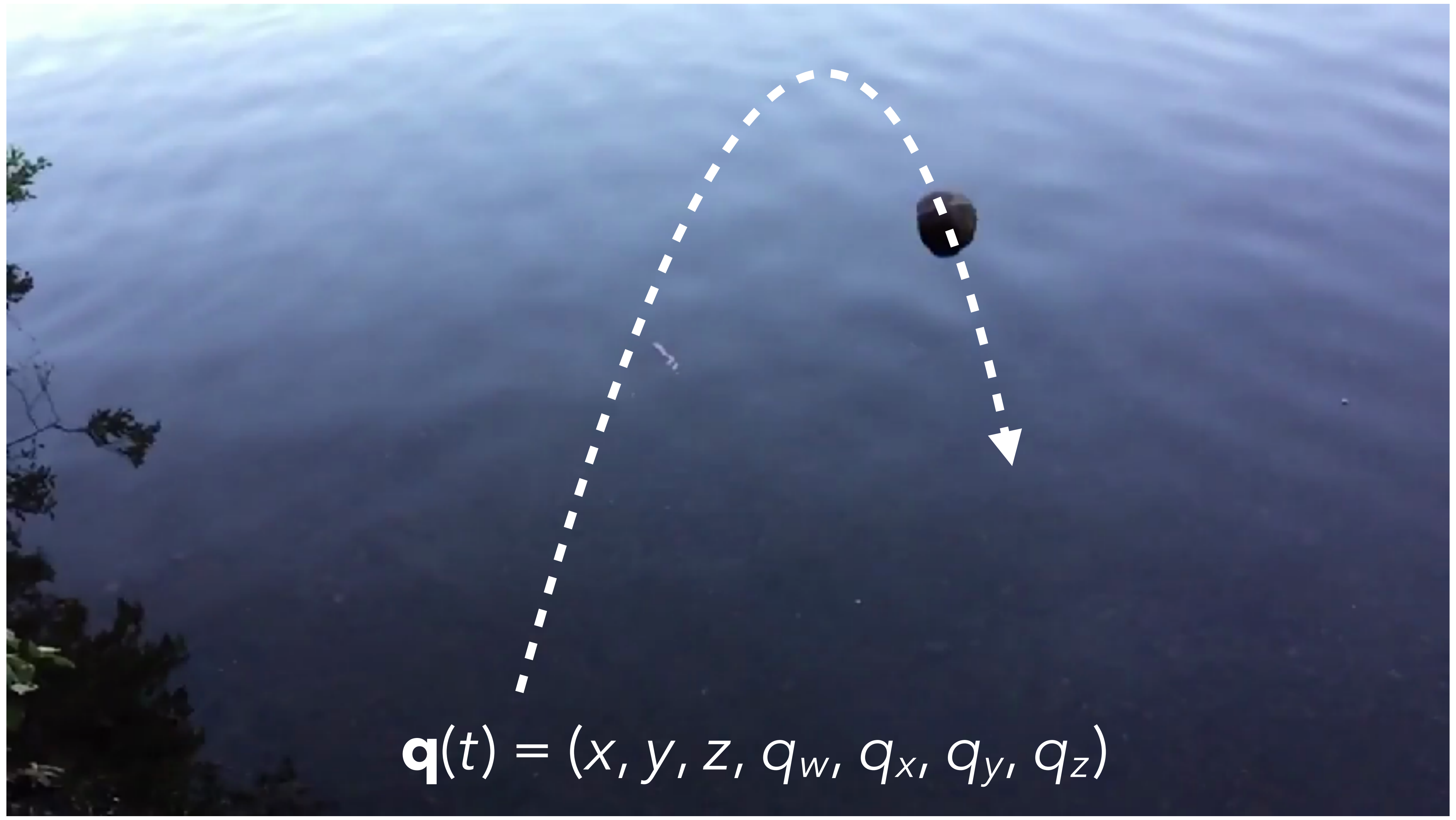
COL781: Computer Graphics

35. Continuum Models

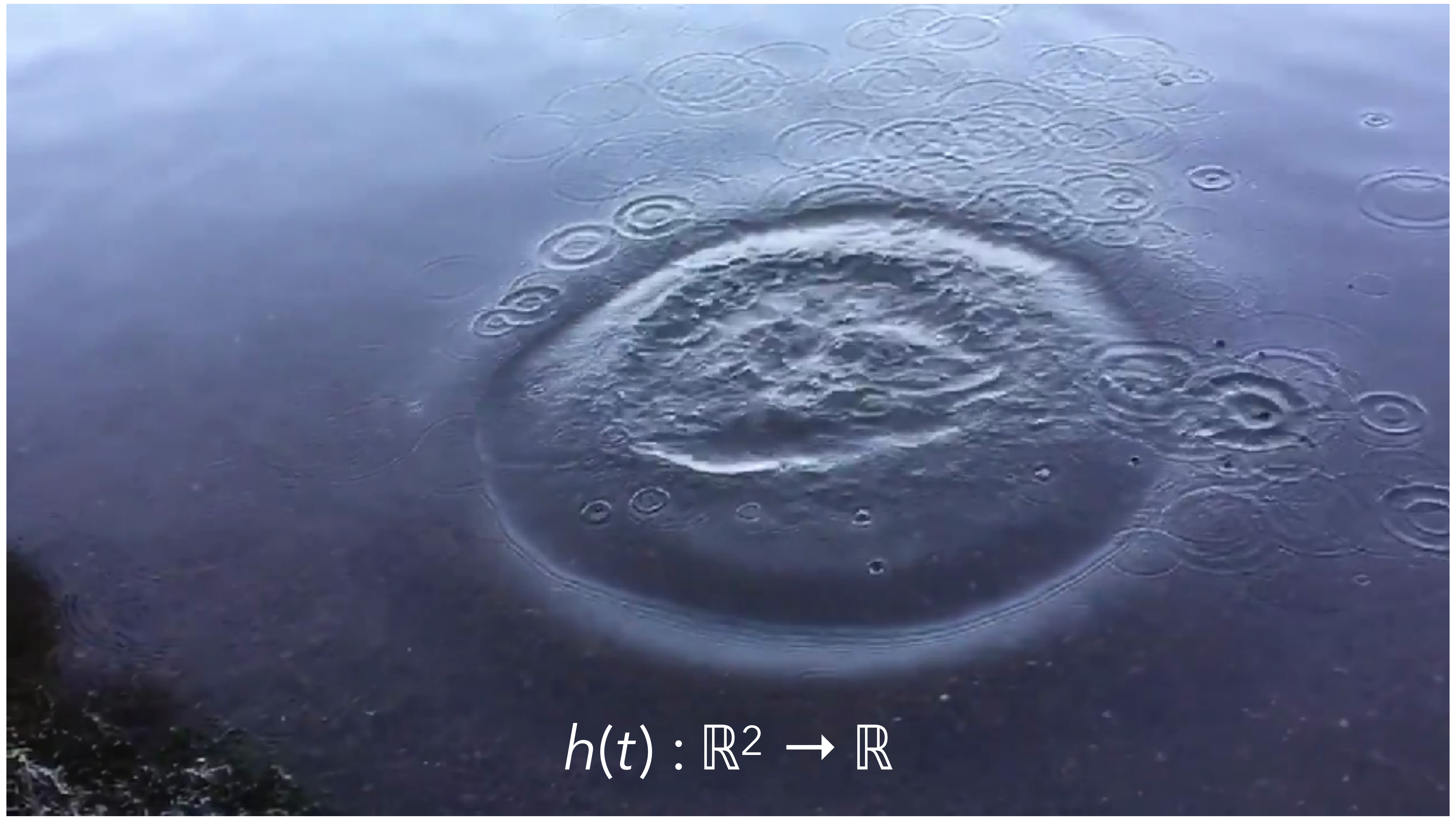
Schreck et al. 2019

So far, we know how to simulate discrete systems of particles and rigid bodies.

But lots of things in real life are not discrete:



Handsans



https://www.youtube.com/watch?v=RRPP73QM_4k

Liquids

https://www.youtube.com/watch?v=jVxYuPeqOPI

Narita & Ando 2022

Smoke

Molemaker et al. 2008

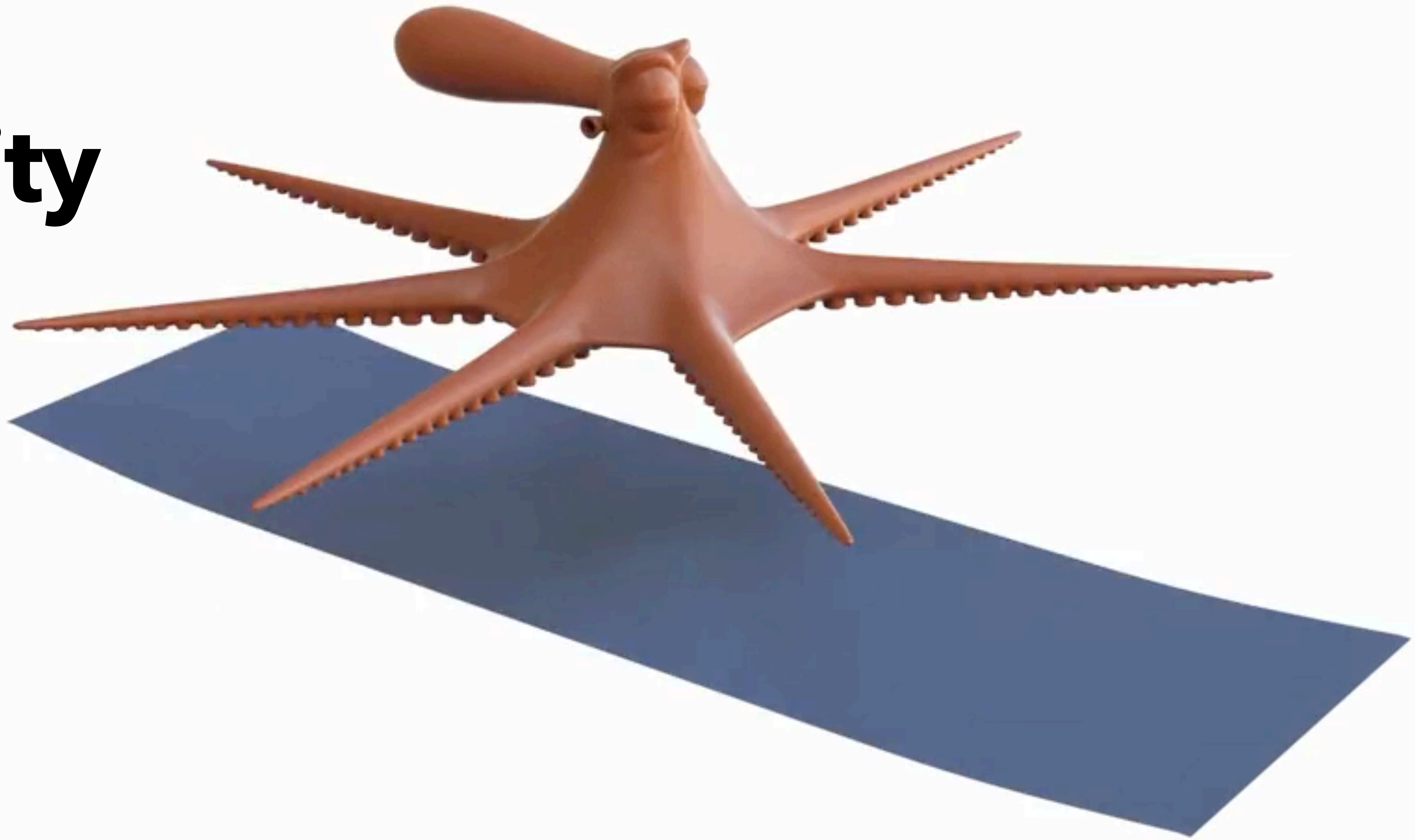
https://research.nvidia.com/publication/2008-07_low-viscosity-flow-simulations-animation

Cloth

https://www.youtube.com/watch?v=TH5g8TuKIkk

Wu et al. 2022

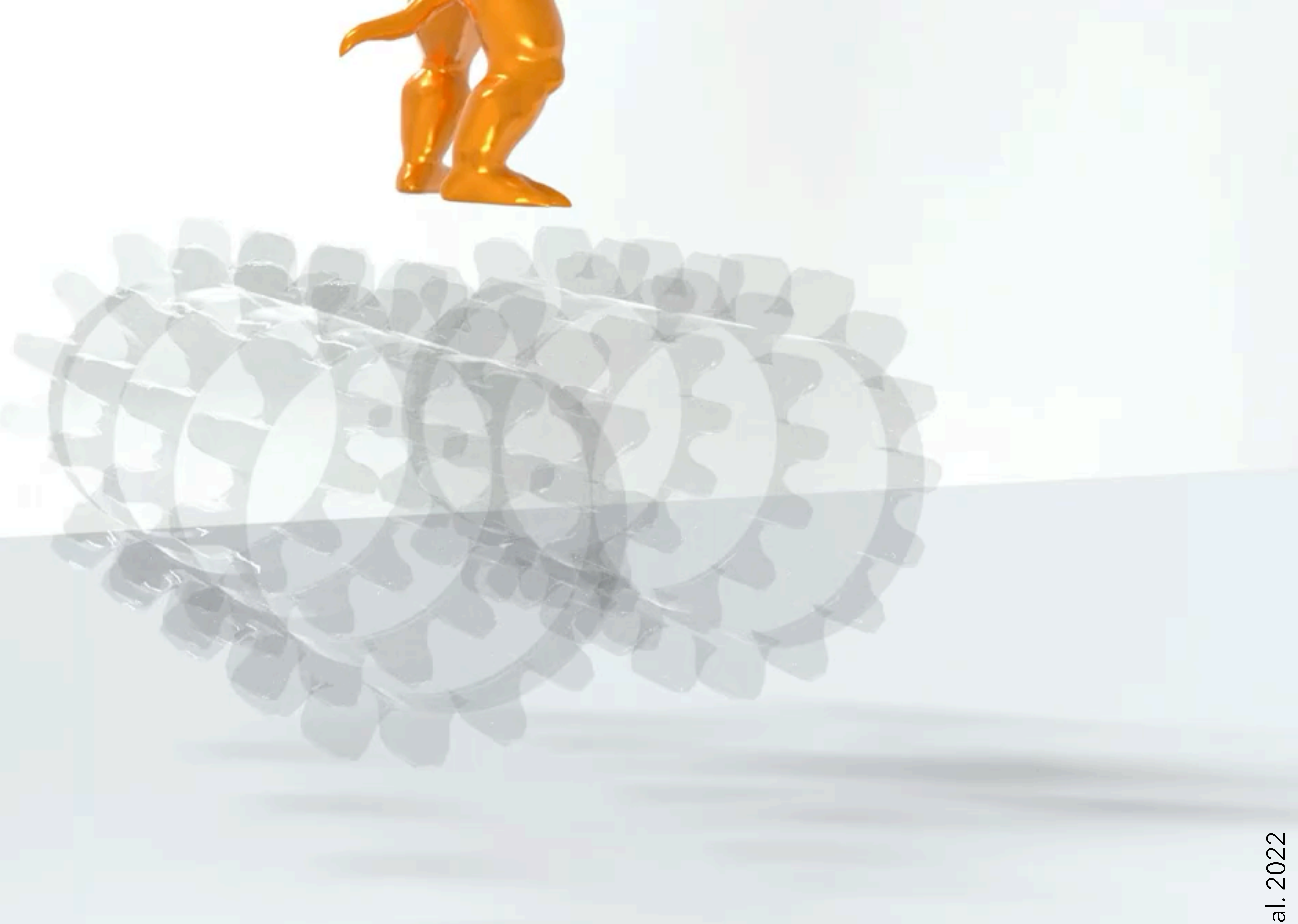
Elasticity



https://vimeo.com/333798247

Kim et al. 2019

Plasticity



https://www.youtube.com/watch?v=YvvoSu8NK3A

Li et al. 2022

Fracture

https://www.youtube.com/watch?v=lNri-x2nK7o

Wolper et al. 2019

Snow

https://vimeo.com/160322962

Stomakhin et al. 2013

Mass-spring chain → continuum rod

- Positions: $(\ldots, \mathbf{x}_i, \ldots)$ where $i \in \{1, 2, \ldots, n\}$
- Mass of particle: m_i
- Stretch of spring: $\|\mathbf{x}_{i+1} - \mathbf{x}_i\|/\ell_0$
- Force on particle: $\mathbf{f}_i - \mathbf{f}_{i-1}$

Equations of motion: ODE

$$\mathrm{d}^2\mathbf{x}_i/\mathrm{d}t^2 = m_i^{-1}\, \mathbf{f}(\mathbf{x}_i, \mathbf{x}_{i-1}, \mathbf{x}_{i+1}, \ldots)$$

- Positions: $\mathbf{x}(s)$ where $s \in [0, L]$
- Mass of differential segment: $\mathrm{d}m = \rho\ \mathrm{d}s$
- Stretch of segment: $\|\mathbf{x}(s+\mathrm{d}s) - \mathbf{x}(s)\|/\mathrm{d}s$ $= \|\partial\mathbf{x}/\partial s\|$
- Differential force on segment: $\mathbf{f}(s+\mathrm{d}s) - \mathbf{f}(s) = (\partial\mathbf{f}/\partial s)\ \mathrm{d}s$

Equations of motion: partial differential eq.

$$\partial^2\mathbf{x}/\partial t^2 = \rho^{-1}\, \mathbf{f}(\mathbf{x}, \partial\mathbf{x}/\partial s, \partial^2\mathbf{x}/\partial s^2, \ldots)$$

$$\frac{\partial^2 \mathbf{x}}{\partial t^2} = \rho^{-1}\,\mathbf{f}\left(\mathbf{x}, \frac{\partial \mathbf{x}}{\partial s}, \frac{\partial^2 \mathbf{x}}{\partial s^2}, \ldots\right)$$

Our goal in this course:

- Not to fully understand the physics (except enough to gain intuition)
- Not to fully understand the mathematics (except enough to gain intuition)
- Understand how to compute numerical solutions to such equations!

We'll start with some simpler examples…

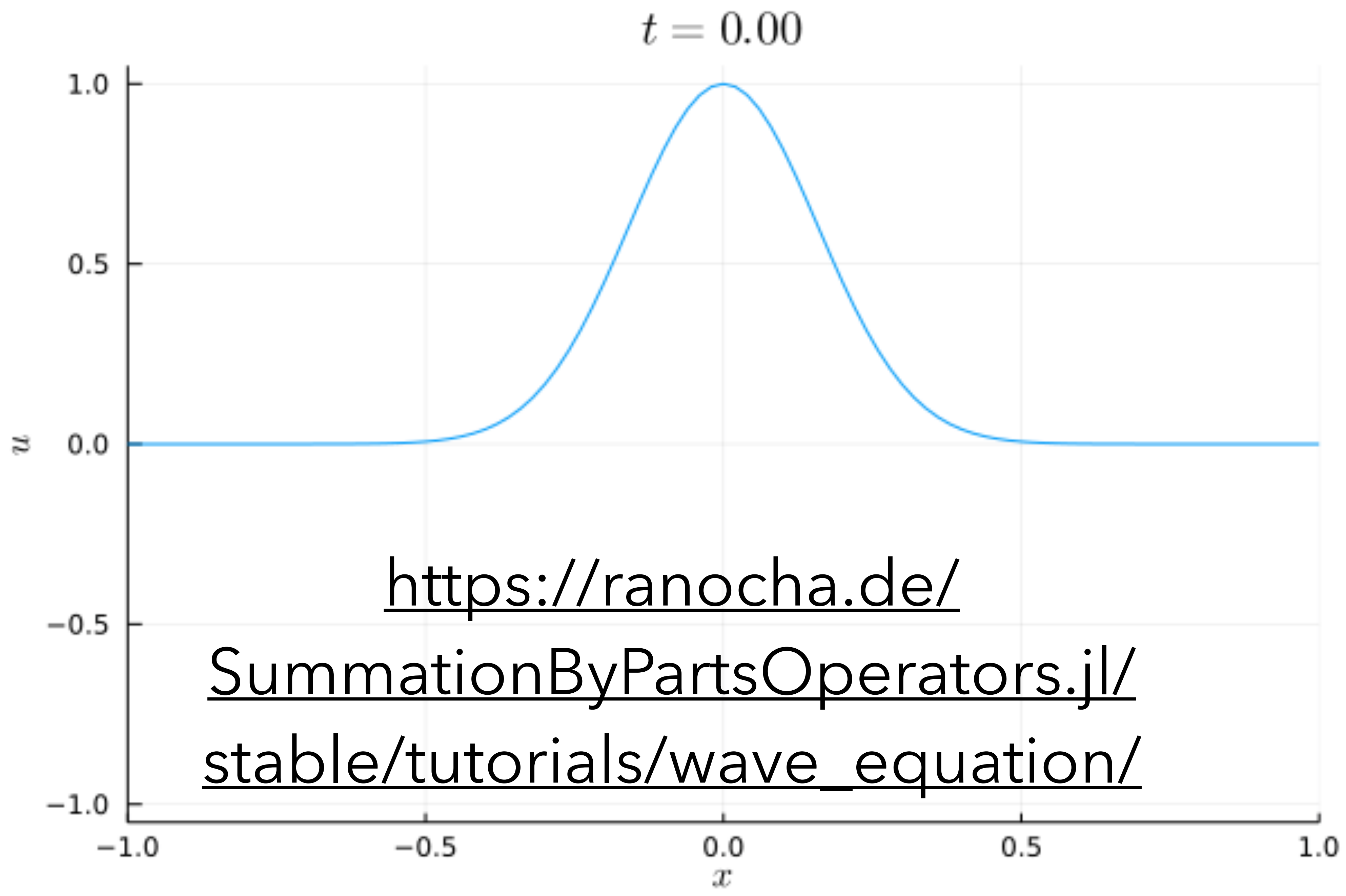
The wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$$

Similar to a harmonic oscillator $\mathrm{d}^2u/\mathrm{d}t^2 = -ku$: leads to oscillations

Right-hand side $\mathrm{d}^2u/\mathrm{d}x^2$ is curvature of graph $u(x)$

- Restoring force tries to straighten curves, flatten extrema



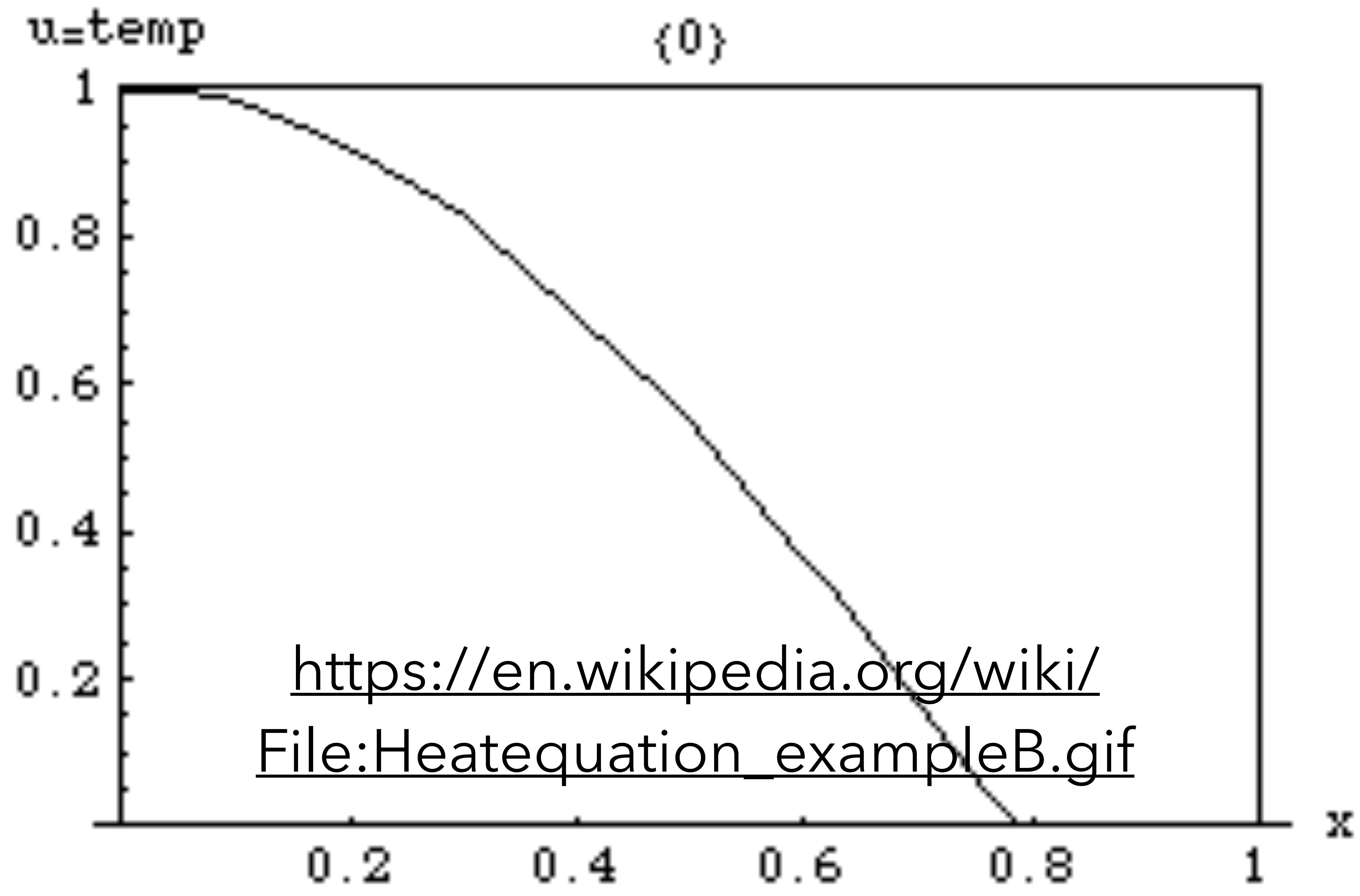
https://ranocha.de/SummationByPartsOperators.jl/stable/tutorials/wave_equation/

Hendrik Ranocha

The heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

- Second derivative → oscillations / waves
- First derivative → decay / smoothing



https://en.wikipedia.org/wiki/File:Heatequation_exampleB.gif

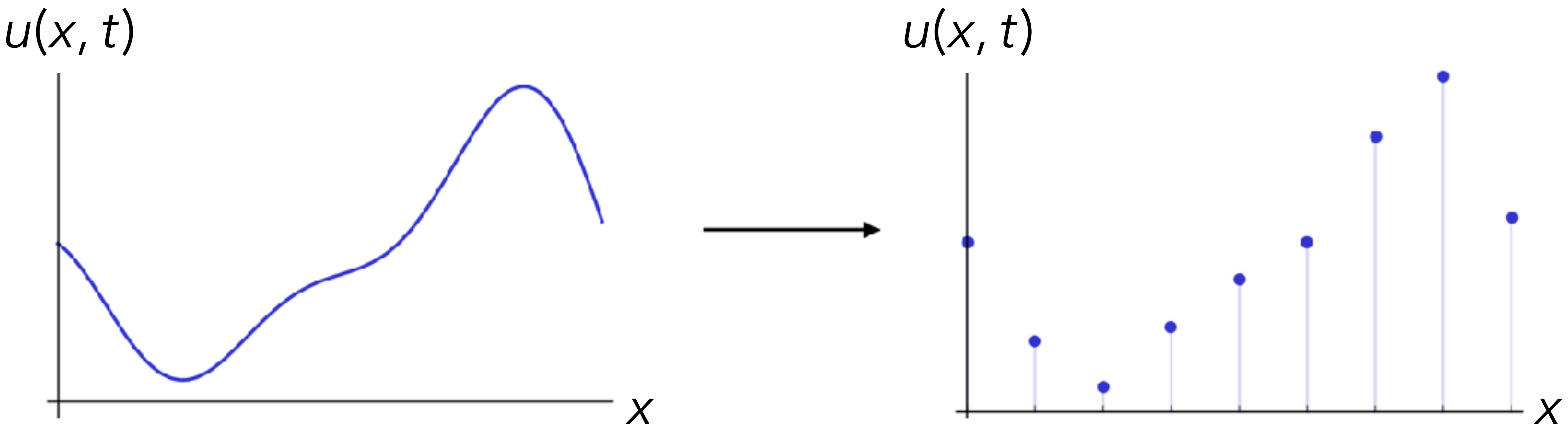
So, here is our problem:

Given the partial differential equation
and the initial conditions $u(x, t=0)$,
find the spatial distribution $u(x, t)$ at all future times t

At any future time t, how to represent the function $u(x, t)$?

- Even if $u(x, 0)$ has an analytical form, $u(x, t)$ probably doesn't for $t > 0$

Simplest representation: store samples $u_i(t) = u(x_i, t)$ at various x_i



Now we need to solve $\dfrac{\mathrm{d}u_i}{\mathrm{d}t} = \dfrac{\partial^2 u}{\partial x^2} \ldots$

How to estimate spatial derivatives from discrete samples?

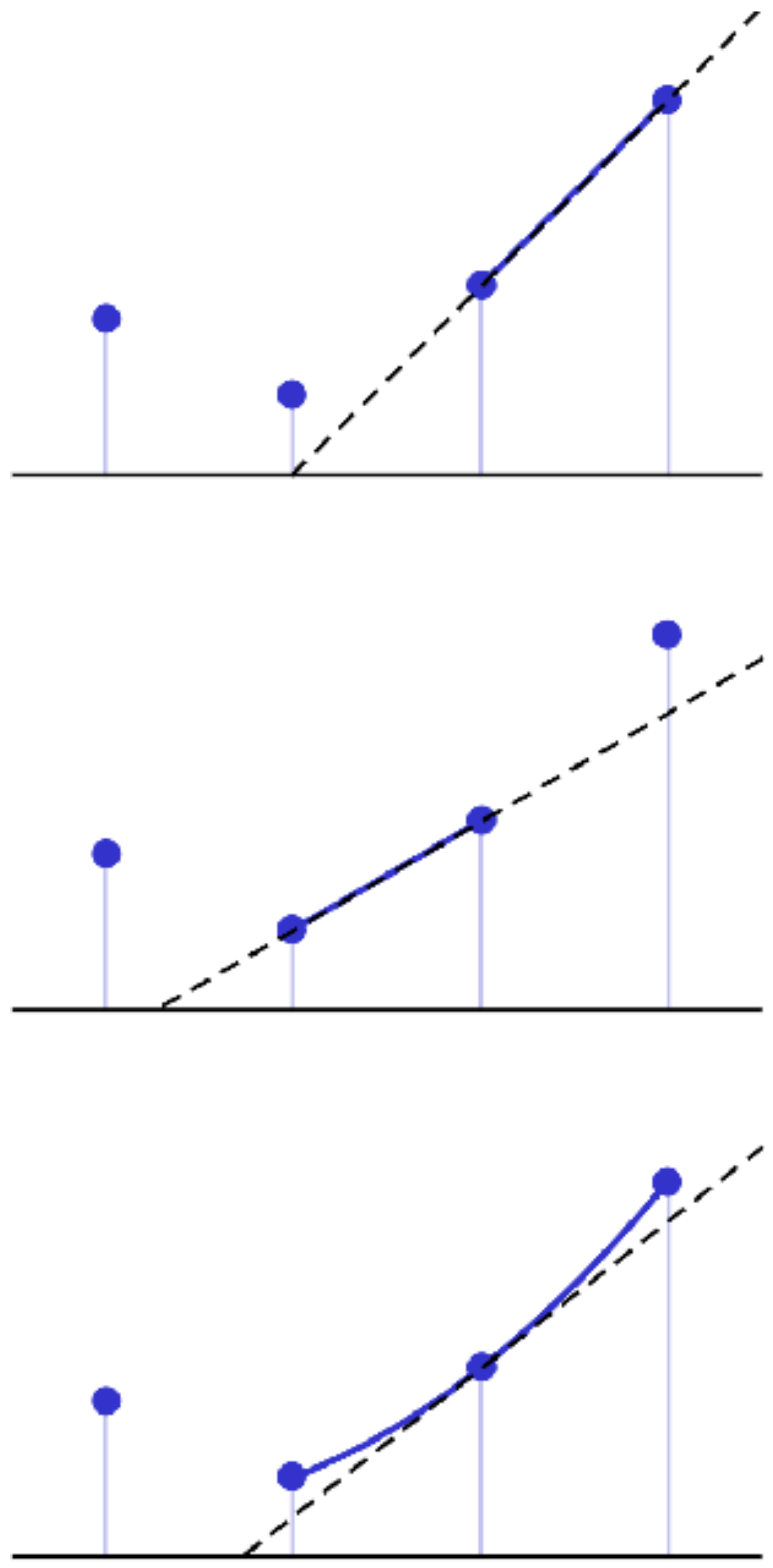
General strategy: reconstruct a sufficiently smooth function, then differentiate it

$$\frac{\mathrm{d}u}{\mathrm{d}x} \approx \frac{u_{i+1} - u_i}{\Delta x} \quad \text{(forward difference)}$$

$$\approx \frac{u_i - u_{i-1}}{\Delta x} \quad \text{(backward difference)}$$

$$\approx \frac{u_{i+1} - u_{i-1}}{2\Delta x} \quad \text{(centered difference)}$$

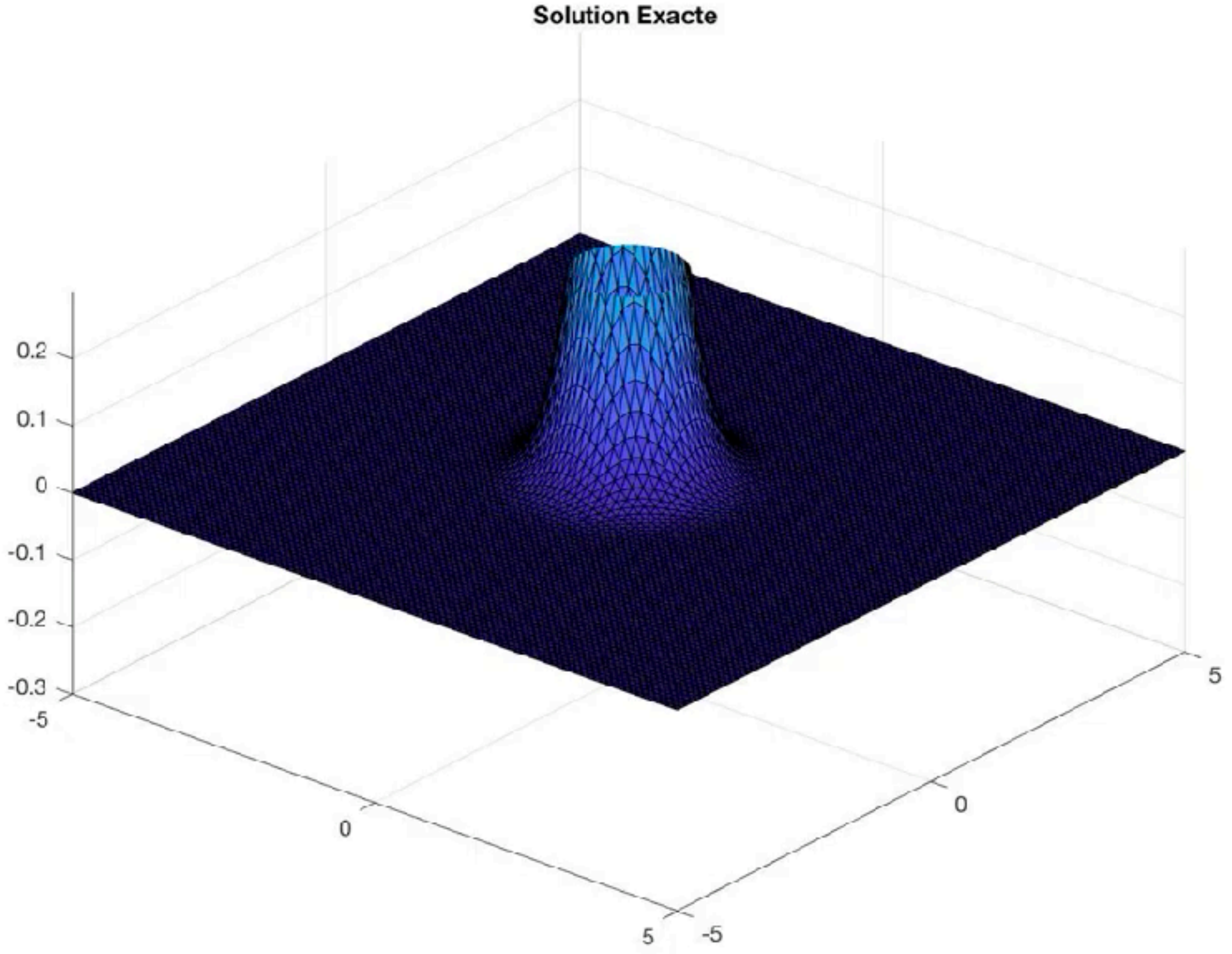
$$\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} \approx \frac{u_{i-1} - 2u_i + u_{i+1}}{\Delta x^2}$$



Also gives intuition: $\mathrm{d}^2u/\mathrm{d}x^2 \propto$ (average of neighbours) – u_i

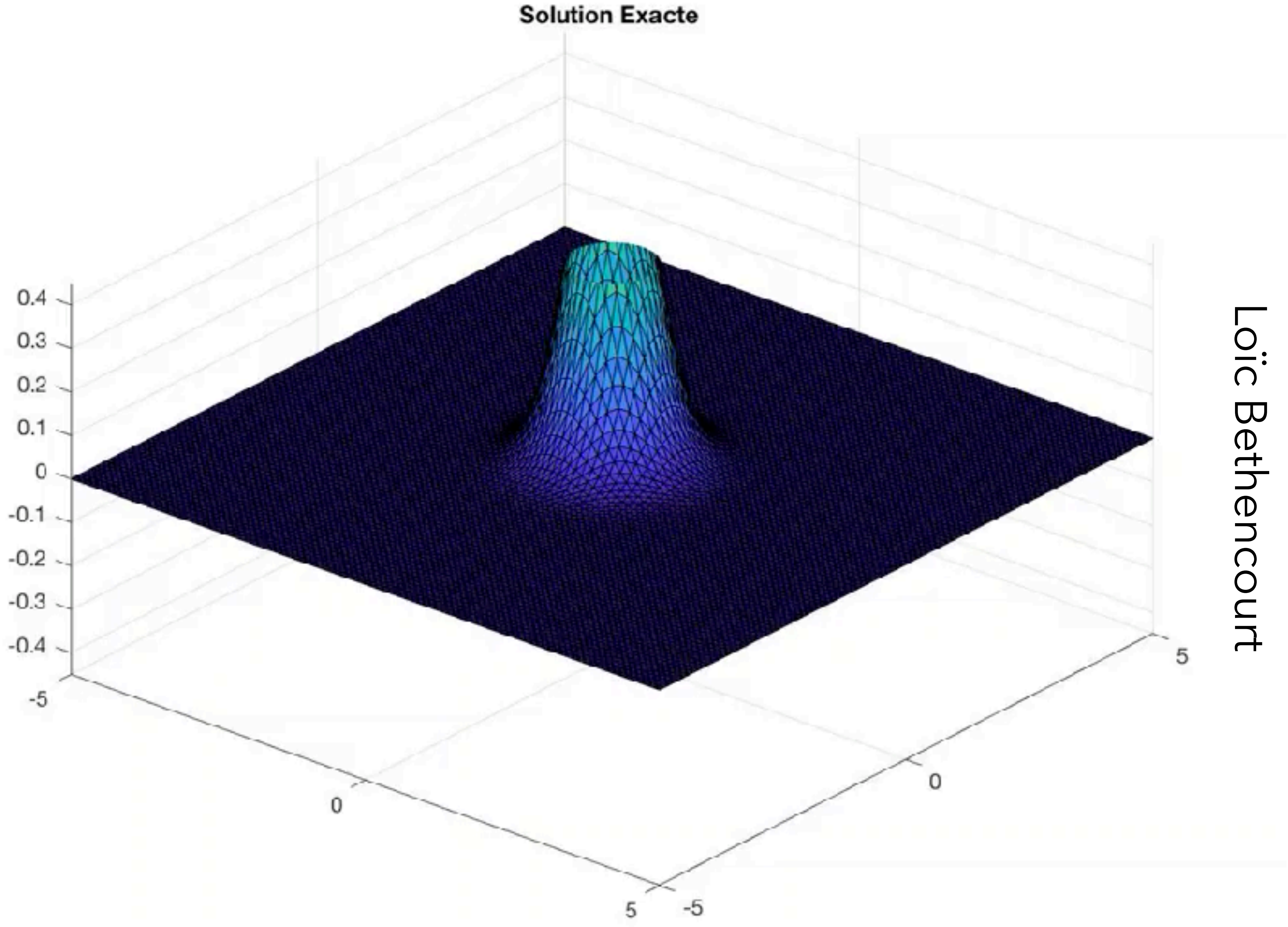
Boundary conditions

Dirichlet boundaries: u = fixed



https://www.youtube.com/watch?v=-chMgHvZxH0

Neumann boundaries: $\mathbf{n} \cdot \nabla u$ = fixed



Loïc Bethencourt

https://www.youtube.com/watch?v=1hsj10dOgt0

Anyway, we can't do things like $\dfrac{u_{i-1} - 2u_i + u_{i+1}}{\Delta x^2}$ if i = 0 or N.

- Dirichlet boundary: u = user-specified f

  Easy: Just fix $u_0 = f$

- Neumann boundary: du/dx = user-specified g

  Create a "ghost node" u_{-1} so that $\dfrac{u_0 - u_{-1}}{\Delta x} = g$, then plug in

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

$$\downarrow$$

$$\frac{\mathrm{d}u_i}{\mathrm{d}t} = \frac{\partial^2 u}{\partial x^2}$$

$$\downarrow$$

$$\frac{\mathrm{d}u_i}{\mathrm{d}t} = \frac{u_{i-1} - 2u_i + u_{i+1}}{\Delta x^2}$$

This is now an ordinary differential equation! And we know how to solve those.

- Choose a time integration scheme, solve the equations, …

PDEs in higher dimensions

In 1D, unknown function is $u = u(x, t)$, equation is something like $\frac{\partial^2 u}{\partial t^2} = f\left(u, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}, \ldots\right)$

In nD, it's $u = u(x, y, \ldots, t)$

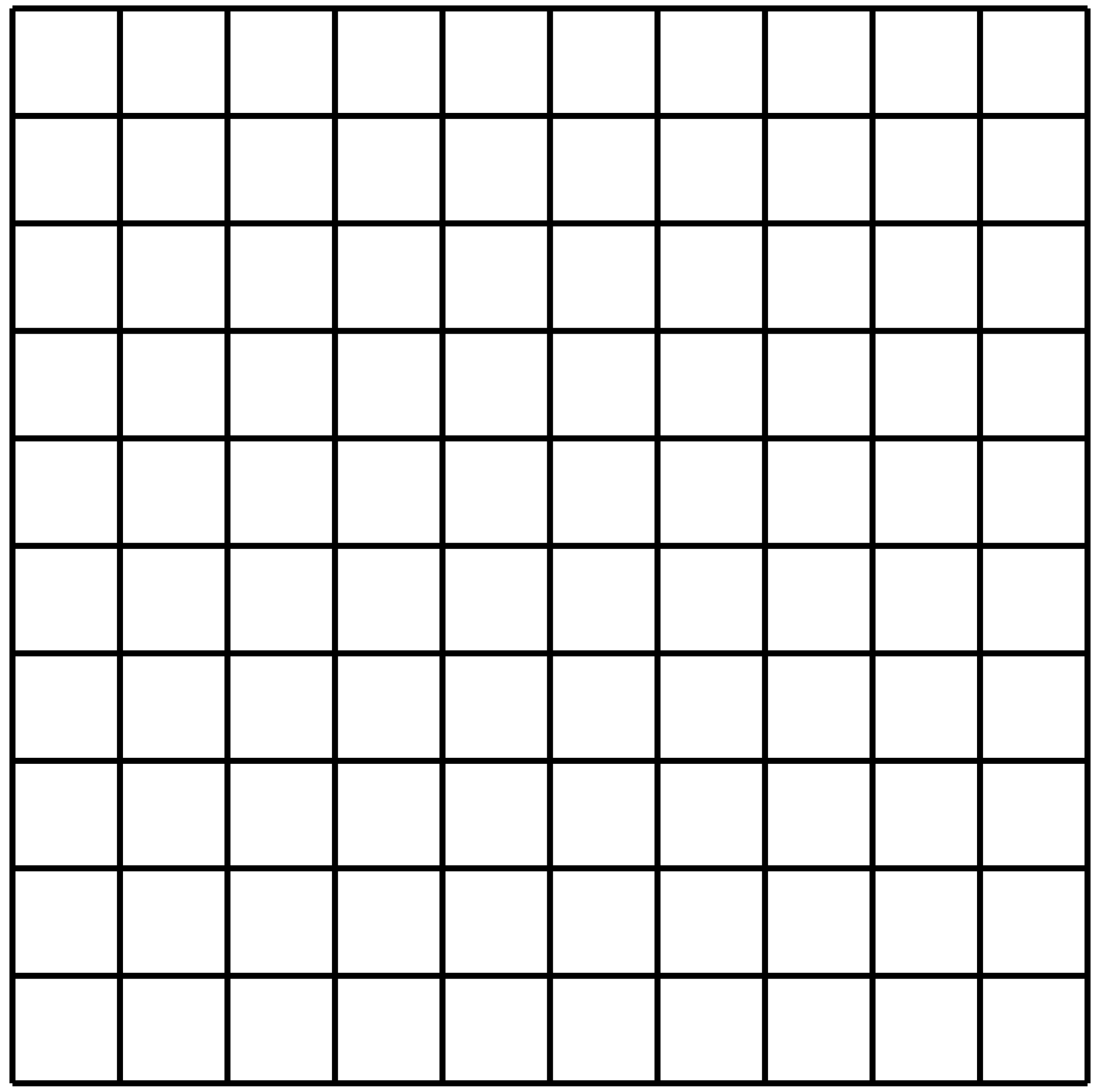
$$\frac{\partial^2 u}{\partial t^2} = f\left(u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \ldots, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial y^2}, \frac{\partial^2 u}{\partial x \partial y}, \ldots\right)$$

In particular, right-hand side of heat equation / wave equation becomes the Laplacian

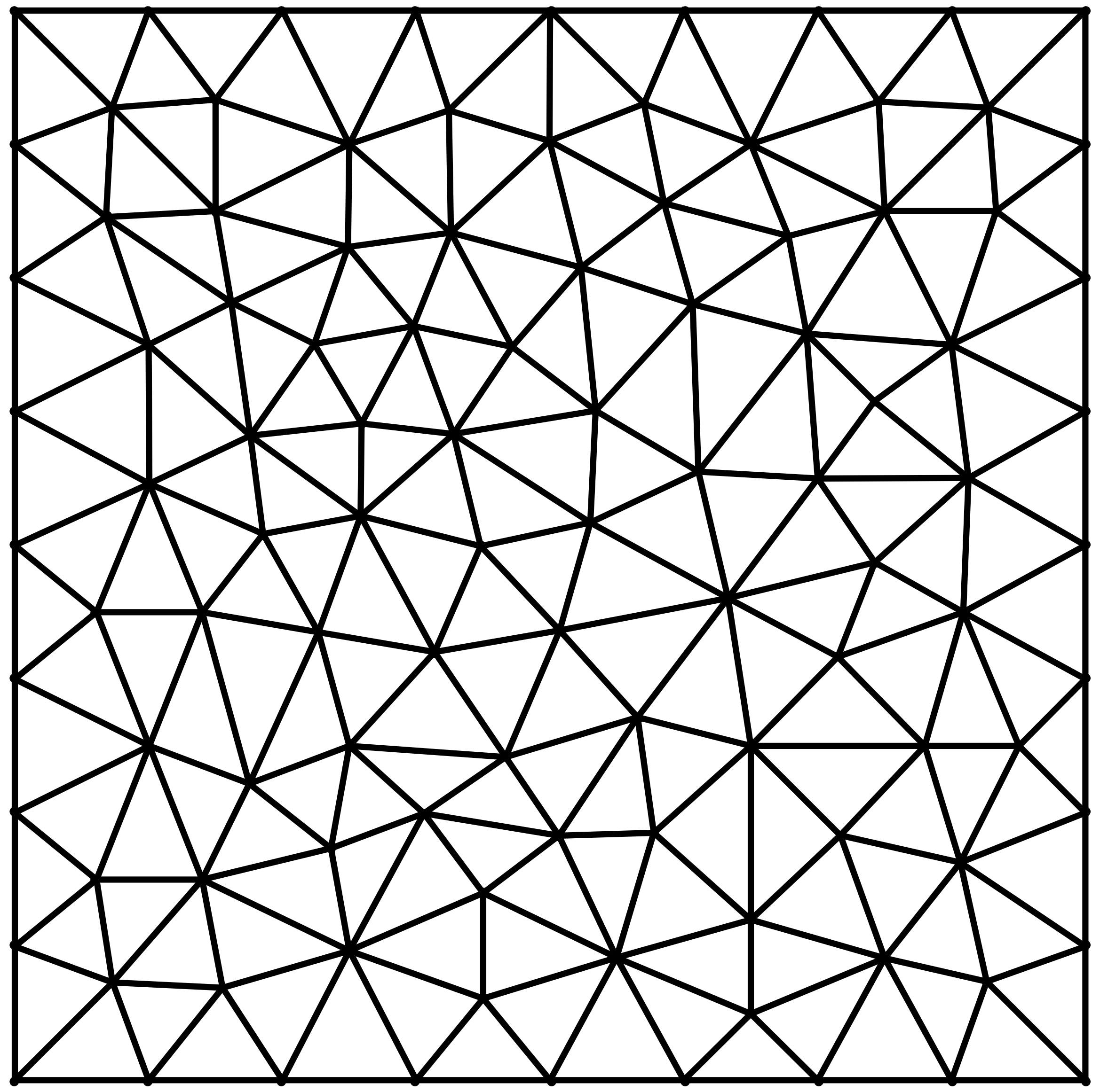
$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \cdots$$

Spatial discretizations

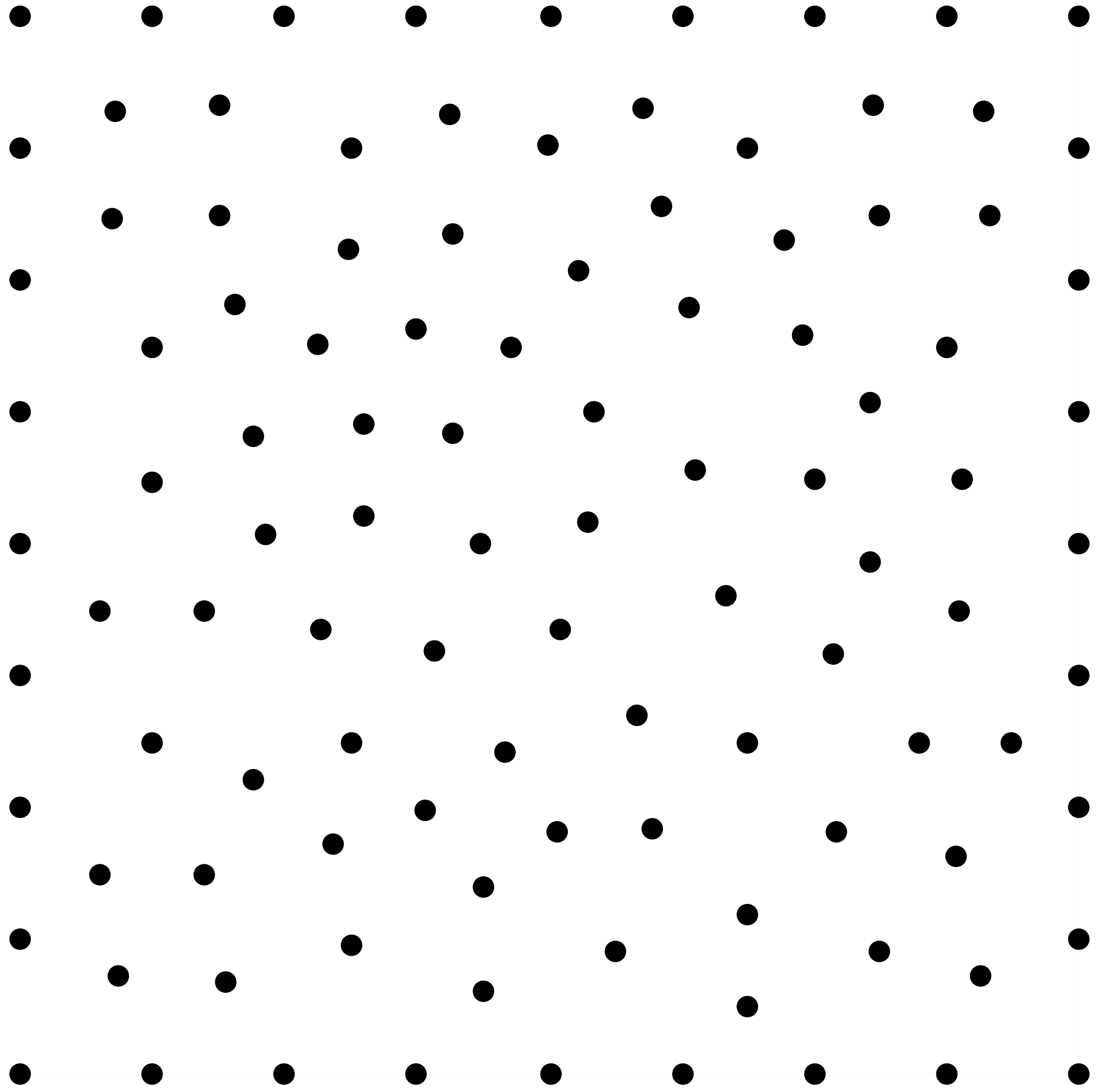
Now we have more choices of how to discretize space…



Grids



Meshes



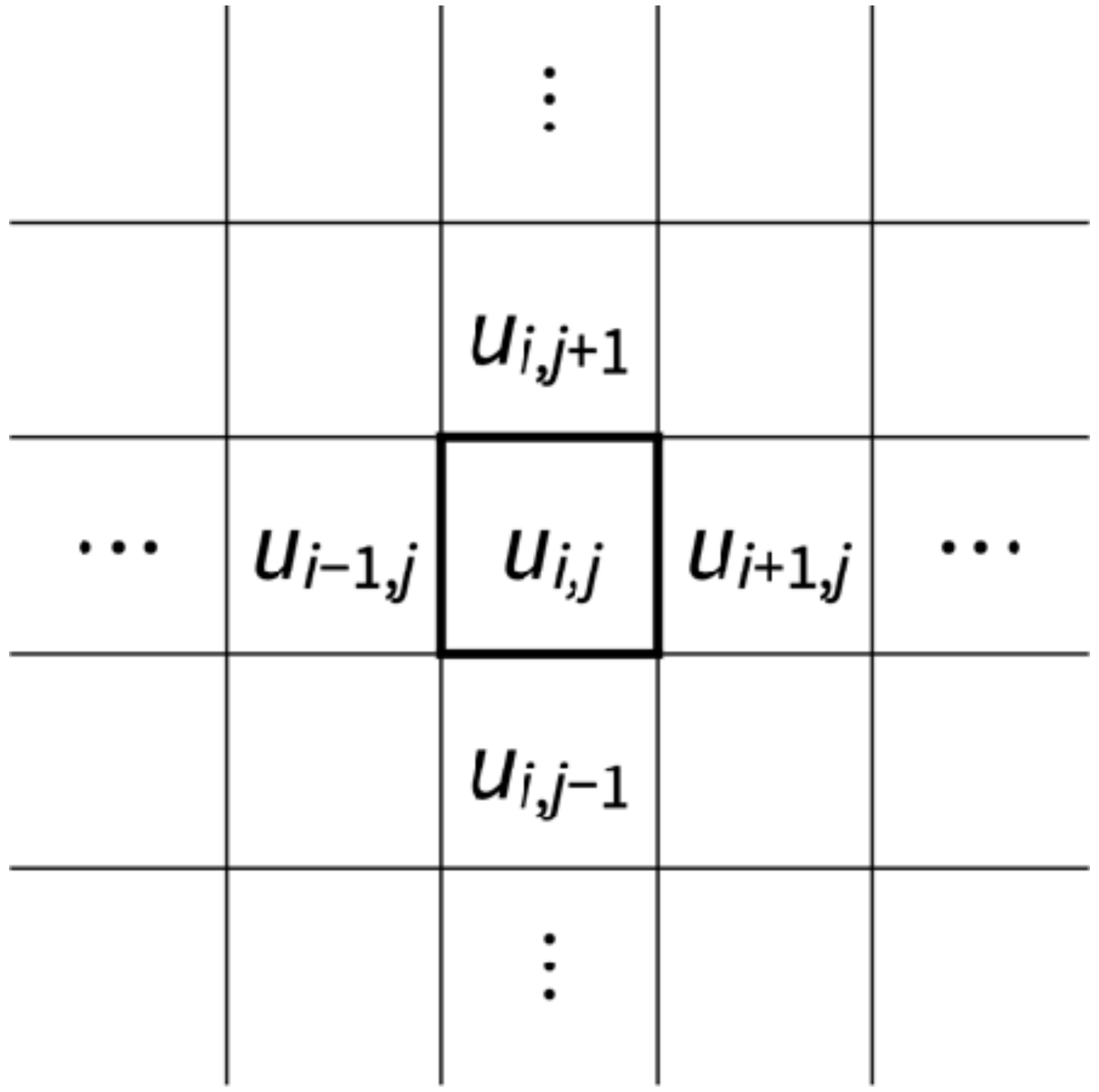
Particles

Easier computation ⟷ More flexibility

Example: Finite differences on grids in 2D

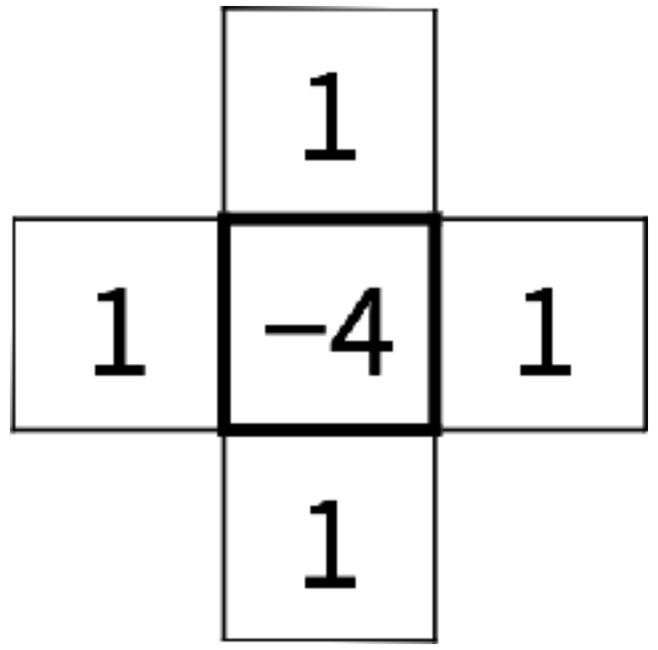
$$\frac{\partial u}{\partial x} \approx \frac{u_{i+1,j} - u_{i,j}}{\Delta x} \;\; \left(\text{or } \frac{u_{i,j} - u_{i-1,j}}{\Delta x} \text{ or } \frac{u_{i+1,j} - u_{i-1,j}}{2\Delta x} \ldots\right)$$

$$\frac{\partial^2 u}{\partial x^2} \approx \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{\Delta x^2}$$



So the Laplacian becomes

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \approx \frac{u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j}}{\Delta x^2}$$



For concreteness, here's one possible spatial discretization of the wave equation in 2D:

$$\frac{\partial^2 u}{\partial t^2} = \nabla^2 u$$

$$\downarrow$$

$$\frac{\mathrm{d}u_{i,j}}{\mathrm{d}t} = v_{i,j}$$

$$\frac{\mathrm{d}v_{i,j}}{\mathrm{d}t} = \frac{u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j}}{\Delta x^2}$$