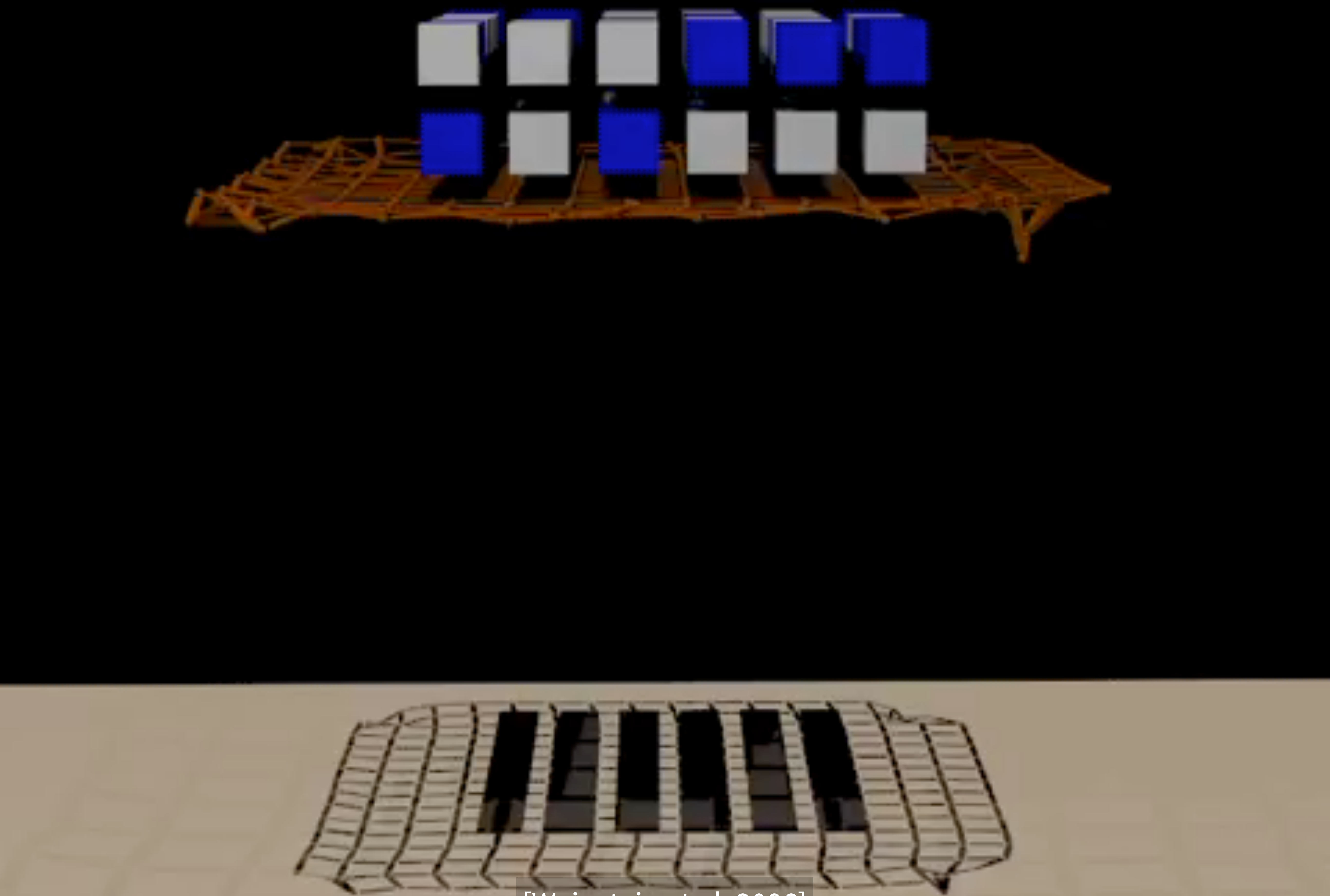
Constrained dynamics

How to describe the motion of a system subject to constraints?

- **Reduced coordinates**
- **Penalty forces**
- **Differentiating the constraints**
- **Projection methods**

Projection methods



[Weinstein et al. 2006]

[Goldenthal et al. 2007]

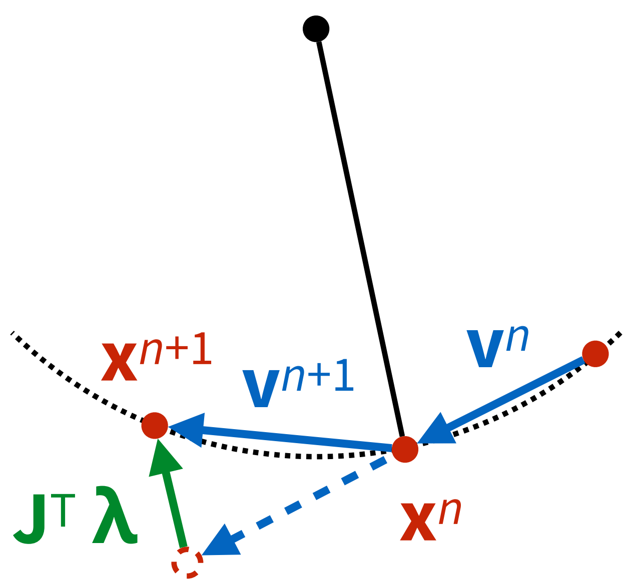
[Goldenthal et al. 2007]

Projection methods

Find constraint forces to make the system exactly satisfy $\mathbf{g}(\mathbf{x}) = \mathbf{0}$ at the end of the time step!

Example: semi-implicit Euler

$$\mathbf{v}^{n+1} = \mathbf{v}^{n} + \mathbf{M}^{-1}\,(\mathbf{f}^{n} + \mathbf{f}_{\mathrm{c}})\,\Delta t$$
$$\mathbf{x}^{n+1} = \mathbf{x}^{n} + \mathbf{v}^{n+1}\,\Delta t$$
$$\mathbf{g}(\mathbf{x}^{n+1}) = \mathbf{0}$$



Set $\mathbf{f}_{\mathrm{c}} = \mathbf{J}^{\mathrm{T}}\,\boldsymbol{\lambda}$, solve for $\boldsymbol{\lambda}$

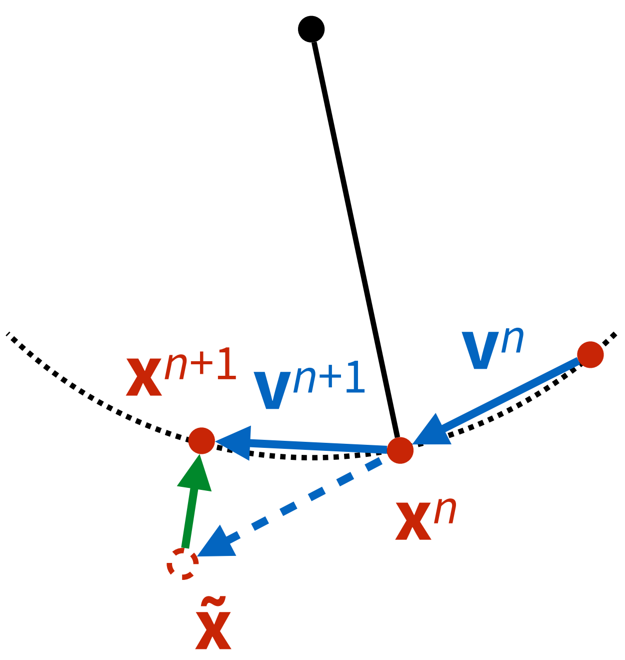
Linearization looks like earlier constraint equation:

$$(\mathbf{J}\,\mathbf{M}^{-1}\,\mathbf{J}^{\mathrm{T}})\,\boldsymbol{\lambda} = -(\mathbf{J}\,\mathbf{M}^{-1}\,\mathbf{f}^{n} + \mathbf{J}\,\mathbf{v}^{n}/\Delta t)$$

Projection methods

Since $\mathbf{g} = \mathbf{0}$ is imposed on $\mathbf{x}^{n+1}$, try to evaluate $\mathbf{J}$ also at $\mathbf{x}^{n+1}$

Various interpretations:



- ***Mass-orthogonal projection:***
  Take time step ignoring constraints, then project to "closest" point on $\mathbf{g}(\mathbf{x}) = \mathbf{0}$ manifold
- ***Impulse-based method:***
  Apply an impulse $\mathbf{J}^\mathsf{T} \boldsymbol{\lambda} \, \Delta t$ at end of time step so that $\mathbf{g}(\mathbf{x}^{n+1}) = \mathbf{0}$
- ***Splitting method:***
  Instead of integrating $\ddot{\mathbf{x}} = \mathbf{M}^{-1} (\mathbf{f} + \mathbf{f}_\mathrm{c})$, first integrate $\ddot{\mathbf{x}} = \mathbf{M}^{-1} \mathbf{f}$ then $\ddot{\mathbf{x}} = \mathbf{M}^{-1} \mathbf{f}_\mathrm{c}$

Projection methods

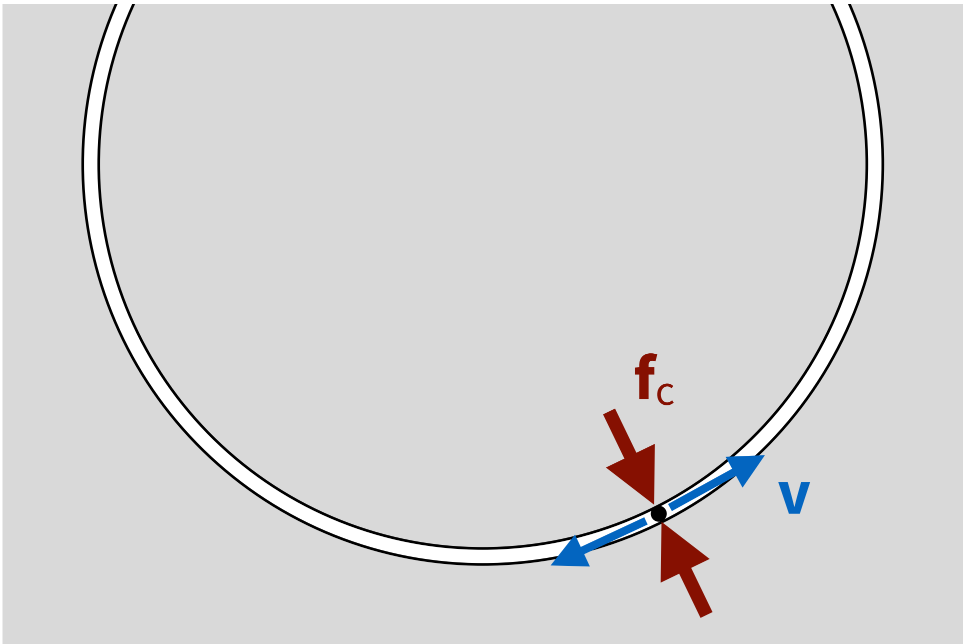
Several types of projection methods are possible:

- Apply constraint impulse at beginning of time step (***pre-stabilization***), or at end (***post-stabilization***)…
- …to satisfy position equation $\mathbf{g}(\mathbf{x}^{n+1}) = \mathbf{0}$, or velocity equation $\mathbf{J}(\mathbf{x}^{n+1})\, \mathbf{v}^{n+1} = \mathbf{0}$

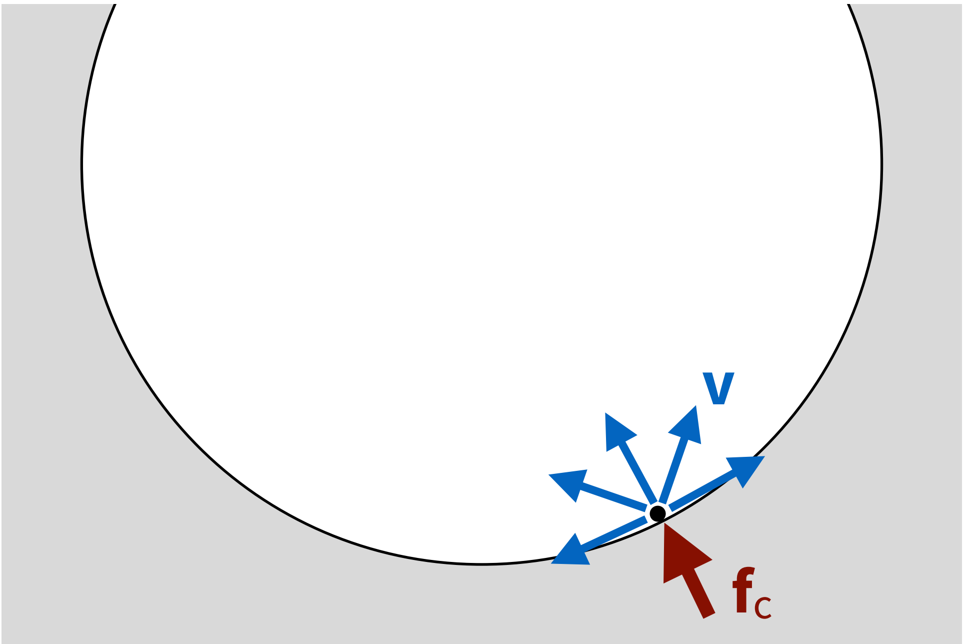
Limitation: Time integration accuracy reduces to first-order (unless you also include $\mathbf{J}^{\mathsf{T}} \boldsymbol{\lambda}$ in prediction step)

Outro: Collisions as inequality constraints

Inequality constraints



$\|\mathbf{x}_1\| - \ell_1 = 0$



$\|\mathbf{x}_1\| - \ell_1 \leq 0$

$$\mathbf{g}(\mathbf{x}) = \mathbf{0}$$

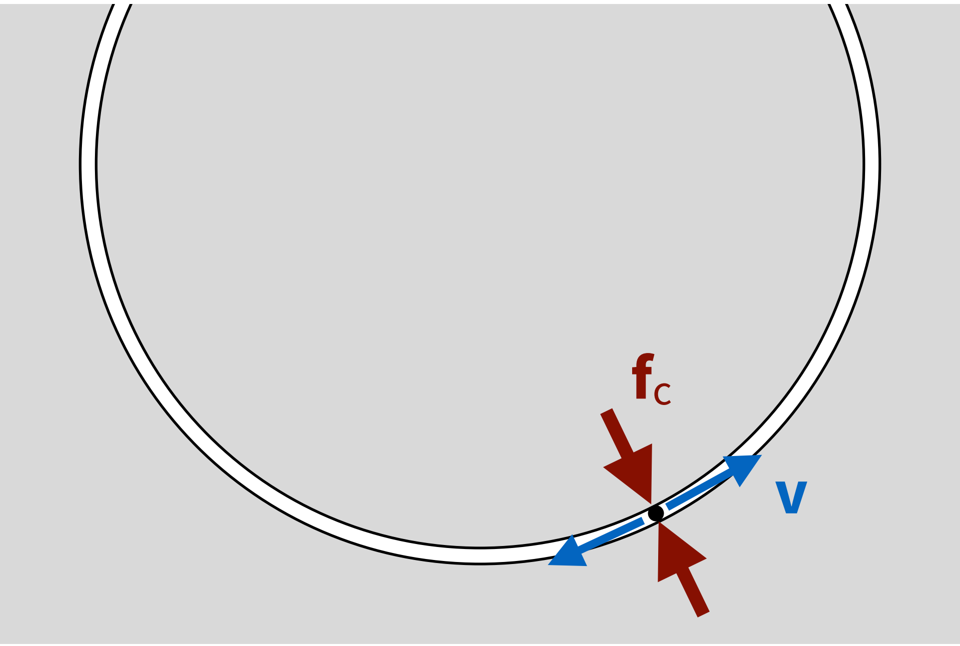
$$\mathbf{J}\,\mathbf{v} = \mathbf{0},\ \mathbf{f}_c = \mathbf{J}^\top \boldsymbol{\lambda}$$

$$\mathbf{g}(\mathbf{x}) \leq \mathbf{0}$$

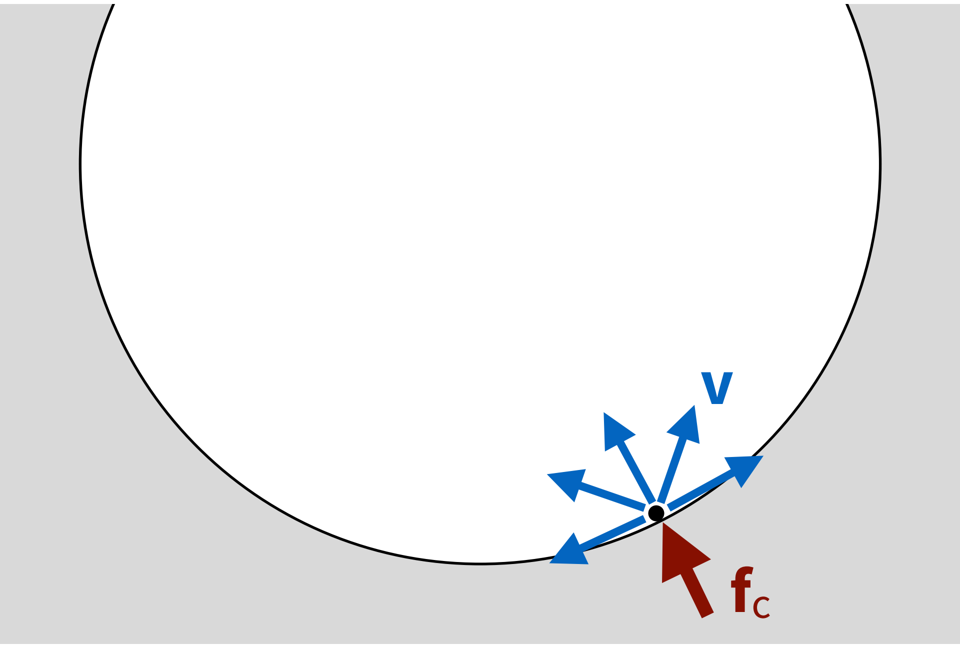
$$\mathbf{g}(\mathbf{x}) = \mathbf{0} \Rightarrow \mathbf{J}\,\mathbf{v} \leq \mathbf{0},\ \ \mathbf{f}_c = \mathbf{J}^\top \boldsymbol{\lambda}$$

$$\mathbf{g}(\mathbf{x}) < \mathbf{0} \Rightarrow \mathbf{J}\,\mathbf{v} = \text{anything},\ \ \mathbf{f}_c = \mathbf{0}$$

Inequality constraints



$\|\mathbf{x}_1\| - \ell_1 = 0$



$\|\mathbf{x}_1\| - \ell_1 \leq 0$

Penalty forces, projection methods can be applied

New issues: constraints can change between inactive & active

- How to detect when constraints become active
- Elastic vs. inelastic response, sustained contact