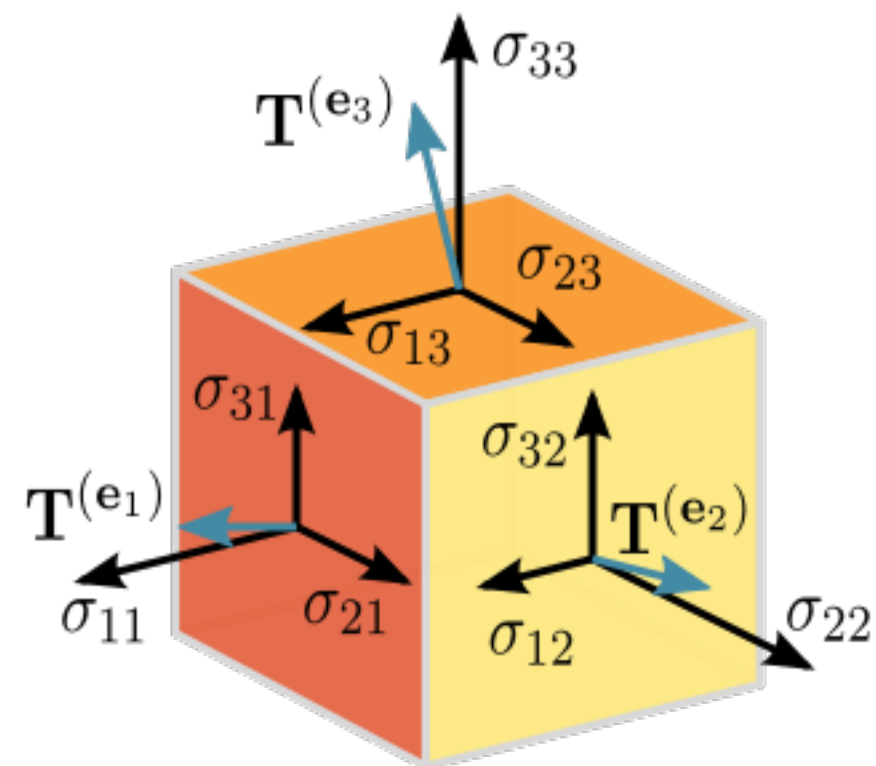
$$\mathbf{R}\,\mathbf{S} = (\mathbf{U}\,\mathbf{V}^T)\,(\mathbf{V}\,\boldsymbol{\Sigma}\,\mathbf{V}^T) = \mathbf{U}\,\boldsymbol{\Sigma}\,\mathbf{V}^T$$



Strain and stress

In continuum elasticity, stretching is measured by ***strain tensor***

Forces will also be described by a ***stress tensor***



Tensor analysis

Tensors

Mathematically, an nth-order tensor in an inner product space V is essentially a multilinear function from $V \times V \times \cdots \times V$ to $\mathbb{R}$

- 0th-order = scalar s: $() \mapsto s$
- 1st-order = vector $\mathbf{b}$: $\mathbf{u} \mapsto \mathbf{b} \cdot \mathbf{u} = \sum b_i u_i$
- 2nd-order = matrix $\mathbf{M}$: $(\mathbf{u}, \mathbf{v}) \mapsto \mathbf{u}^\mathsf{T} \mathbf{M} \mathbf{v} = \sum m_{ij} u_i v_j$

Computationally, a 2nd-order tensor is just a matrix…

Useful operations

Inner product for 2nd-order tensors: ***double contraction***

$$\mathbf{A} : \mathbf{B} = \mathrm{tr}\,(\mathbf{A}^\mathrm{T}\mathbf{B}) = \sum A_{ij}\, B_{ij}$$

- Commutativity: $\mathbf{A} : \mathbf{B} = \mathbf{B} : \mathbf{A} = \mathbf{A}^\mathrm{T} : \mathbf{B}^\mathrm{T} = \mathbf{B}^\mathrm{T} : \mathbf{A}^\mathrm{T}$
- Frobenius norm: $\|\mathbf{A}\|^2 = \mathbf{A} : \mathbf{A} = \sum A_{ij}^2$

Symmetric and skew-symmetric tensors:

- $\mathbf{S} : \mathbf{W} = 0$ for all $\mathbf{S}$ symmetric, $\mathbf{W}$ skew-symmetric
- $\mathbf{A} = \mathbf{S} + \mathbf{W}$ where $\mathbf{S} = \frac{1}{2}\,(\mathbf{A} + \mathbf{A}^\mathrm{T})$, $\mathbf{W} = \frac{1}{2}\,(\mathbf{A} - \mathbf{A}^\mathrm{T})$

Tensor invariants

For a symmetric 2nd-order tensor **S**:

- $I_{\mathbf{S}} = \text{tr}\,\mathbf{S} \quad = \lambda_1 + \lambda_2 + \lambda_3$
- $II_{\mathbf{S}} = \|\mathbf{S}\|^2 \quad = \lambda_1^2 + \lambda_2^2 + \lambda_3^2$
- $III_{\mathbf{S}} = \det \mathbf{S} \quad = \lambda_1 \lambda_2 \lambda_3$

These are invariant to rotation: same for **S** and $\mathbf{RSR}^T$

Differentiation

What is the derivative of a real-valued $f(\mathbf{A})$?
e.g. $f(\mathbf{A}) = \|\mathbf{A} - \mathbf{I}\|^2$. Consider infinitesimal change $\delta\mathbf{A}$:

$$\begin{aligned} f(\mathbf{A} + \delta\mathbf{A}) &= \|\mathbf{A} + \delta\mathbf{A} - \mathbf{I}\|^2 \\ &= (\mathbf{A} + \delta\mathbf{A} - \mathbf{I}) : (\mathbf{A} + \delta\mathbf{A} - \mathbf{I}) \\ &= \|\mathbf{A} - \mathbf{I}\|^2 + 2\,(\mathbf{A} - \mathbf{I}) : \delta\mathbf{A} + \|\delta\mathbf{A}\|^2 \end{aligned}$$

$$\Rightarrow \quad \delta f = 2(\mathbf{A} - \mathbf{I}) : \delta\mathbf{A}$$

So we define $\mathrm{d}f/\mathrm{d}\mathbf{A} = 2(\mathbf{A} - \mathbf{I})$

Differentiation

$$\nabla f(\mathbf{u}) = \mathbf{v} \quad \iff \quad f(\mathbf{u} + \delta\mathbf{u}) = f(\mathbf{u}) + \mathbf{v} \cdot \delta\mathbf{u}$$

$$\iff \quad \mathbf{v} = \begin{bmatrix} \partial f / \partial u_1 \\ \vdots \\ \partial f / \partial u_n \end{bmatrix}$$

$$\frac{\mathrm{d}f(\mathbf{A})}{\mathrm{d}\mathbf{A}} = \mathbf{B} \quad \iff \quad f(\mathbf{A} + \delta\mathbf{A}) = f(\mathbf{A}) + \mathbf{B} : \delta\mathbf{A}$$

$$\iff \quad \mathbf{B} = \begin{bmatrix} \partial f / \partial A_{11} & \cdots & \partial f / \partial A_{1n} \\ \vdots & \ddots & \vdots \\ \partial f / \partial A_{m1} & \cdots & \partial f / \partial A_{mn} \end{bmatrix}$$

Differentiation

If f maps matrices to matrices, derivative is a 4th-order tensor…
Easier to express via differentials instead

e.g. $\mathbf{E} = \frac{1}{2}(\mathbf{F}^\mathsf{T}\mathbf{F} - \mathbf{I})$. Show that $\delta\mathbf{E} = \frac{1}{2}(\mathbf{F}^\mathsf{T}\delta\mathbf{F} + \delta\mathbf{F}^\mathsf{T}\mathbf{F})$.

Tensor analysis

Strains, stresses vary over space: ***tensor field***

If $\mathbf{S}(\mathbf{x})$ is a tensor field, its ***divergence*** is a vector:

$$(\mathrm{div}\,\mathbf{S})_i = \sum \partial S_{ij}/\partial x_j$$

- Treat each row as vector field, compute divergences

Divergence theorem for tensors:

$$\oiint \mathbf{S}\,\mathbf{n}\,\mathrm{d}A = \iiint \mathrm{div}\,\mathbf{S}\,\mathrm{d}V$$

Notational conveniences

Einstein notation: implied summation over repeated indices

- $\mathbf{u} \cdot \mathbf{v} = u_i \, v_i$
- $\mathbf{A} : \mathbf{B} = A_{ij} \, B_{ij}$
- $\mathbf{y} = \mathbf{A}\,\mathbf{x} \;\Leftrightarrow\; y_i = A_{ij} \, x_j$

Comma derivative for partial differentiation:

- $(\nabla f)_i = f_{,i}$
- $\nabla \cdot \mathbf{v} = v_{i,i}$
- $\operatorname{div} \mathbf{S} = S_{ij,j}$

Mechanics

Traction and stress

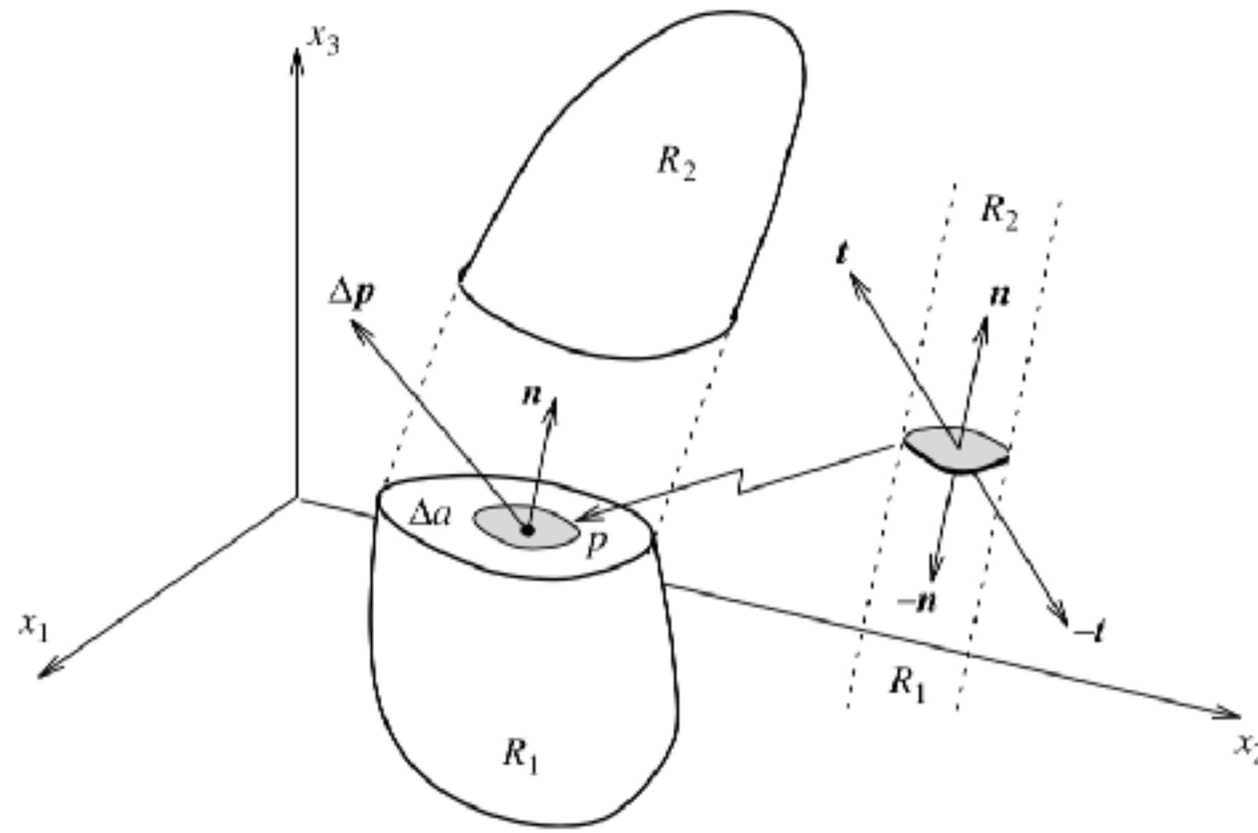
Traction = force per unit area (not necessarily normal)

$$d\mathbf{f} = \mathbf{t}\, dA$$

Force is linear in $\mathbf{n}\, dA$, so

$$d\mathbf{f} = \boldsymbol{\sigma}\, \mathbf{n}\, dA,$$
$$\mathbf{t} = \boldsymbol{\sigma}\, \mathbf{n}$$

for some 2nd-order tensor $\boldsymbol{\sigma}(\mathbf{x})$:
Cauchy stress tensor

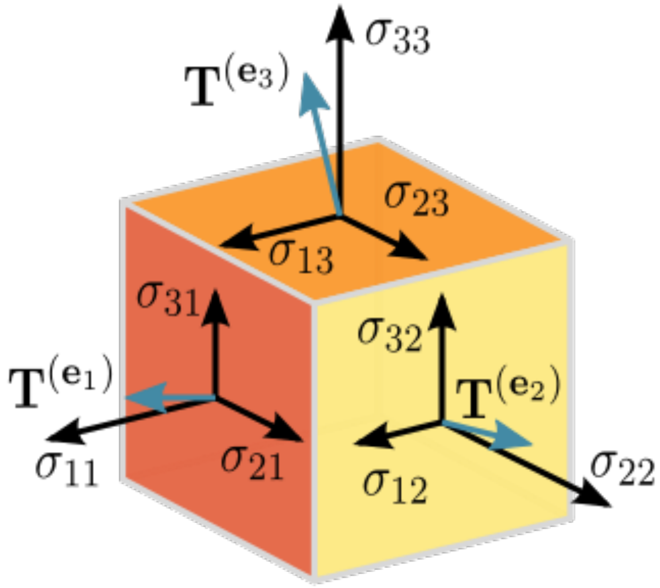


[Bonet & Wood]

Stress

Force on each face: $\mathbf{t}\,\mathrm{d}A = \boldsymbol{\sigma}\,\mathbf{n}\,\mathrm{d}A$

- Net force = div $\boldsymbol{\sigma}\,\mathrm{d}V$
- Angular momentum conservation $\Rightarrow \boldsymbol{\sigma} = \boldsymbol{\sigma}^\mathsf{T}$



Equations of motion:

$$\mathrm{d}m\,\ddot{\mathbf{x}} = \mathrm{div}_{\mathbf{x}}\,\boldsymbol{\sigma}\,\mathrm{d}V + \mathrm{d}\mathbf{f}^{\mathrm{ext}}$$

At boundaries:

$$\mathbf{t}^{\mathrm{ext}}\,\mathrm{d}A - \boldsymbol{\sigma}\,\mathbf{n}\,\mathrm{d}A = 0$$

e.g. Pressure in fluid: $\mathbf{t} = -p\,\mathbf{n} \;\Rightarrow\; \boldsymbol{\sigma} = -p\,\mathbf{I}$. What is net force?

Stress vs. material space

Cauchy stress is defined in terms of axis-aligned faces in world space

Consider faces in material space:

$$d\mathbf{f} = \mathbf{t}_0 \, dA_0 = \mathbf{P} \, \mathbf{n}_0 \, dA_0$$

$\mathbf{n}_0$, dA_0: normal, area in material space

$\mathbf{P}(\mathbf{X})$: ***1st Piola-Kirchhoff stress tensor*** (not symmetric)

$$dm \, \ddot{\mathbf{x}} = \mathrm{div}_{\mathbf{X}} \, \mathbf{P} \, dV_0 + d\mathbf{f}^{ext}$$

$$\mathbf{t}_0^{ext} \, dA_0 - \mathbf{P} \, \mathbf{n}_0 \, dA_0 = 0$$