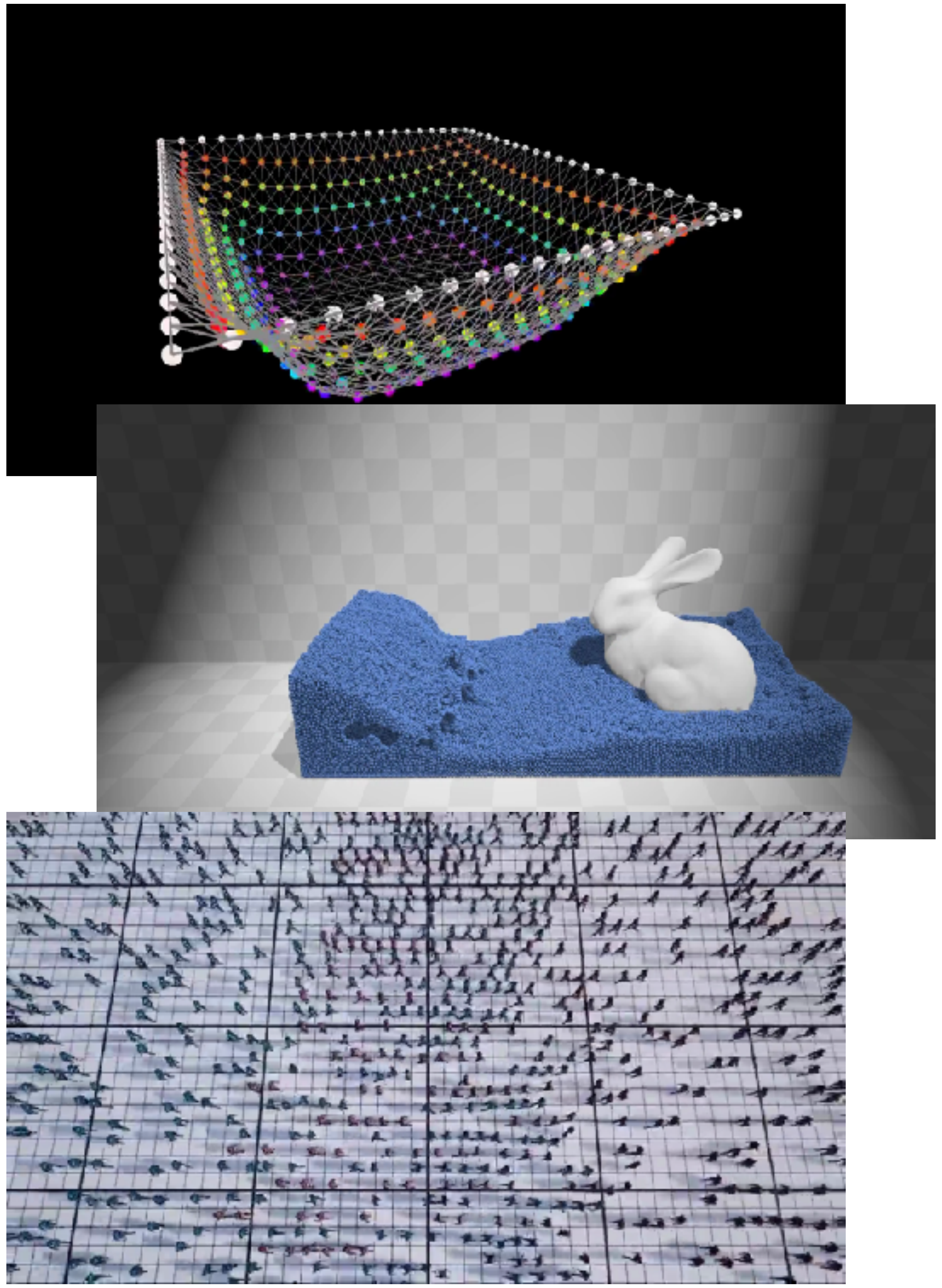
COL781: Computer Graphics

34. Mass-Spring Systems

World of Goo

Interacting particles give us much more interesting dynamics…



We can easily implement this with time stepping:

- For each particle i, compute total force $\mathbf{f}_i(t)$
- For each particle i, compute new state

$$\mathbf{v}_i(t + \Delta t) = \mathbf{v}_i(t) + m_i^{-1}\,\mathbf{f}_i(t)\,\Delta t$$
$$\mathbf{x}_i(t + \Delta t) = \mathbf{x}_i(t) + \mathbf{v}_i(t + \Delta t)\,\Delta t$$

But this may be a bit inconvenient to analyze mathematically:

Each $\mathbf{f}_i(t)$ could depend on $\mathbf{x}_1(t)$, $\mathbf{v}_1(t)$, $\mathbf{x}_2(t)$, $\mathbf{v}_2(t)$, …

Simpler with generalized coordinates:

$$\mathbf{q} = \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \vdots \\ \mathbf{x}_n \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \vdots \\ \mathbf{v}_n \end{bmatrix}$$

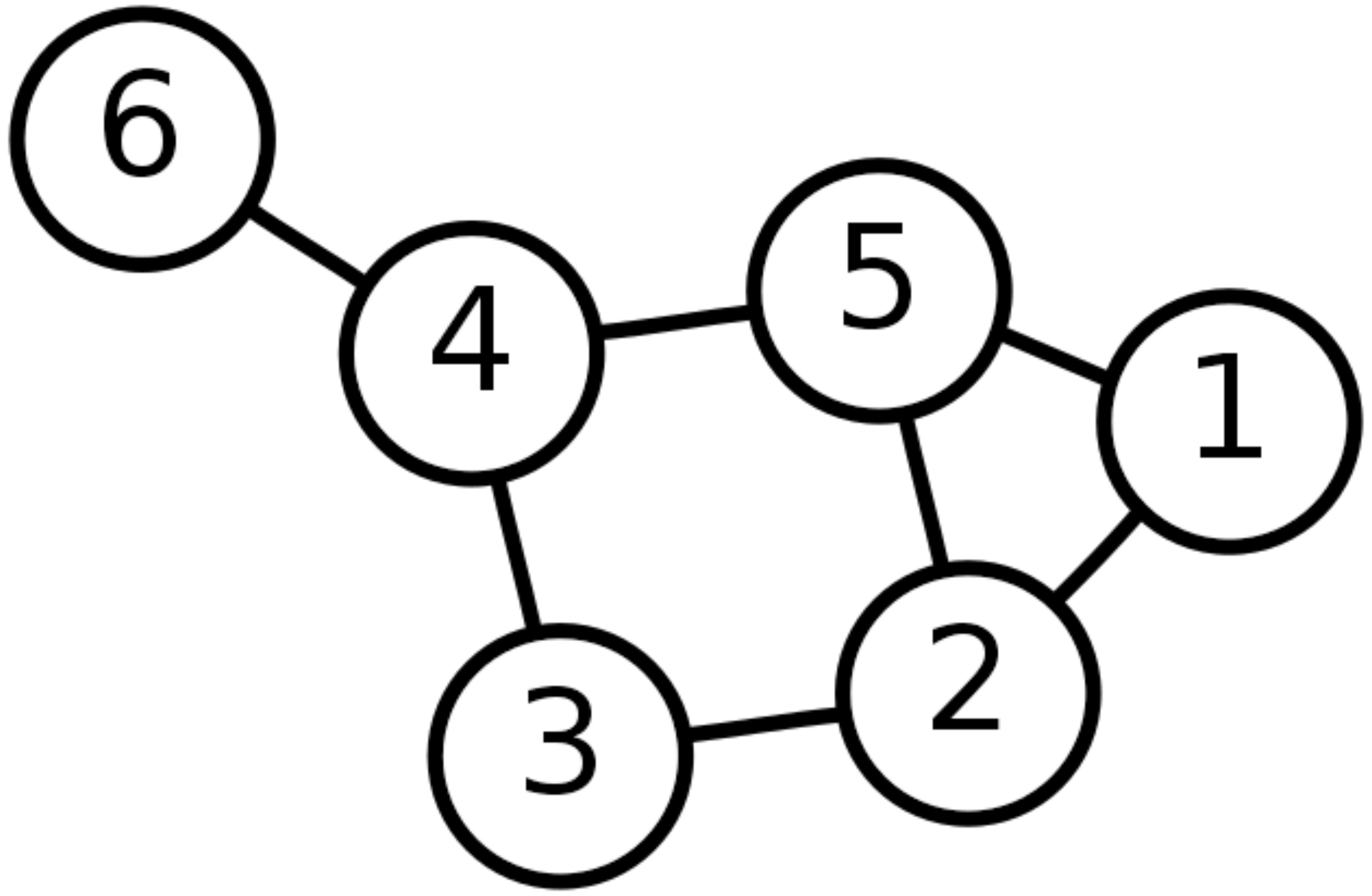
Then

$$\frac{\mathrm{d}^2\mathbf{q}(t)}{\mathrm{d}t^2} = \begin{bmatrix} m_1^{-1}\mathbf{f}_1(t, \mathbf{q}, \mathbf{v}) \\ m_2^{-1}\mathbf{f}_2(t, \mathbf{q}, \mathbf{v}) \\ \vdots \\ m_n^{-1}\mathbf{f}_n(t, \mathbf{q}, \mathbf{v}) \end{bmatrix} = \begin{bmatrix} m_1\mathbf{I} & & & \\ & m_2\mathbf{I} & & \\ & & \ddots & \\ & & & m_n\mathbf{I} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{f}_1(t, \mathbf{q}, \mathbf{v}) \\ \mathbf{f}_2(t, \mathbf{q}, \mathbf{v}) \\ \vdots \\ \mathbf{f}_n(t, \mathbf{q}, \mathbf{v}) \end{bmatrix}$$

Now we're solving for the evolution of a single (though $3n$-dimensional!) vector

Example: A small mass-spring system

$$\begin{bmatrix} \mathbf{f}_1(t, \mathbf{q}, \mathbf{v}) \\ \mathbf{f}_2(t, \mathbf{q}, \mathbf{v}) \\ \vdots \\ \mathbf{f}_6(t, \mathbf{q}, \mathbf{v}) \end{bmatrix} = \begin{bmatrix} \mathbf{f}_{12} + \mathbf{f}_{15} \\ \mathbf{f}_{21} + \mathbf{f}_{23} + \mathbf{f}_{25} \\ \vdots \\ \mathbf{f}_{64} \end{bmatrix}$$



Force due to spring between particles 2 and 3: $\begin{bmatrix} 0 \\ \mathbf{f}_{23} \\ \mathbf{f}_{32} \\ 0 \\ \vdots \end{bmatrix}$

(of course, $\mathbf{f}_{32} = -\mathbf{f}_{23}$)

Total force on system = $\sum$ force due to each spring

Per-particle formulation:

$$\frac{\mathrm{d}^2\mathbf{x}_i(t)}{\mathrm{d}t^2} = m_i^{-1}\,\mathbf{f}_i(t, \ldots) \quad \forall i = 1, 2, \ldots$$

$$\Downarrow$$

$$\mathbf{v}_i(t + \Delta t) = \mathbf{v}_i(t) + m_i^{-1}\,\mathbf{f}_i(t, \ldots)\,\Delta t \quad \forall i = 1, 2, \ldots$$
$$\mathbf{x}_i(t + \Delta t) = \mathbf{x}_i(t) + \mathbf{v}_i(t + \Delta t)\,\Delta t \quad \forall i = 1, 2, \ldots$$

Careful not to update $\mathbf{x}_1$, $\mathbf{v}_1$ before computing $\mathbf{f}_2$, in case it depends on them

Generalized coordinates:

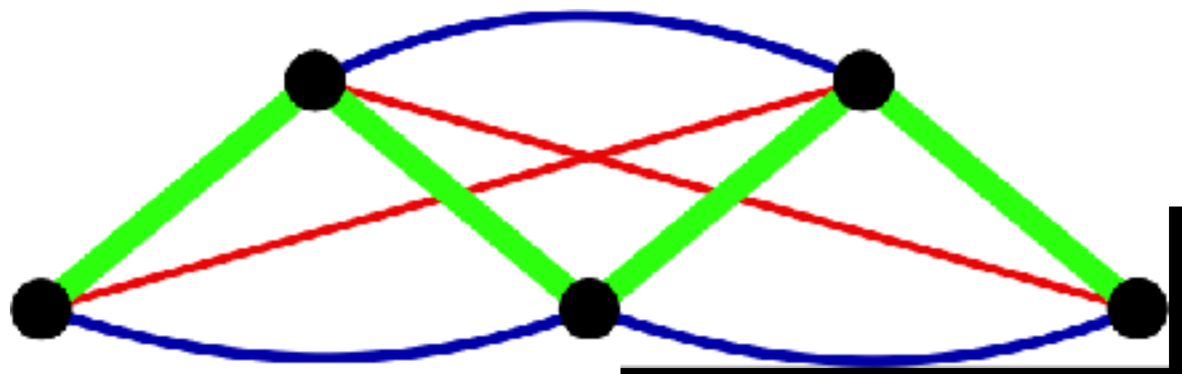
$$\frac{\mathrm{d}^2\mathbf{q}(t)}{\mathrm{d}t^2} = \mathbf{M}^{-1}\,\mathbf{f}(t, \mathbf{q}, \mathbf{v})$$

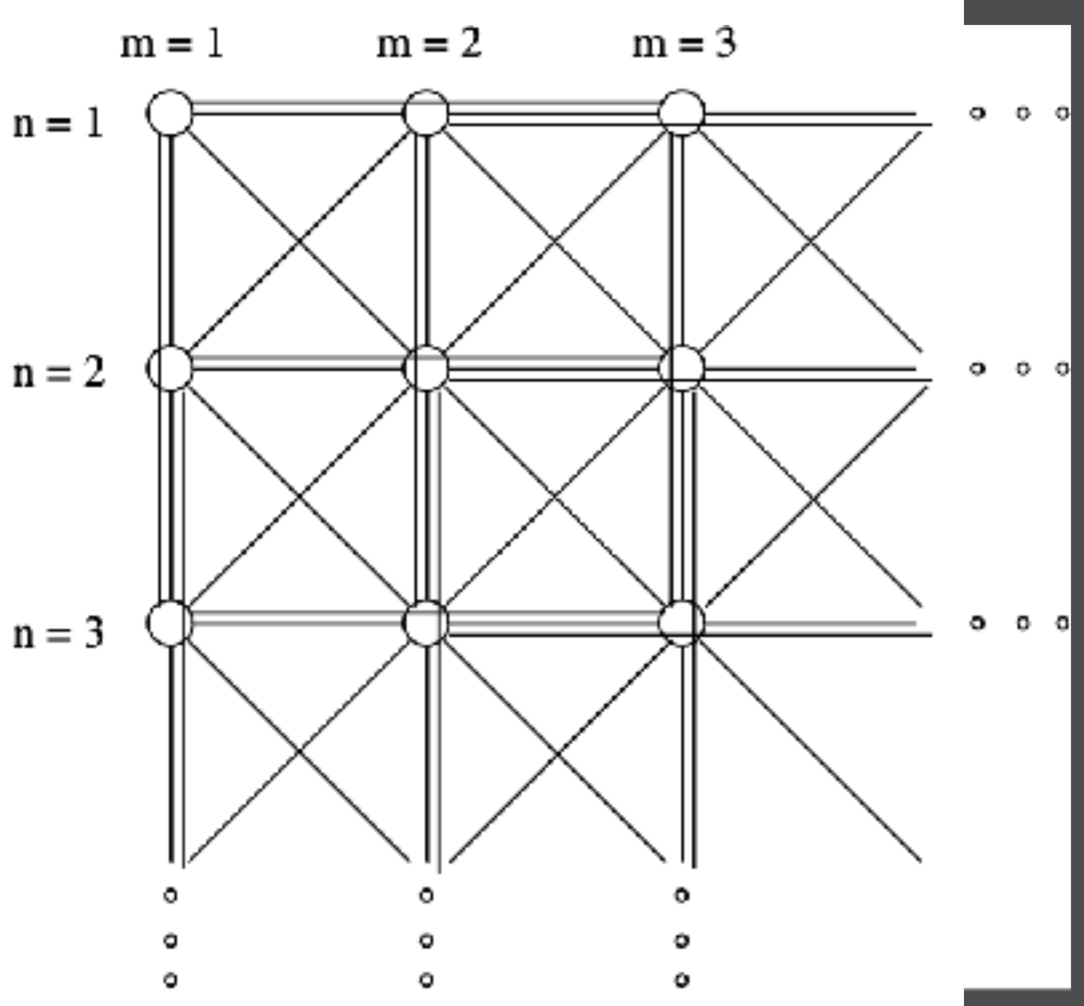
$$\Downarrow$$

$$\mathbf{v}(t + \Delta t) = \mathbf{v}(t) + \mathbf{M}^{-1}\,\mathbf{f}(t, \mathbf{q}, \mathbf{v})\,\Delta t$$
$$\mathbf{q}(t + \Delta t) = \mathbf{q}(t) + \mathbf{v}(t + \Delta t)\,\Delta t$$

Simple! And generalizes to other things (e.g. rigid bodies) with few changes

Mass-spring systems





Recall springs in 1 dimension from physics classes.

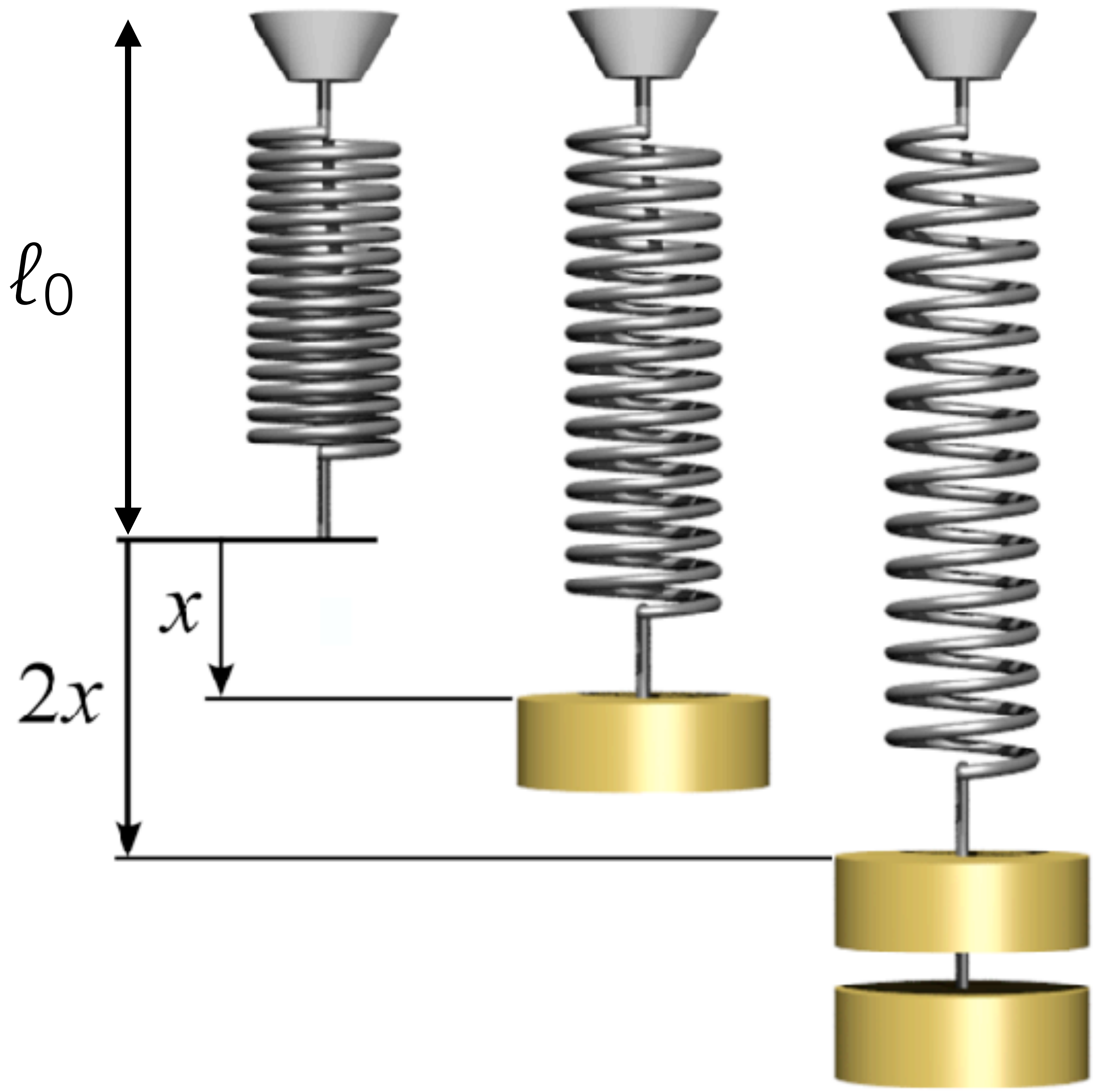
Hooke's law: force is proportional to displacement

$$F = -k\,x = -k\,(\ell - \ell_0)$$

Potential energy:

$$U = ½\,k\,(\ell - \ell_0)^2$$

In fact $F = -\mathrm{d}U/\mathrm{d}\ell$



In 3D, suppose a spring connects particles i and j. What should be the force $\mathbf{f}_{ij}$ on i due to j?

Let's first define the potential:

$$U = ½\, k\, (\|\mathbf{x}_i - \mathbf{x}_j\| - \ell_0)^2$$

Then $\mathbf{f}_{ij} = -\partial U/\partial \mathbf{x}_i \Rightarrow$

$$\mathbf{f}_{ij} = -k\,(\|\mathbf{x}_i - \mathbf{x}_j\| - \ell_0)\,\frac{\mathbf{x}_i - \mathbf{x}_j}{\|\mathbf{x}_i - \mathbf{x}_j\|}$$

$$= -k\,(\|\mathbf{x}_{ij}\| - \ell_0)\,\hat{\mathbf{x}}_{ij}$$

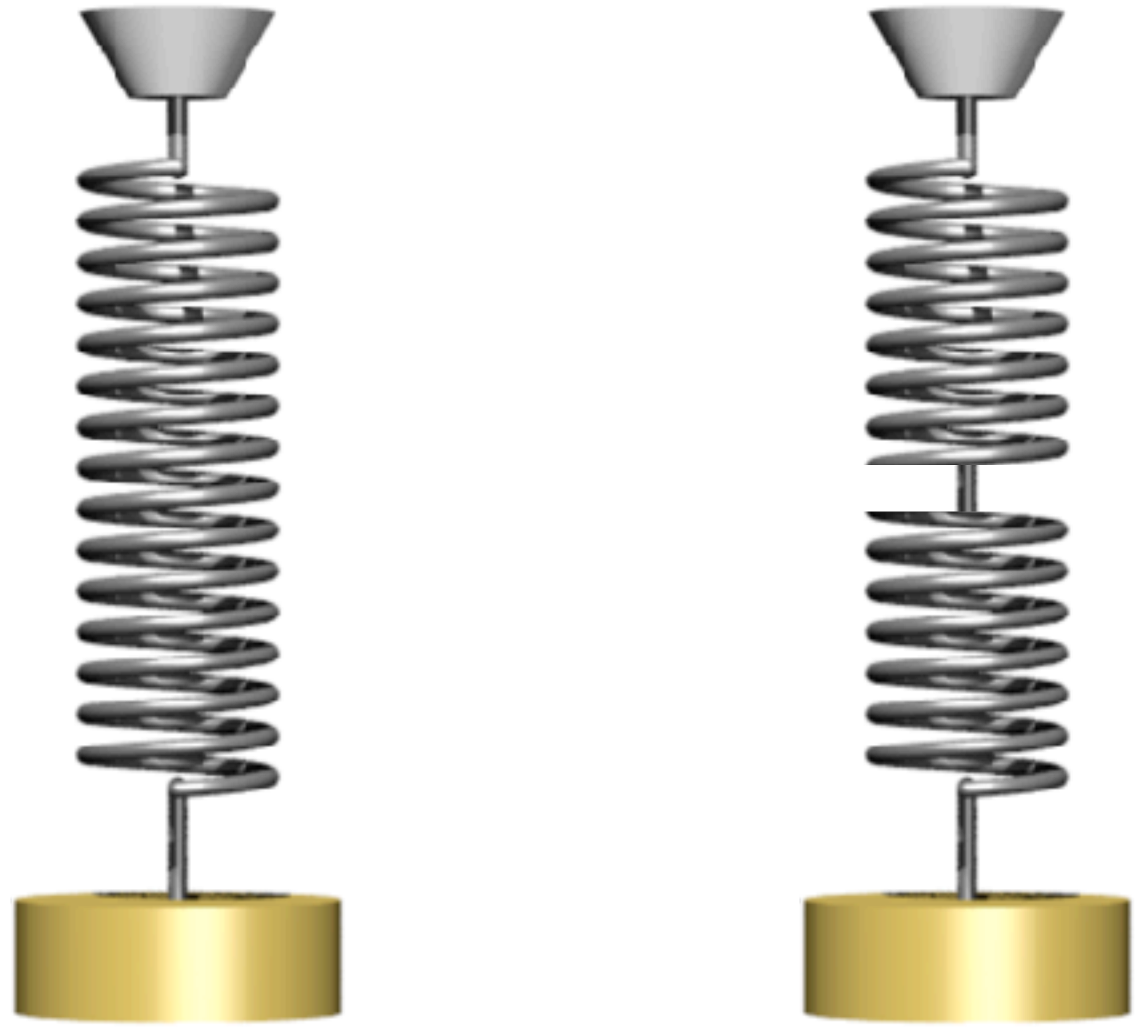
Similarly $\mathbf{f}_{ji} = -\partial U/\partial \mathbf{x}_j$ (but it's also just $-\mathbf{f}_{ij}$)

Exercise: 1. Derive this expression from $-\partial U/\partial \mathbf{x}_i$.
2. (Optional) Look for high-level steps so you don't have to differentiate componentwise.

Puzzle:

Suppose I model a stretchy rope (e.g. a rubber band) as a spring of stiffness k.

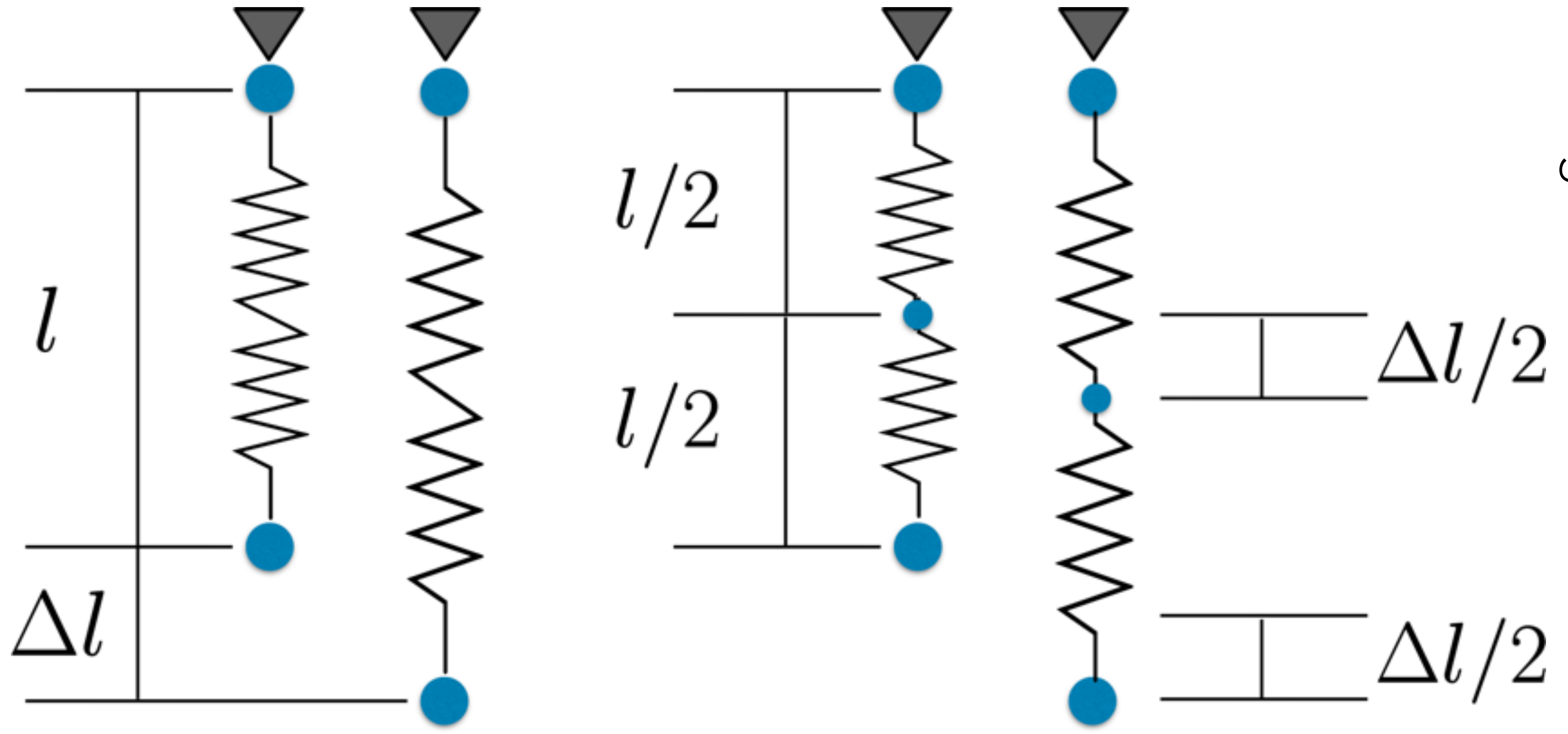
Now I want to allow it bend, so I replace it with two springs of half the length.
What should be the stiffness of these springs to get the same stretchiness?



Different "resolutions" have different absolute change in length…

but same relative length change!
This is called **strain**

$$\varepsilon = \frac{\Delta l}{l_0} = \frac{\|\mathbf{x}_i - \mathbf{x}_j\|}{l_0} - 1$$



Ren Ng

For more consistent behaviour, define spring force in terms of strain:

$$U = \tfrac{1}{2}\, k\, \ell_0\, \varepsilon^2$$

$$\mathbf{f}_{ij} = -k\, \varepsilon\, \hat{\mathbf{x}}_{ij}$$

Problem: Real springs dissipate energy and don't keep oscillating forever!

Bad idea: Just add a force that opposes all velocities

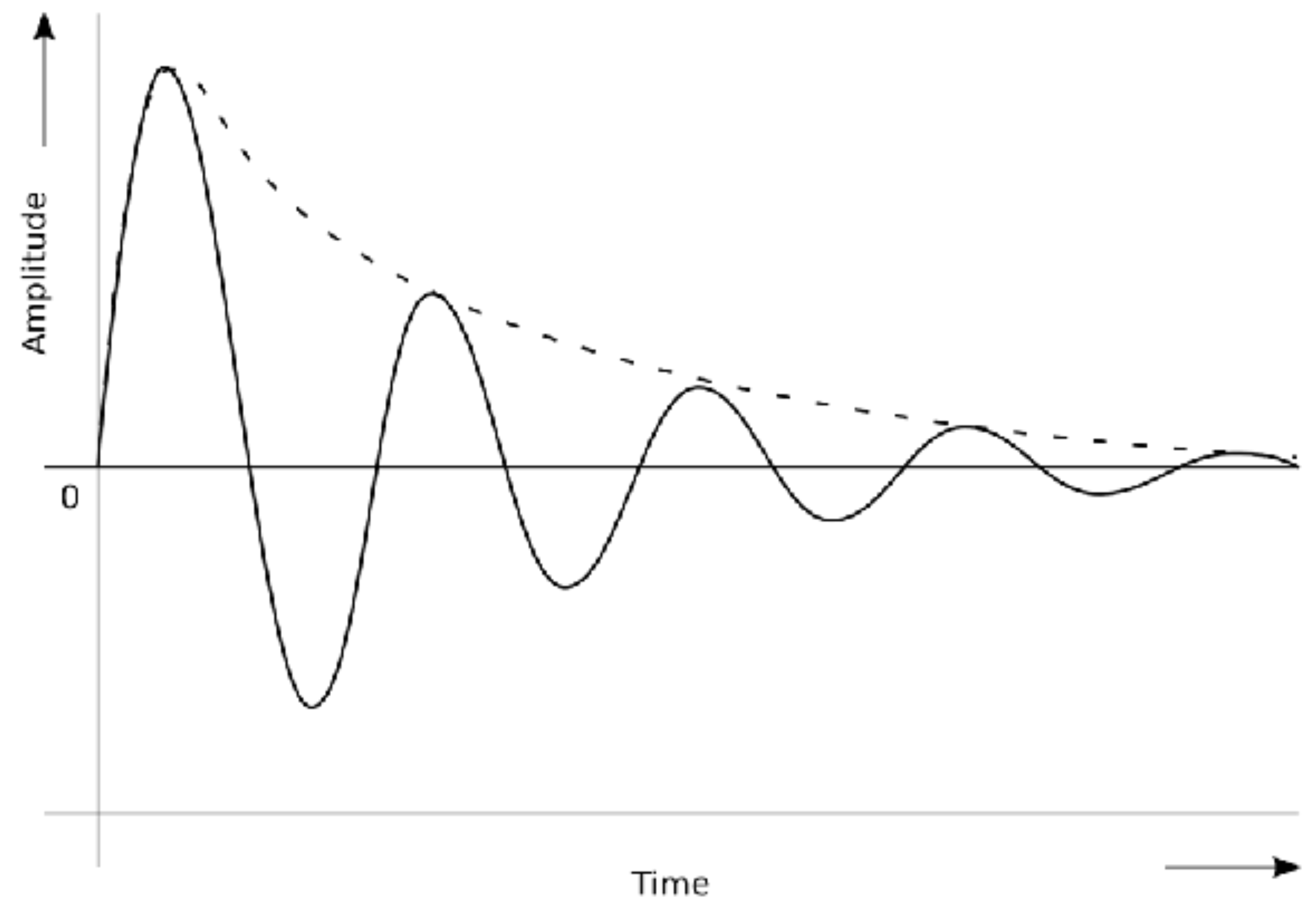
$$\mathbf{f}_i = -k_d\,\mathbf{v}_i$$

Sometimes called "**ether drag**"

- Particles look like they're suspended in a viscous medium
- Should a rusty spring fall slower than a clean spring?

Good idea: Only oppose relative velocities along the spring

$$\mathbf{f}_{ij} = -k_d \left(\frac{\mathbf{v}_i - \mathbf{v}_j}{l_0} \cdot \hat{\mathbf{x}}_{ij} \right) \hat{\mathbf{x}}_{ij}$$



Force due to a spring, finally:

$$\mathbf{f}_{ij} = -k_s\, \varepsilon\, \hat{\mathbf{x}}_{ij} - k_d\, \dot{\varepsilon}\, \hat{\mathbf{x}}_{ij}$$

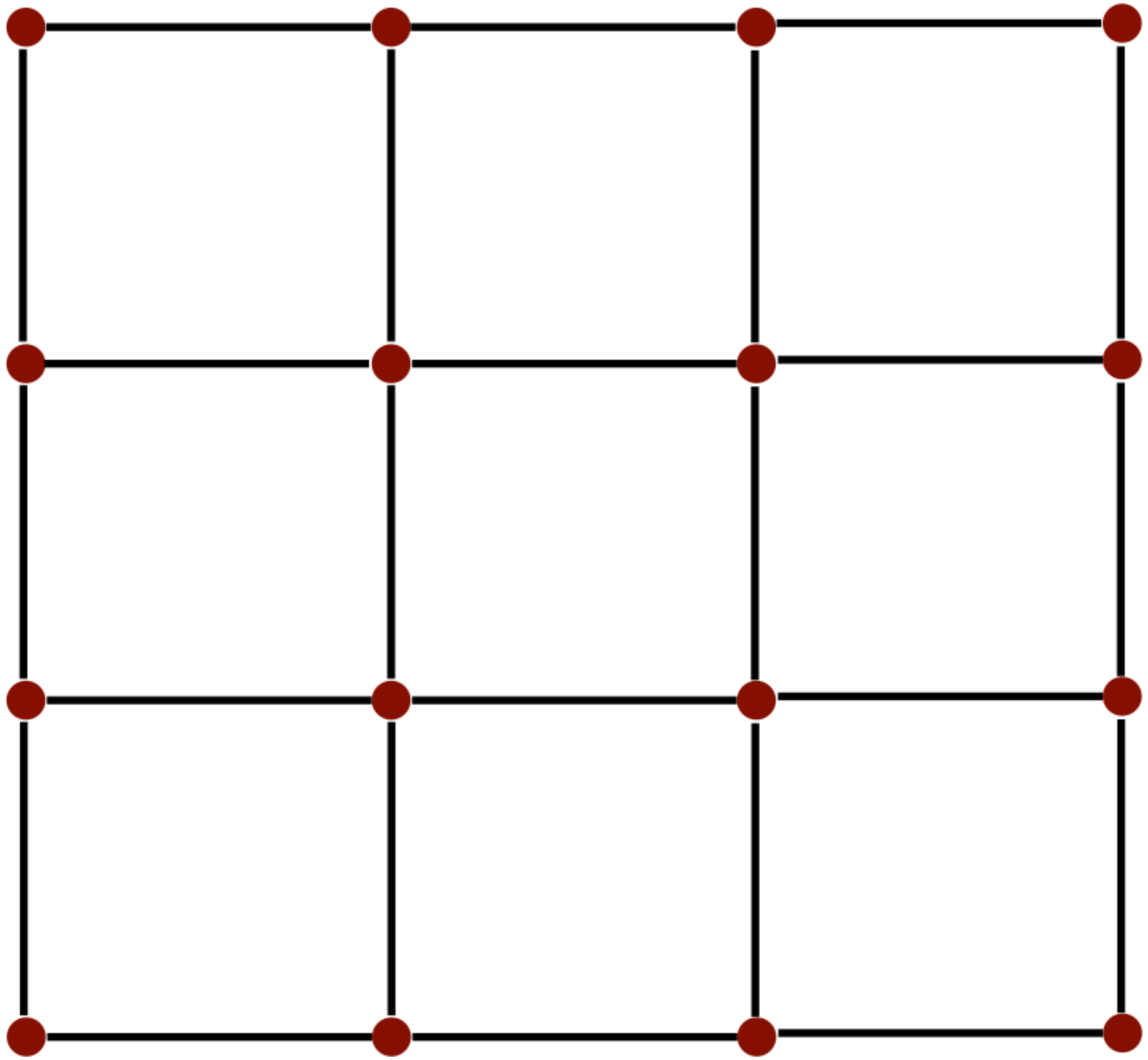
Actually, this term can also be derived as $-\partial R/\partial \mathbf{v}_i$ for some **dissipation potential** $R = ½\, k_d\, \dot{\varepsilon}^2$!

where

- Strain $\varepsilon = \dfrac{\|\mathbf{x}_i - \mathbf{x}_j\|}{l_0} - 1$
- Strain rate $\dot{\varepsilon} = \dfrac{\mathrm{d}\varepsilon}{\mathrm{d}t} = \dfrac{\mathbf{v}_i - \mathbf{v}_j}{l_0} \cdot \hat{\mathbf{x}}_{ij}$
- Spring constant $k_s \geq 0$
- Damping constant $k_d \geq 0$

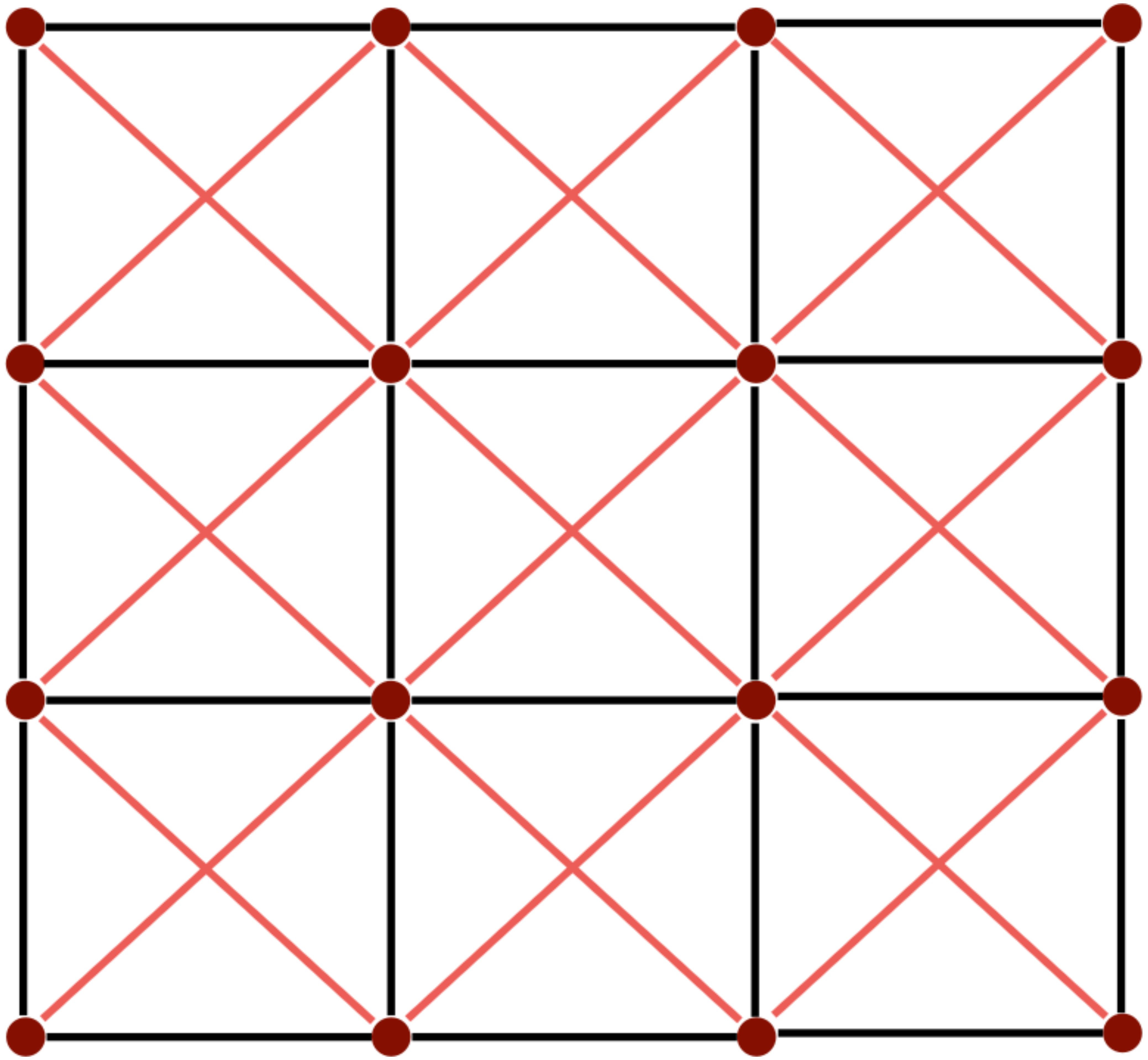
How to model a rectangular sheet of cloth?

- Structural springs



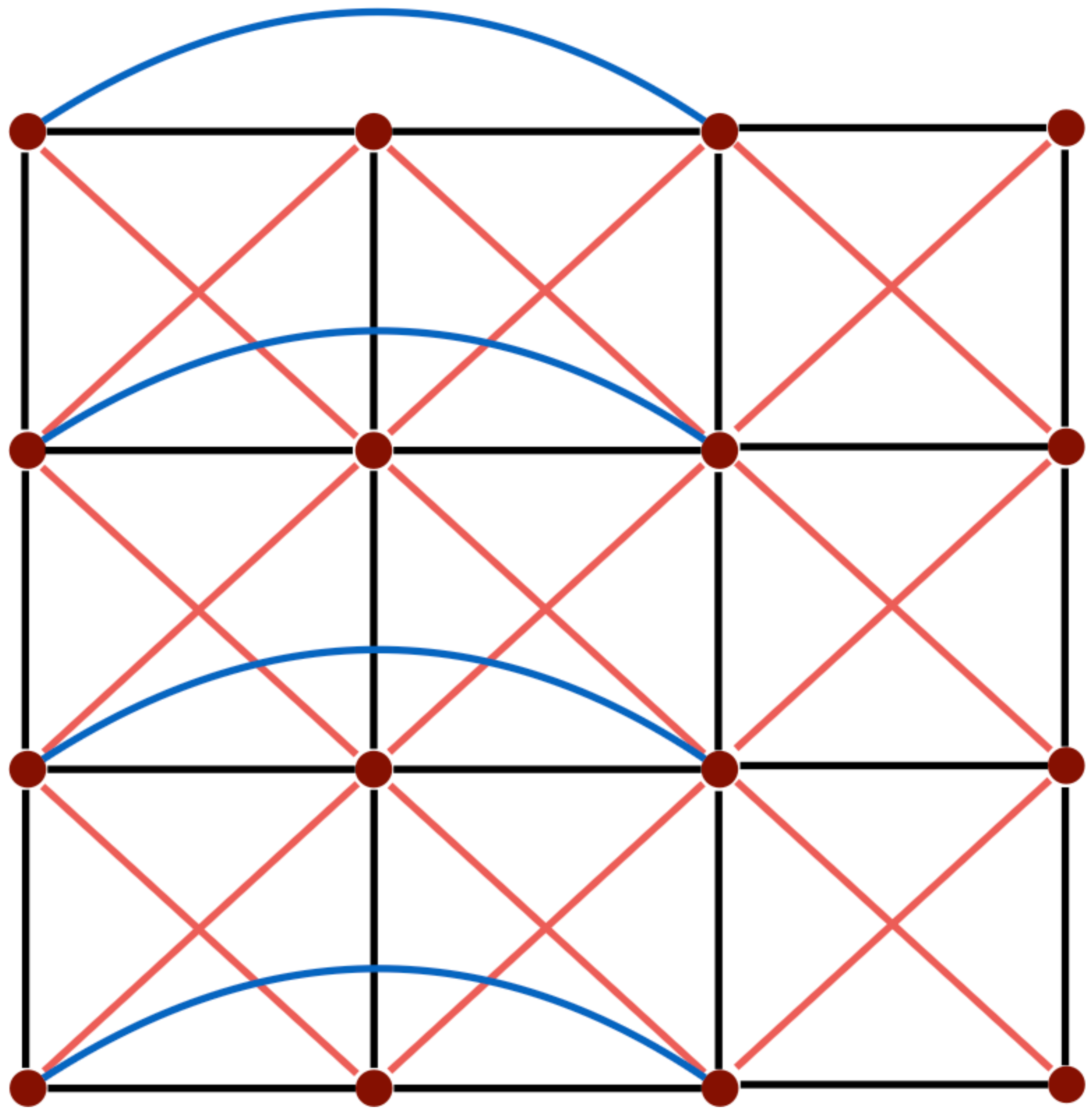
How to model a rectangular sheet of cloth?

- Structural springs
- Shear springs



How to model a rectangular sheet of cloth?

- Structural springs
- Shear springs
- Bending springs



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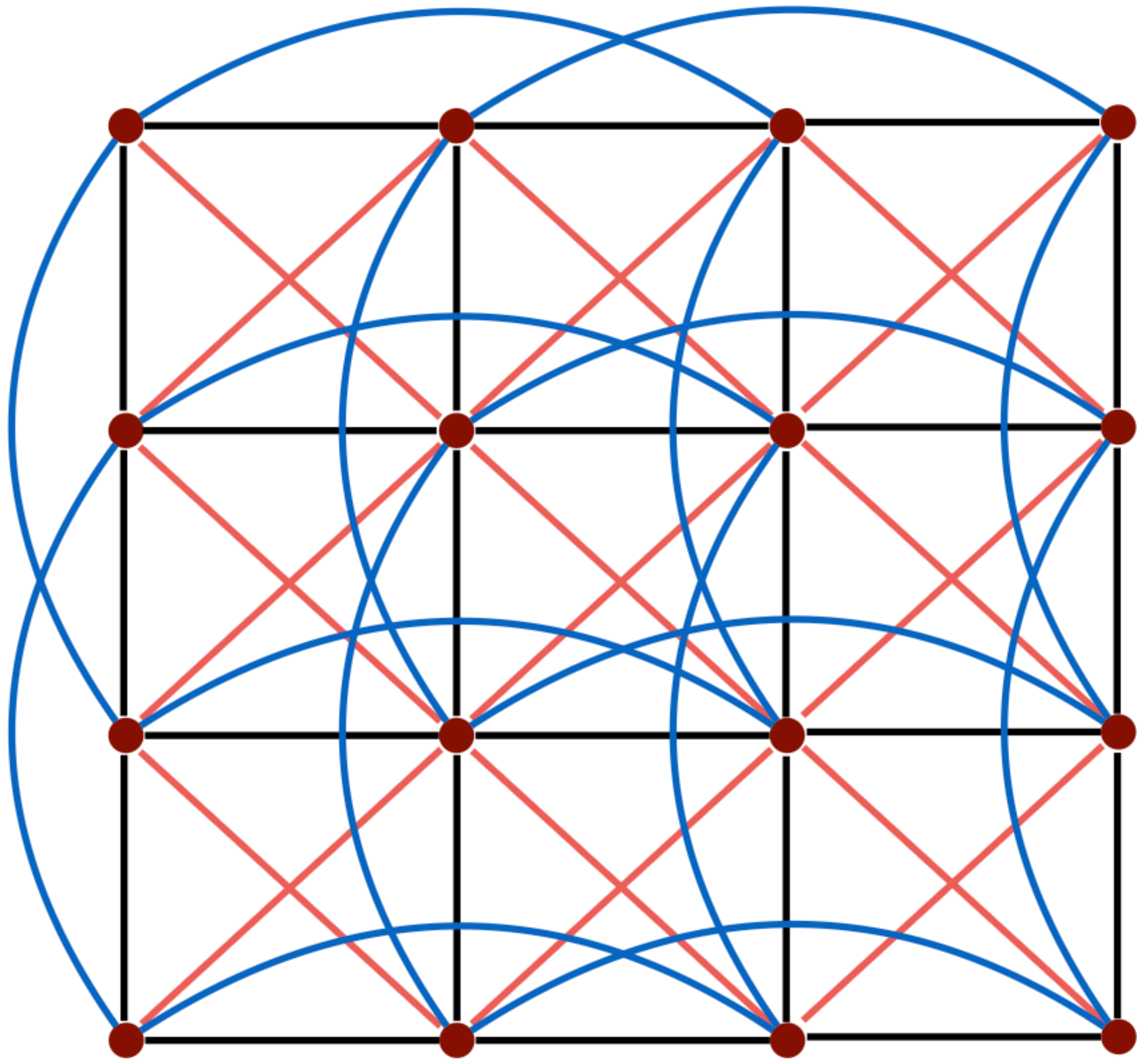
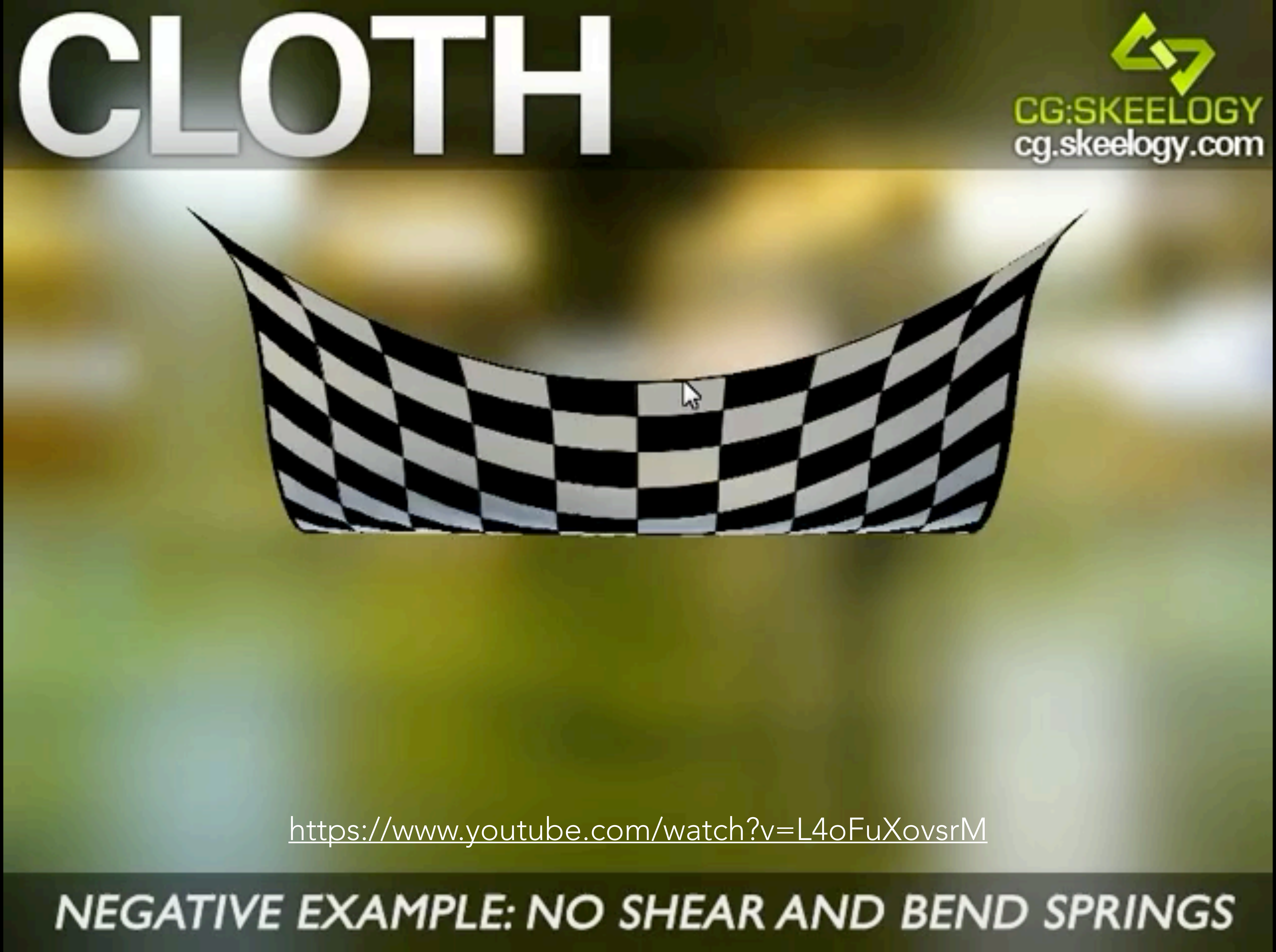
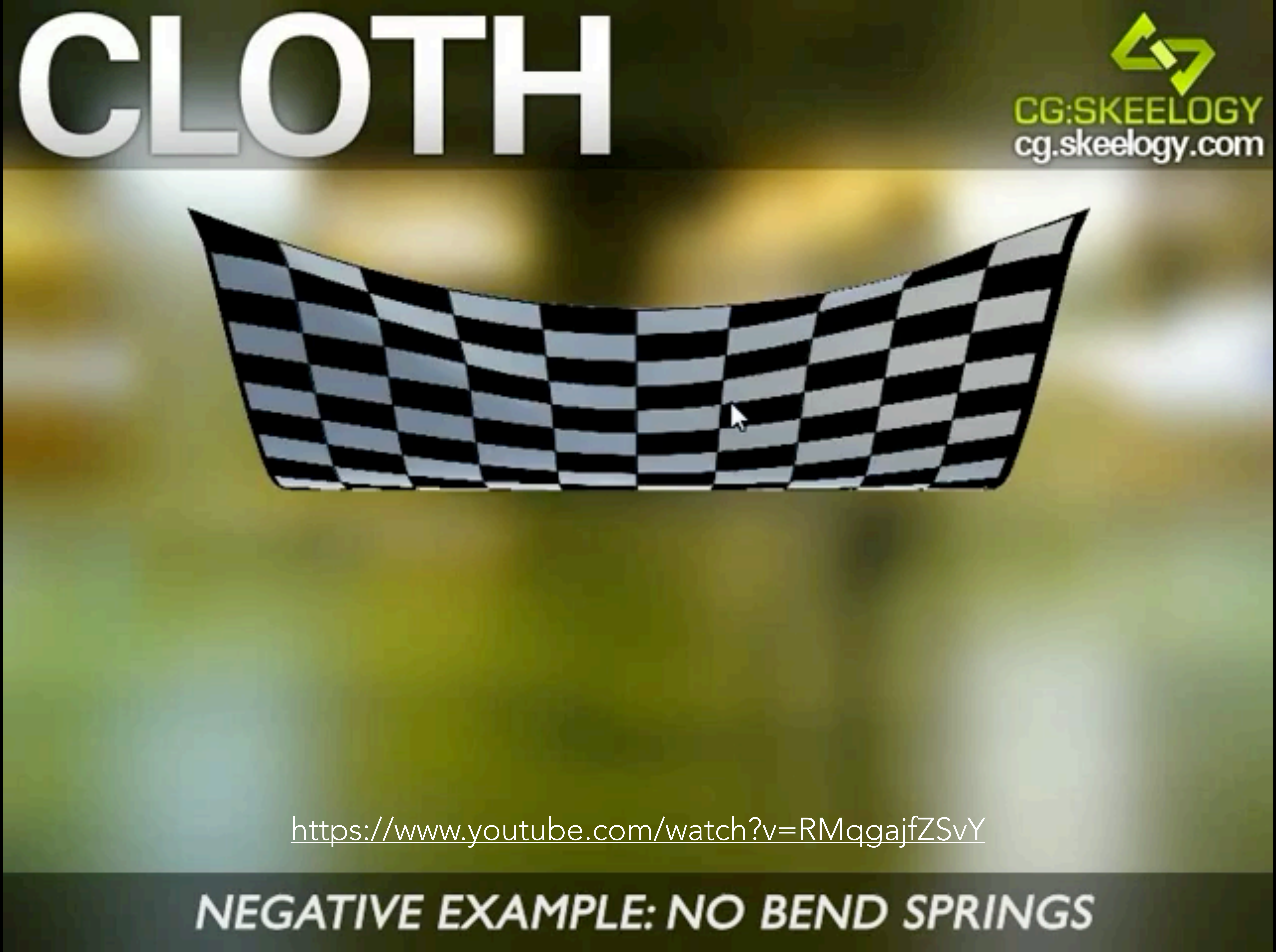




https://www.youtube.com/watch?v=L4oFuXovsrM

Skeel Lee



https://www.youtube.com/watch?v=RMqgajfZSvY

Skeel Lee

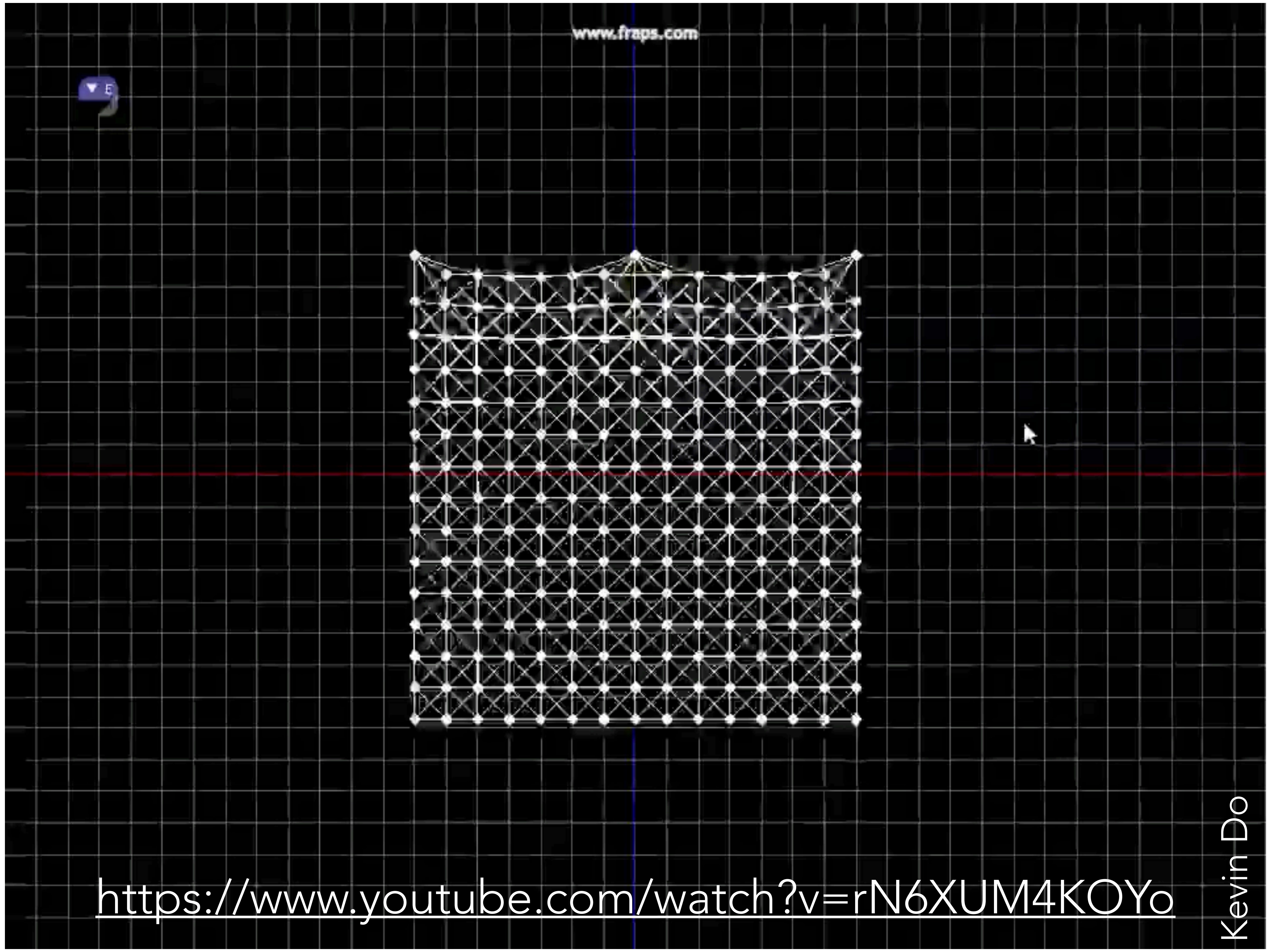
https://www.youtube.com/watch?v=JncQOCg9FS0

Skeel Lee

Here's a problem you'll encounter:

Sometimes your simulation **blows up** for no apparent reason!

Why?



Time integration

We have an ordinary differential equation

$$\ddot{\mathbf{q}} = \mathbf{M}^{-1}\,\mathbf{f}(t, \mathbf{q}, \dot{\mathbf{q}})$$

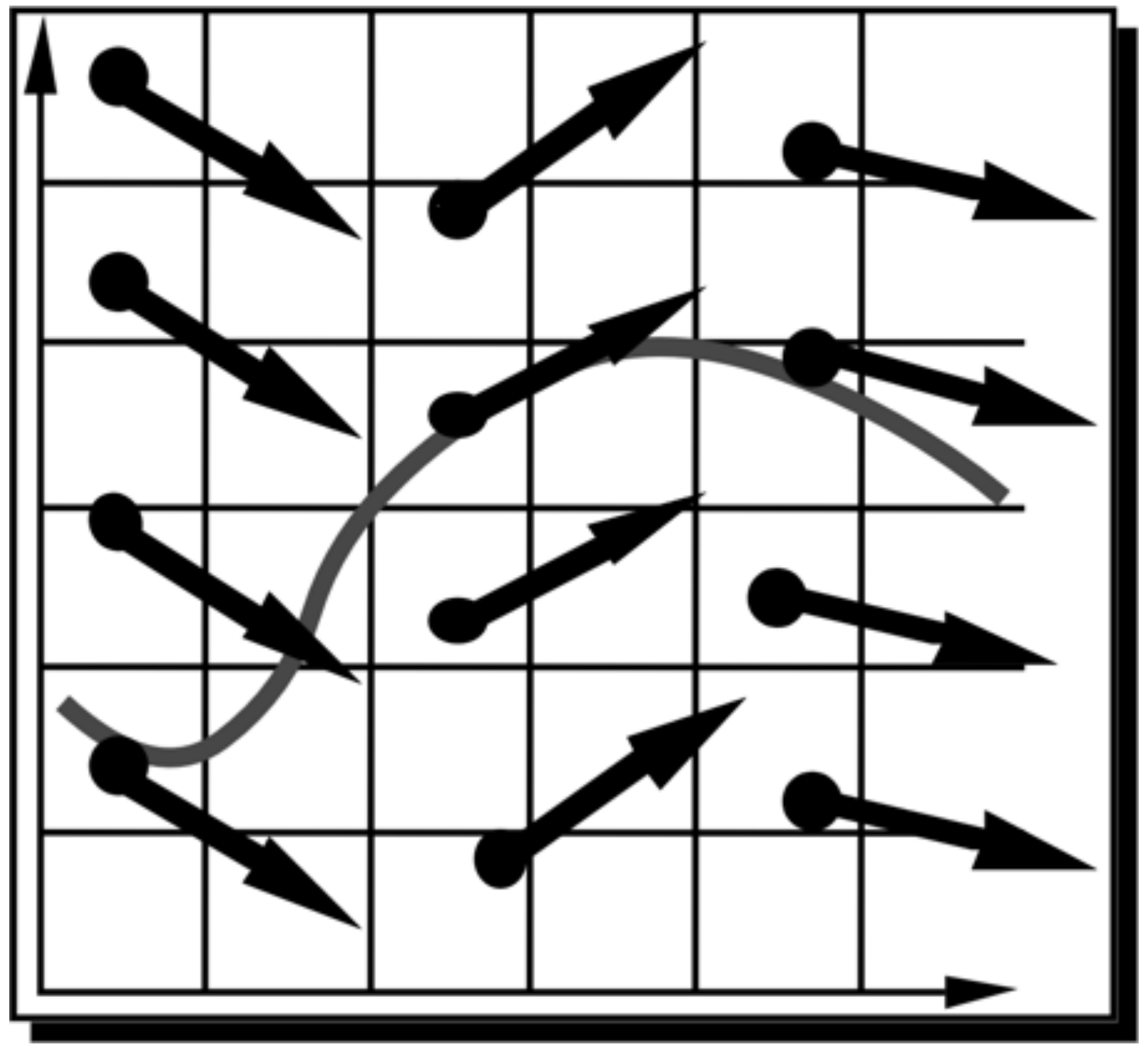
and are trying to solve an initial value problem:

Given $\mathbf{q}(0)$, $\dot{\mathbf{q}}(0)$, find $\mathbf{q}(t)$, $\dot{\mathbf{q}}(t)$ for $t > 0$.

Let's start by understanding this for a simple 1st-order ODE:

$$\dot{x}(t) = \phi(t, x(t))$$

Like a leaf in a river: if you are at position x at time t, your velocity is $\phi(t, x)$



Witkin & Baraff 2001

Taylor series: $x(t + \Delta t) = x(t) + \dot{x}(t)\,\Delta t + O(\Delta t^2)$

$\dot{x}(t) = \phi(t, x(t))$, so

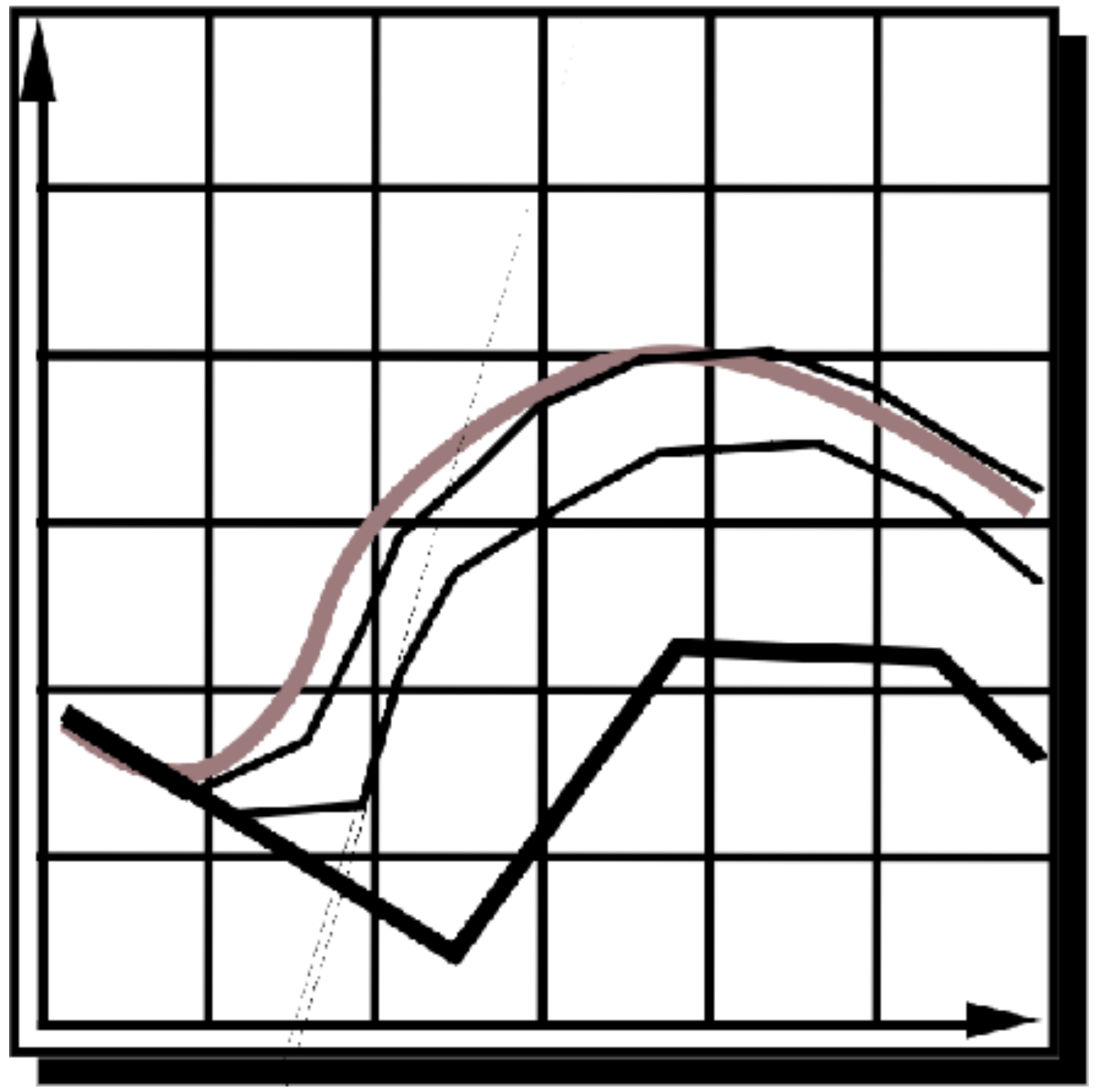
$$x_{n+1} = x_n + \phi(t_n, x_n)\,\Delta t$$

where $t_n = n\,\Delta t$, and x_n = computed estimate of $x(t_n)$

This is called the **(forward) Euler method**

Idea: measure your current velocity $\phi(t_n, x_n)$, then just move forward with that velocity for time Δt

Error in each time step: $O(\Delta t^2)$
Error in solution at fixed time $T = O(\Delta t)$: first-order accurate



Witkin & Baraff 2001

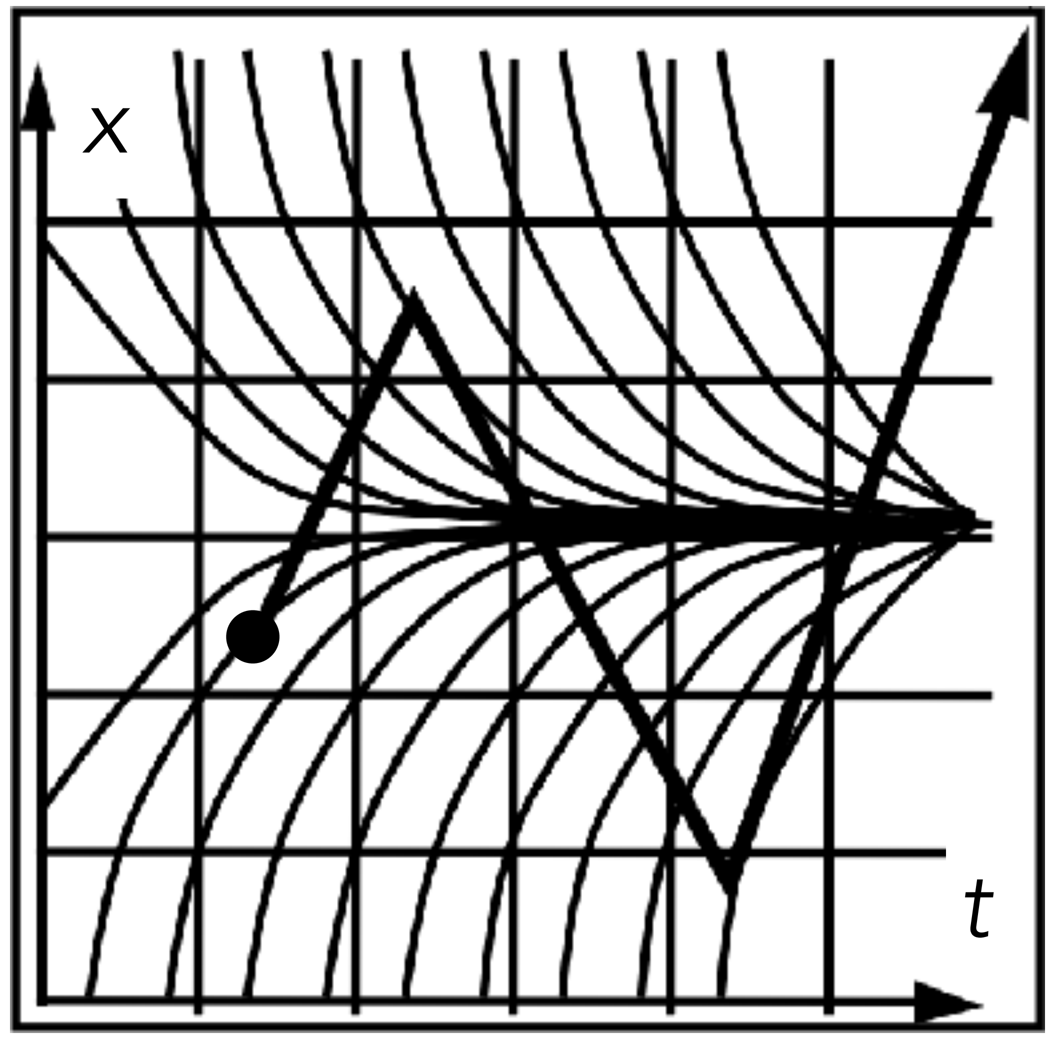
Suppose the ODE is $\dot{x} = a\,x$.

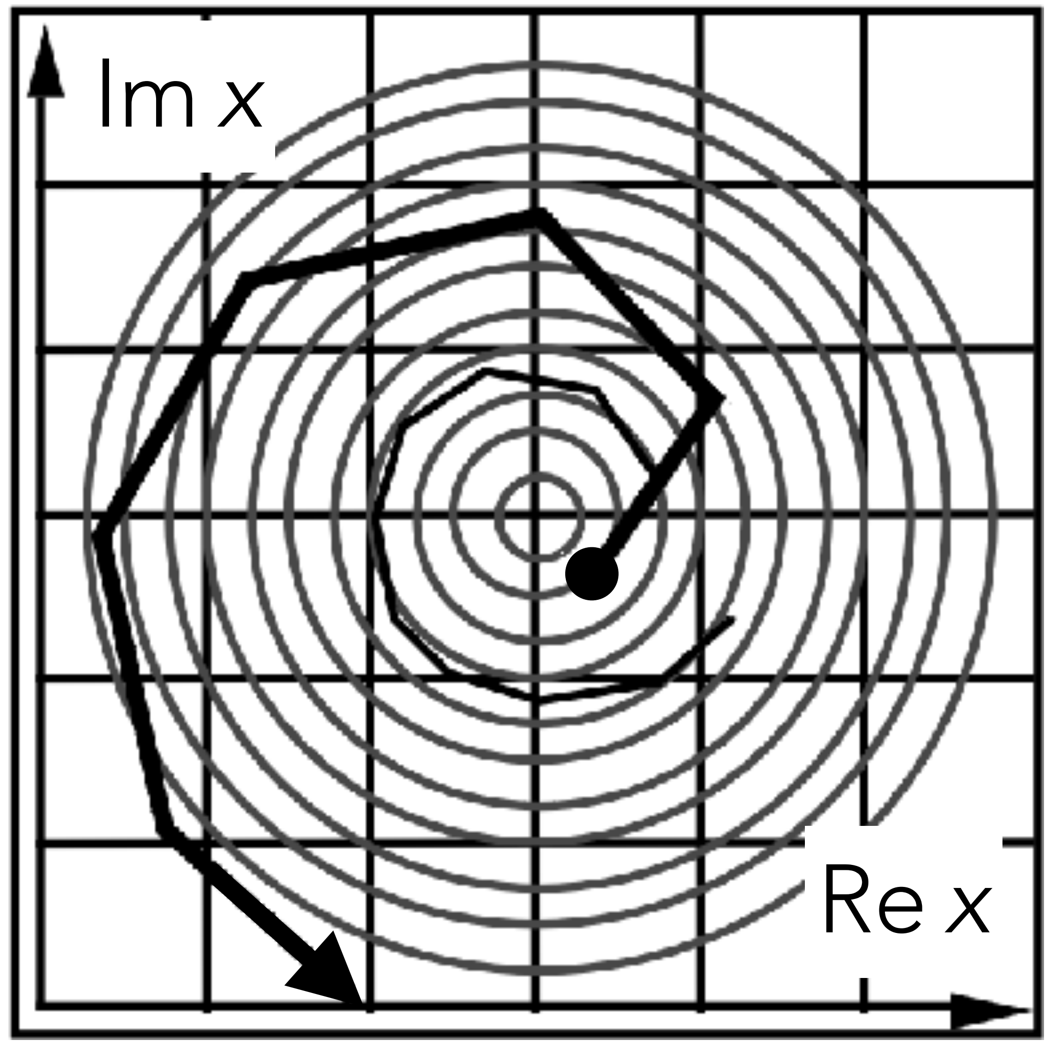
- Exact solution: $x(t) = \exp(a\,t)\,x(0)$

- FE solution: $x_{n+1} = x_n + a\,x_n\,\Delta t = (1 + a\,\Delta t)\,x_n$

For any $a < 0$, exact solution decays smoothly.
But if $|a\,\Delta t| > 2$, FE solution diverges!

If a is imaginary, exact solution moves in a circle.
But for any Δt, FE solution spirals outward!



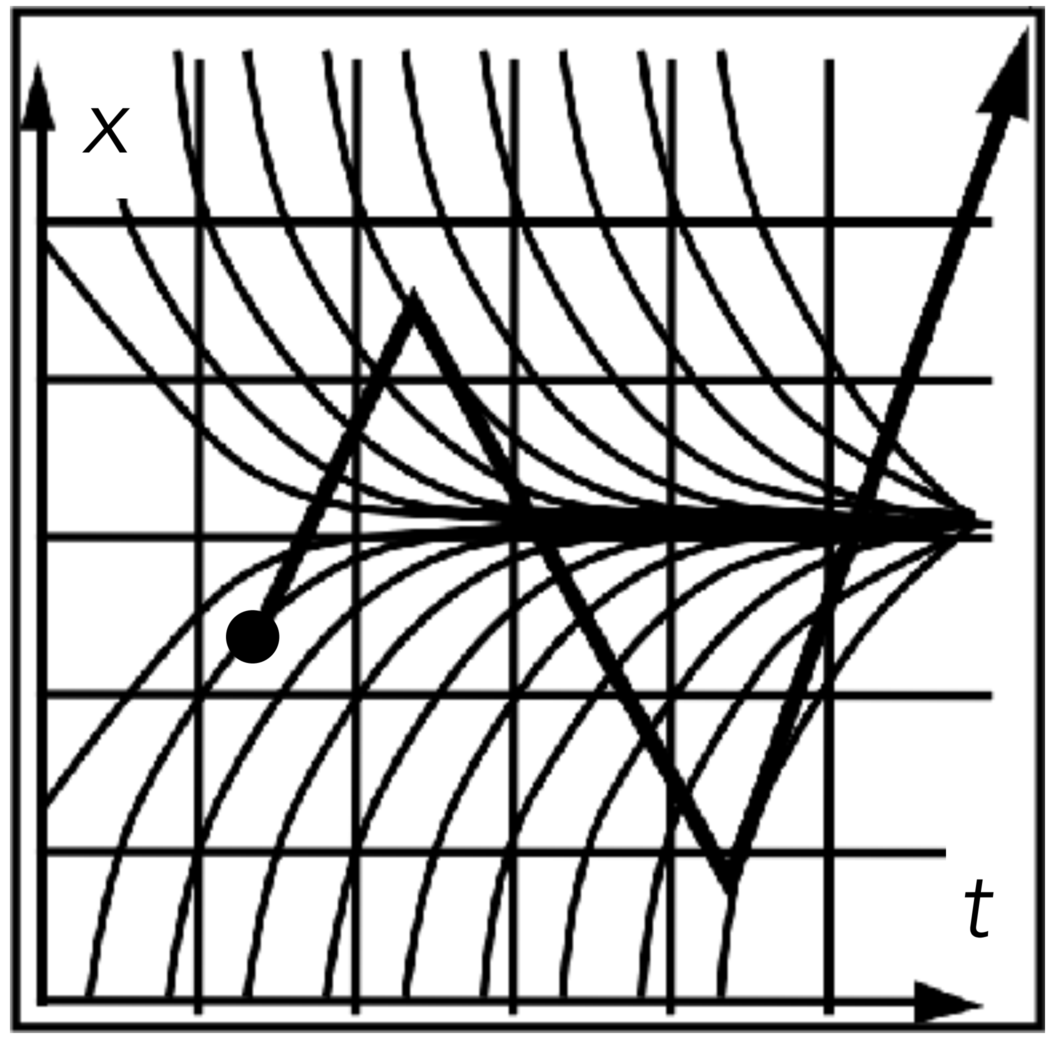


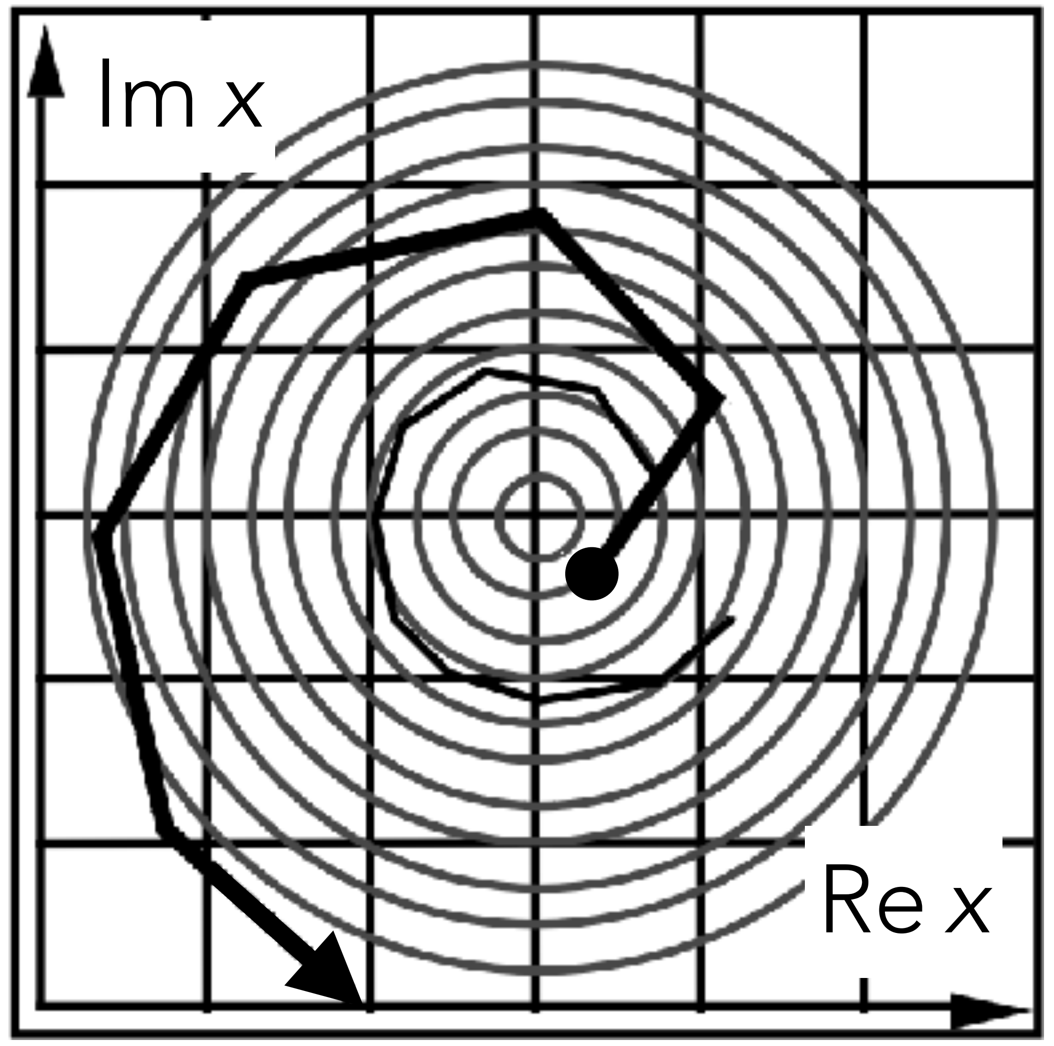
Witkin & Baraff 2001

What does this mean in practice?

- The problem is not accuracy (error after n steps) but stability (whether $|x_n|$ stays bounded)
- When forces vary rapidly with x, Δt needs to be much smaller
- **Rule of thumb:** If time scale of decay / oscillations ≪ Δt, expect problems!

Not good: if we have even a single stiff spring in the system (k_s or k_d very large), we will have to take tiny time steps





Witkin & Baraff 2001