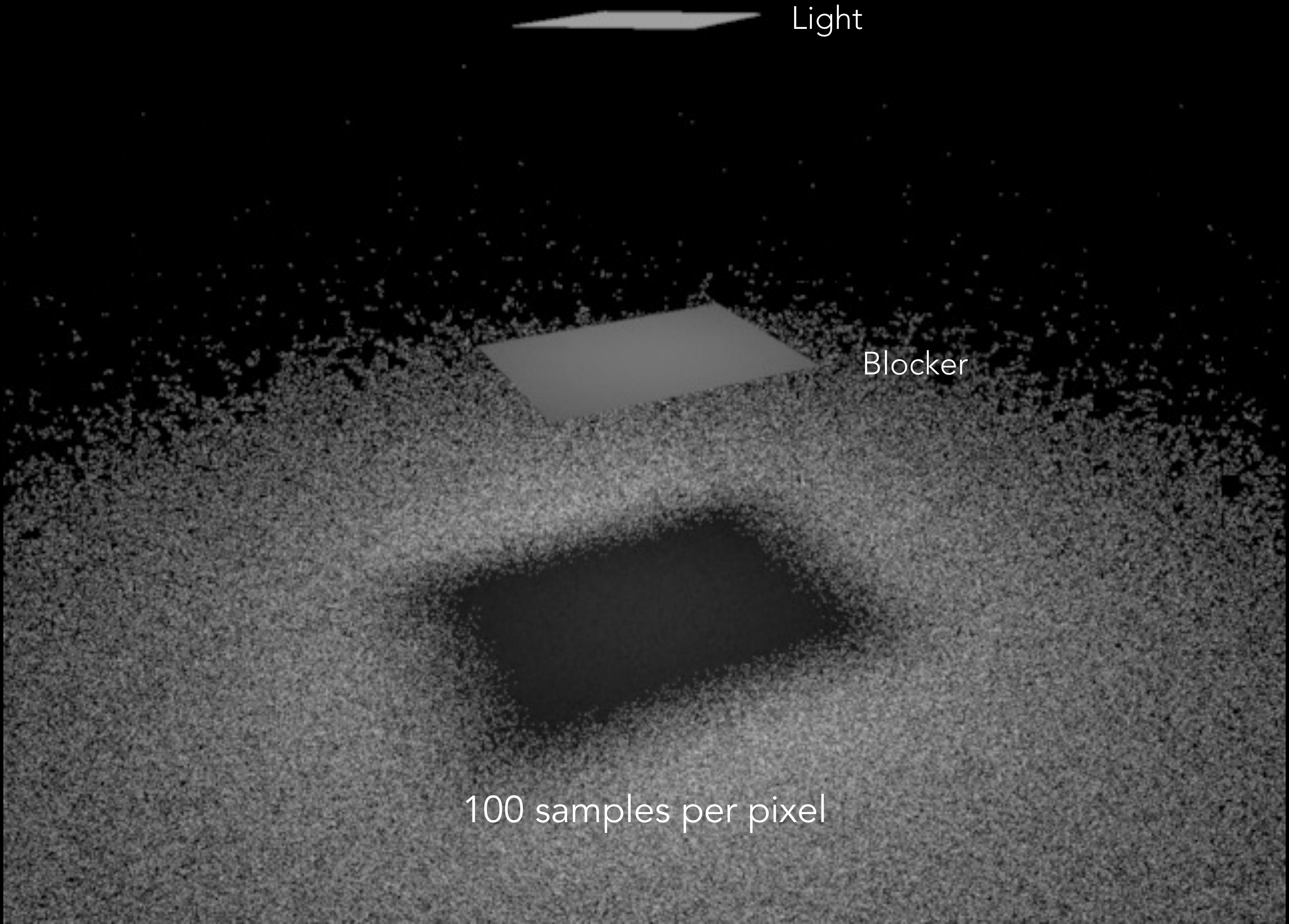
COL781: Computer Graphics

25. Variance Reduction

Veach and Guibas 1997

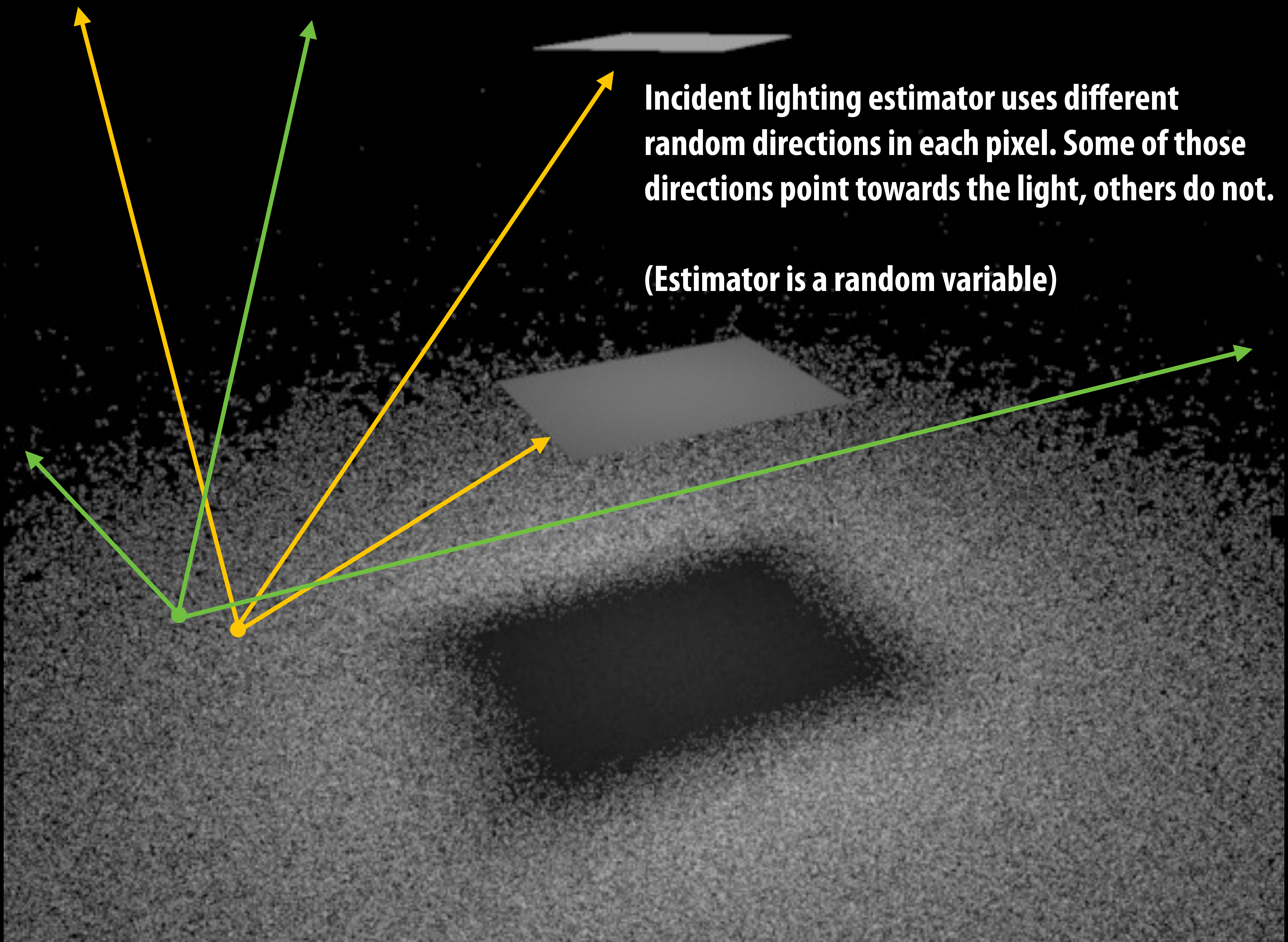
Here's the Monte Carlo path tracer we have so far:

```
incidentRadiance(x, ω):
   p = intersectScene(x, ω)
   L = p.emittedLight(−ω)
   ωi = sampleDirection(p.normal)
   pc = continuationProbability(p, ωi, −ω)
   if random() < pc:
      L += incidentRadiance(p, ωi) * p.BRDF(ωi, −ω) * cos_theta_i * two_pi / pc
   return L
```


Light
Blocker
100 samples per pixel

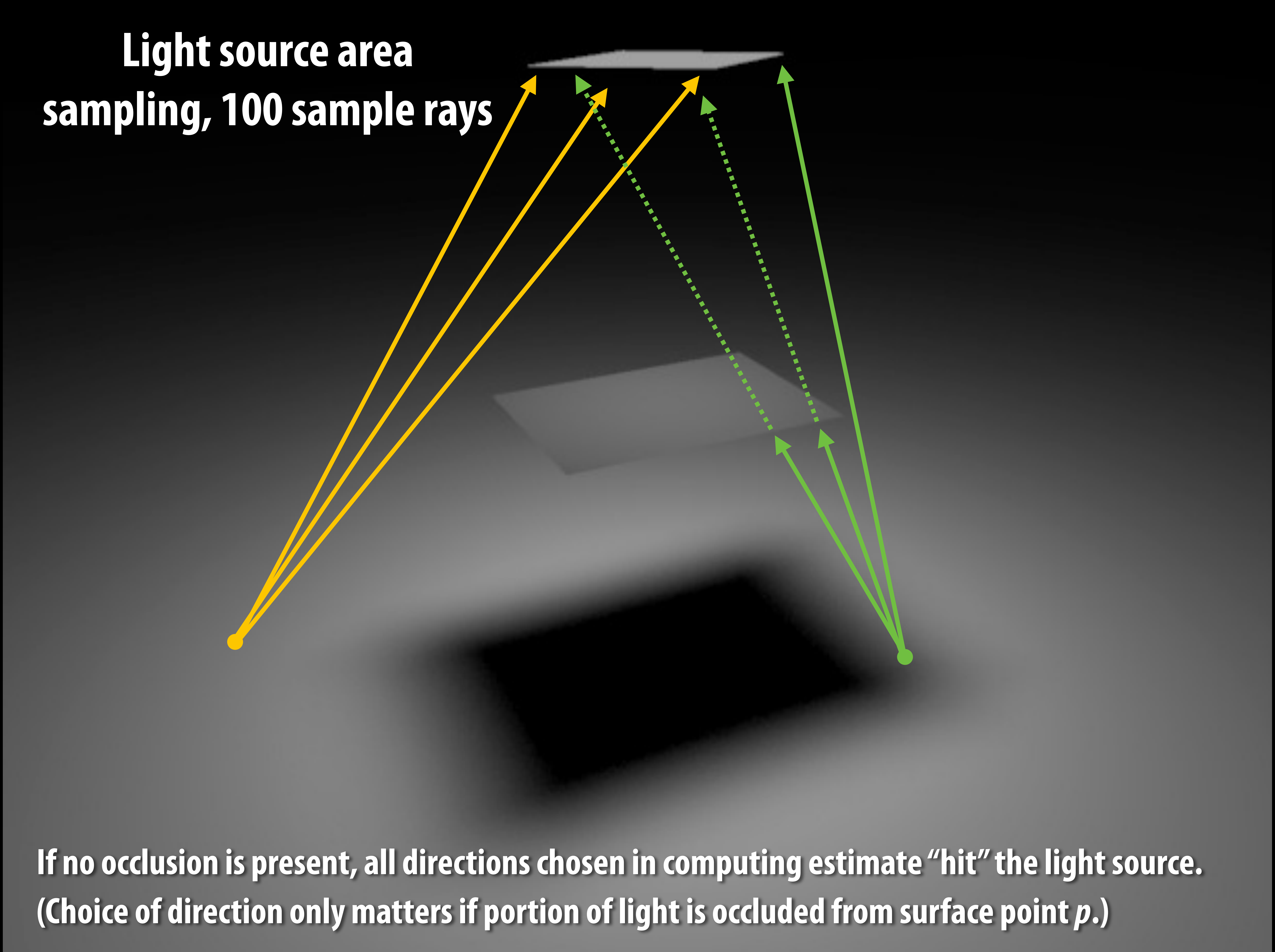
Incident lighting estimator uses different random directions in each pixel. Some of those directions point towards the light, others do not.

(Estimator is a random variable)



Keenan Crane

Light source area sampling, 100 sample rays



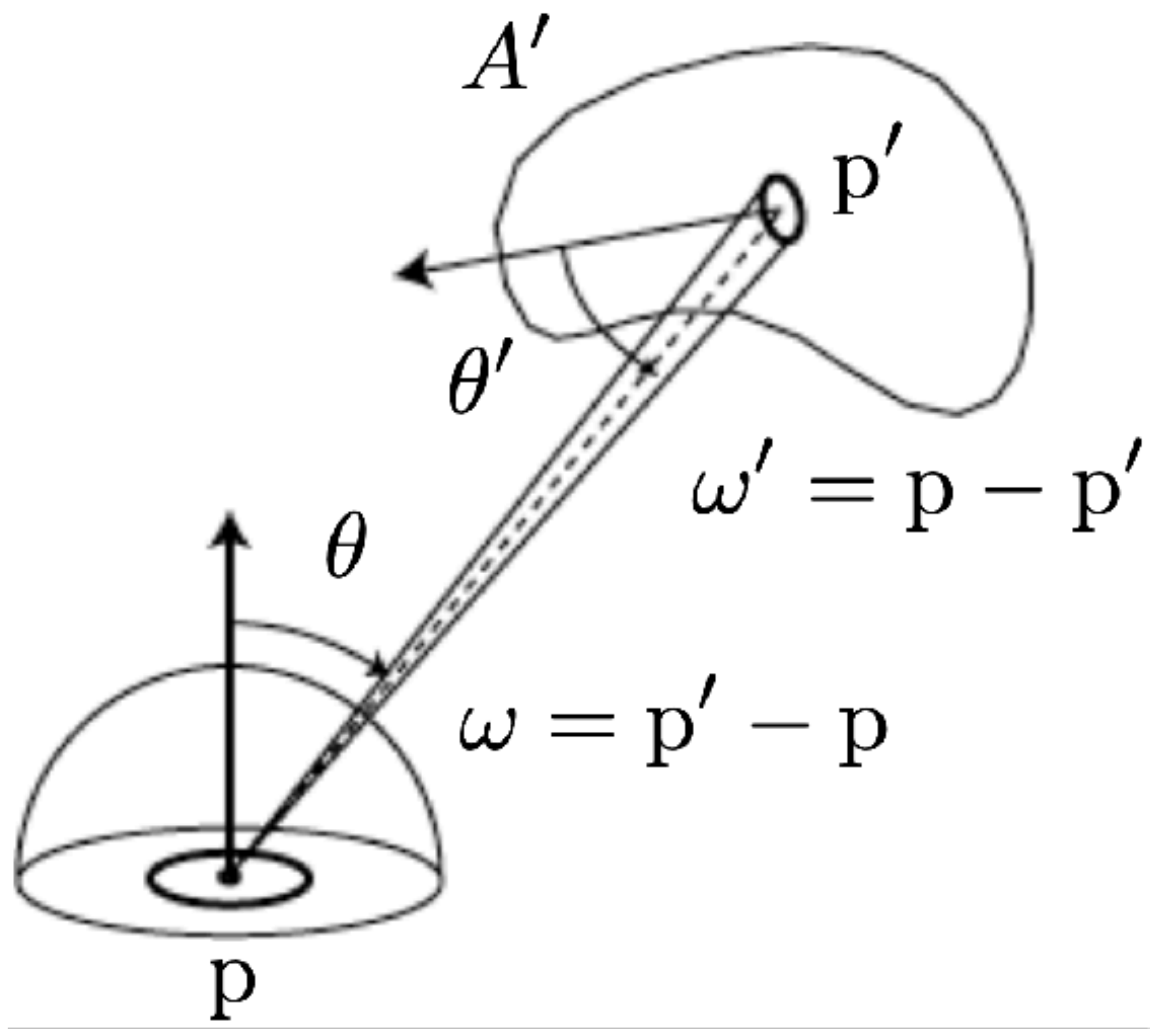
If no occlusion is present, all directions chosen in computing estimate “hit” the light source.
(Choice of direction only matters if portion of light is occluded from surface point *p*.)

Assuming the surface is diffuse,

$$L_o(\mathbf{p}) = f_r \int_{H^2} L_i(\mathbf{p}, \boldsymbol{\omega}) \cos(\theta) \, \mathrm{d}\boldsymbol{\omega}$$

For direct illumination, only need to integrate over directions coming from the light source:

$$L_o(\mathbf{p}) = f_r \int_{A'} L_o(\mathbf{p}', \boldsymbol{\omega}') \, V(\mathbf{p}, \mathbf{p}') \cos(\theta) \cos(\theta') \frac{\mathrm{d}A'}{\|\mathbf{p} - \mathbf{p}'\|^2}$$

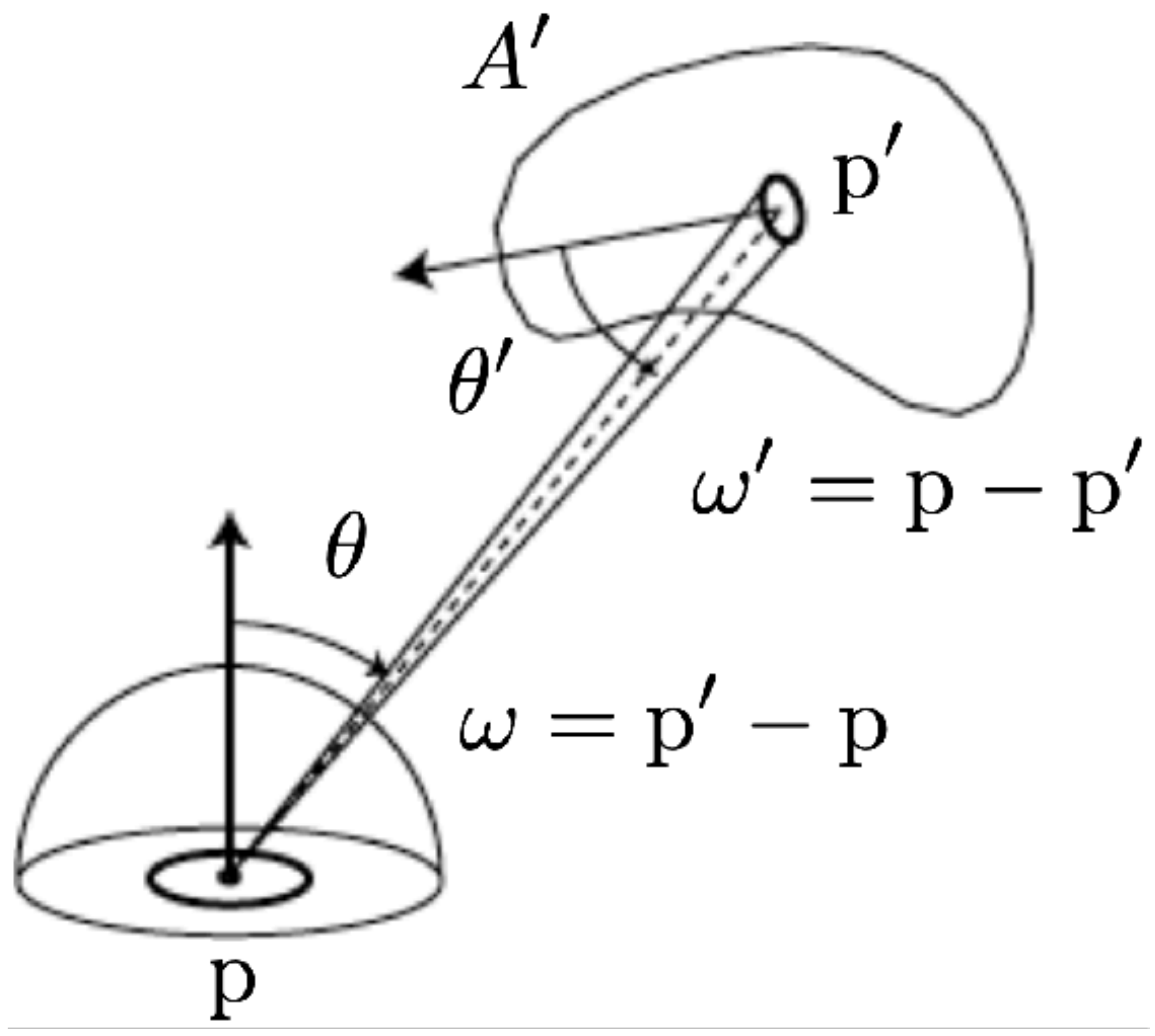


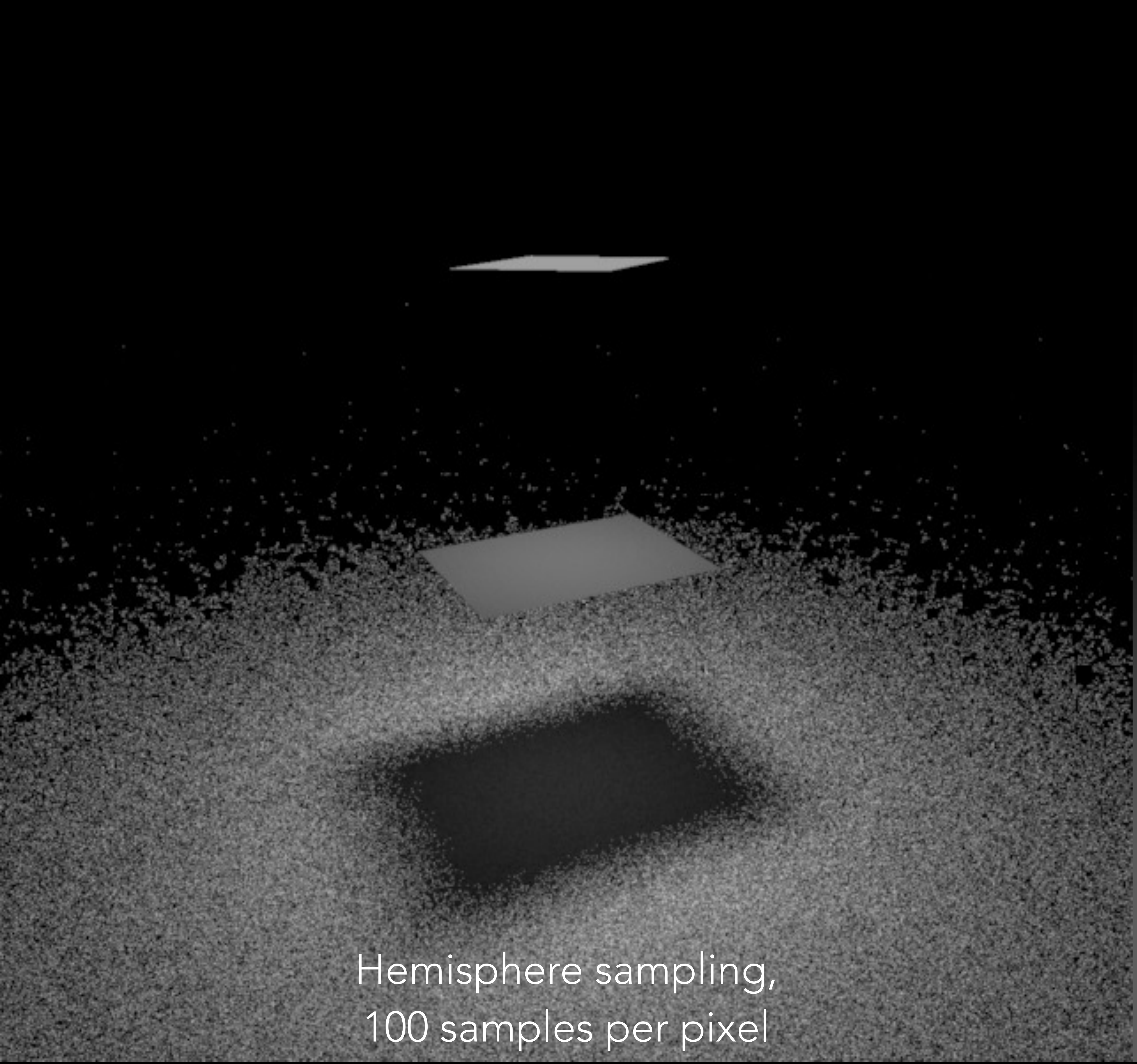
- Differential solid angle $\mathrm{d}\boldsymbol{\omega} = \mathrm{d}A' \cos(\theta') / \|\mathbf{p} - \mathbf{p}'\|^2$
- $V(\mathbf{p}, \mathbf{p}')$: visibility function, 1 if $\mathbf{p}'$ is visible from $\mathbf{p}$ else 0

$$L_o(\mathbf{p}) = f_r \int_{A'} L_o(\mathbf{p}', \boldsymbol{\omega}')\, V(\mathbf{p}, \mathbf{p}')\, \cos(\theta)\, \cos(\theta')\, \frac{\mathrm{d}A'}{\|\mathbf{p} - \mathbf{p}'\|^2}$$

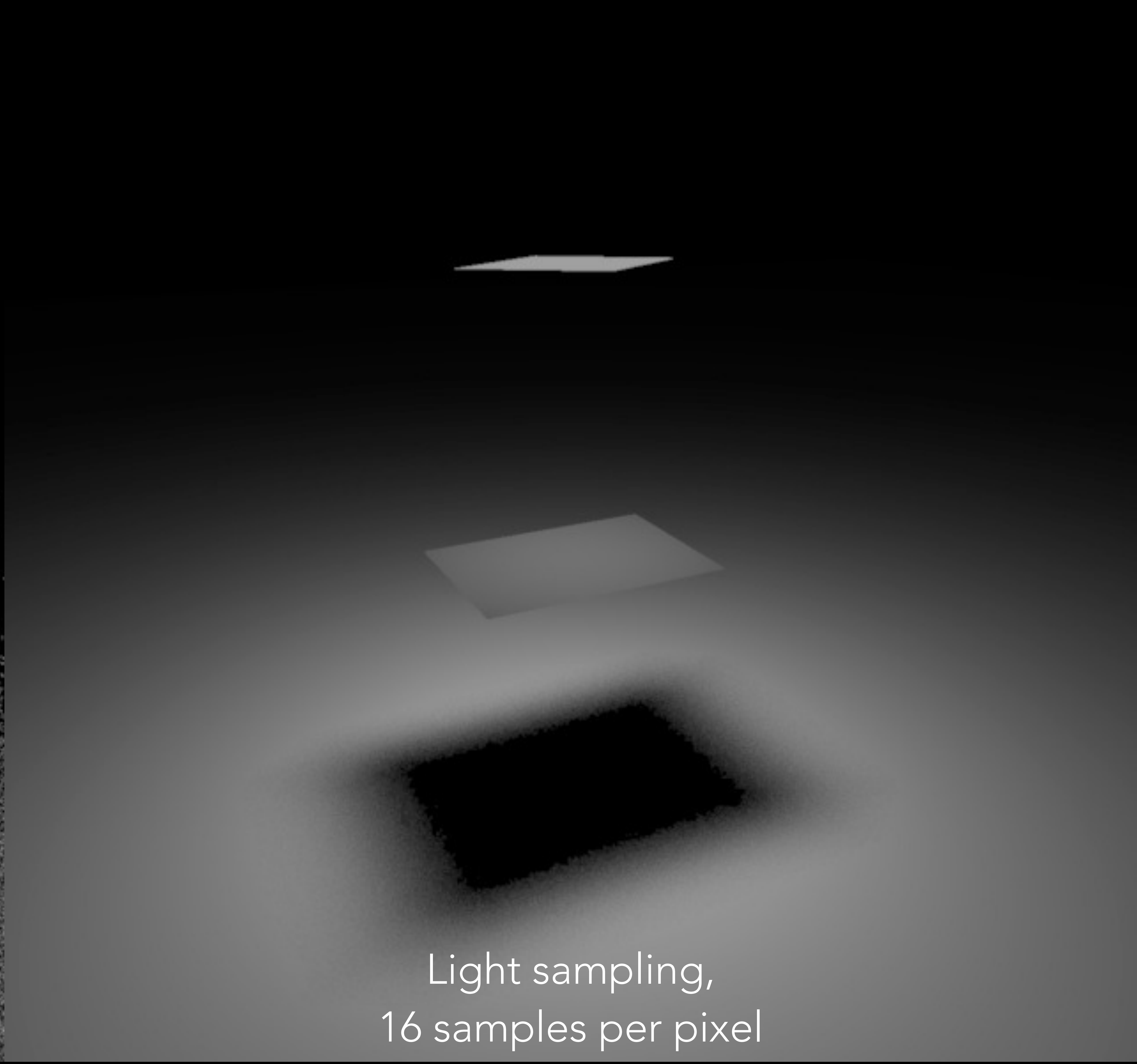
Monte Carlo estimator:

- Uniformly sample area of light source: $\mathbf{p}_1', \ldots, \mathbf{p}_N' \sim 1/A'$
- Evaluate integrand $Y_i = L_o(\mathbf{p}_i', \boldsymbol{\omega}_i')\, V(\mathbf{p}, \mathbf{p}_i')\, \frac{\cos(\theta_i)\, \cos(\theta_i')}{\|\mathbf{p} - \mathbf{p}_i'\|^2}$
- MC estimator is $A'/N \sum Y_i$





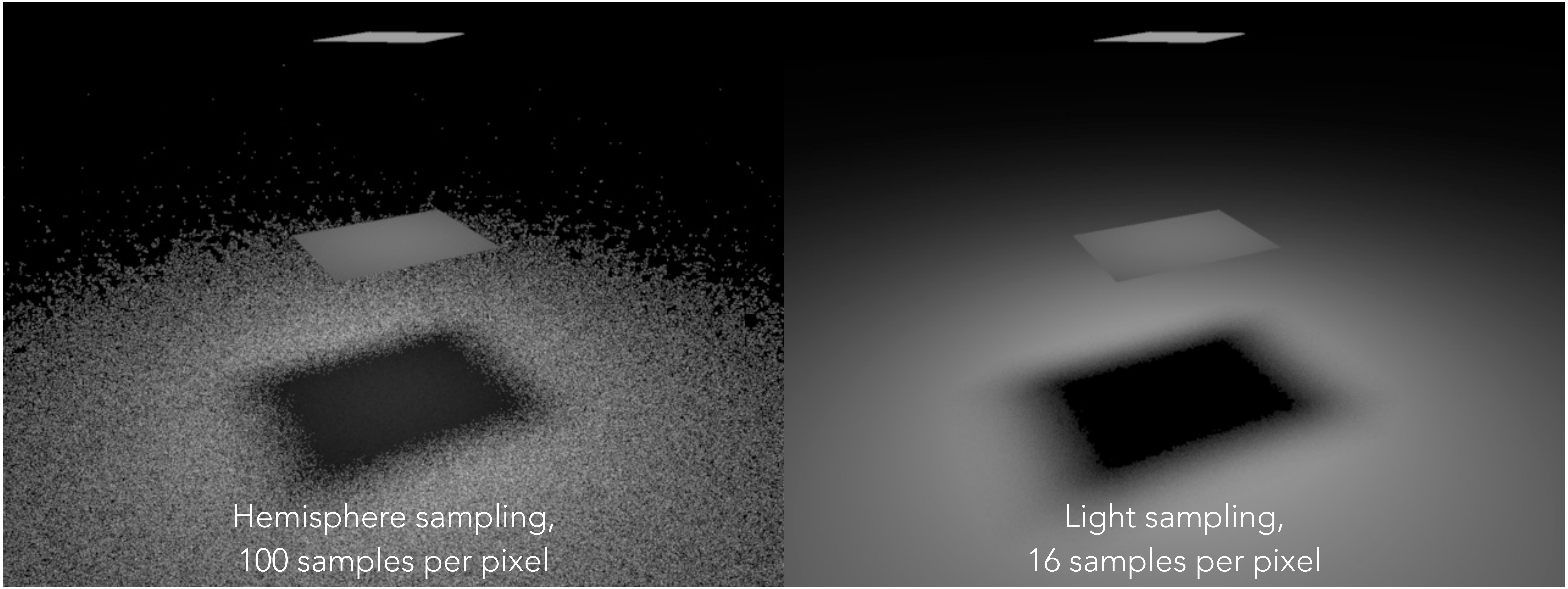
Hemisphere sampling,
100 samples per pixel



Light sampling,
16 samples per pixel

Question: With hemisphere sampling, if I make the light source smaller and smaller so it approaches a point, the image gets noisier and noisier.

What happens if I do the same with light sampling, and why?

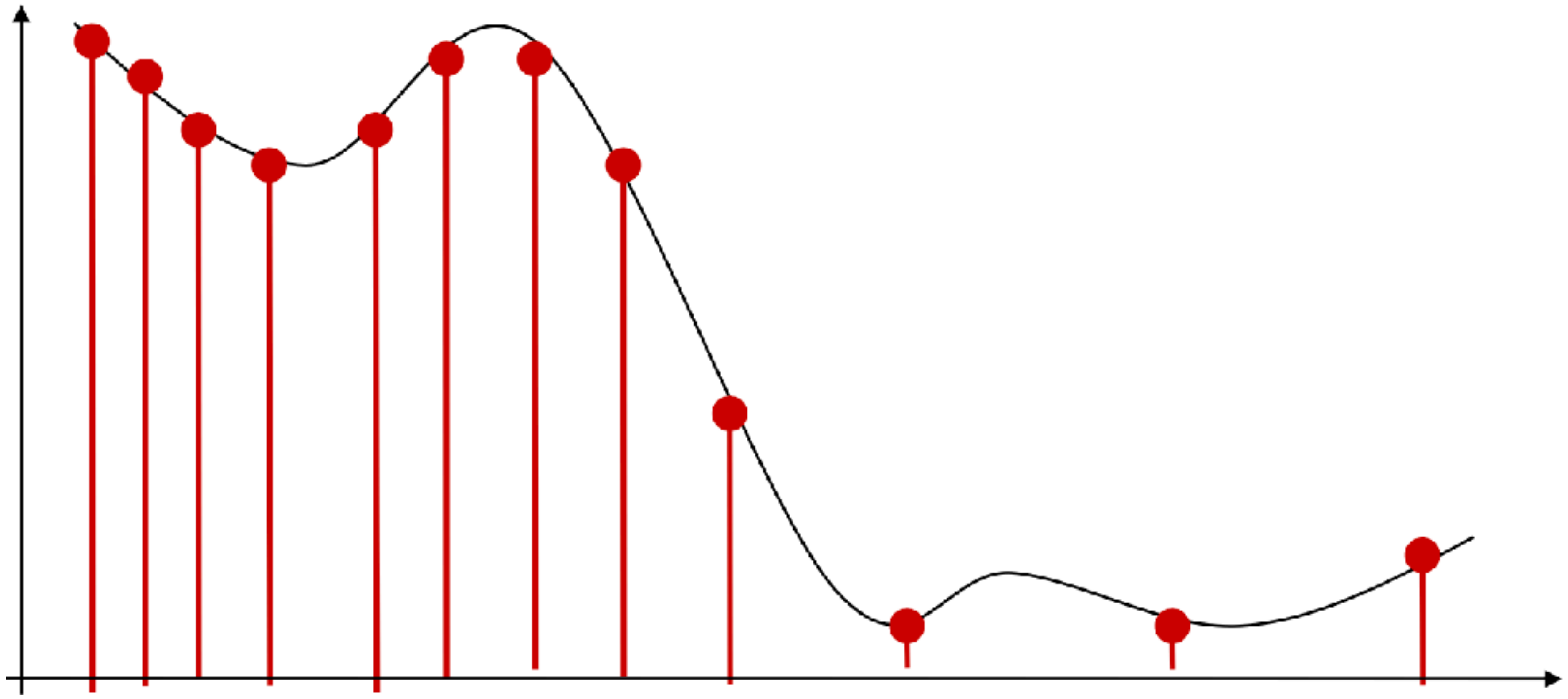


This is an instance of a very general idea:

It can be inefficient to take uniformly distributed samples.

We should take more samples where the integrand is large (i.e. where they will contribute more to the integral), and less samples where it is small!

How to make sure we still get an unbiased estimate?



Wojciech Matusik

Basic Monte Carlo method for $\int_a^b f(x)\,\mathrm{d}x$:

- *X* is uniformly distributed in [*a*, *b*]
- $E[f(X)] = \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x$

What if I sample from a different probability distribution *p*(*x*) on [*a*, *b*]?

- $E[f(X)] = \int_a^b f(x)\,p(x)\,\mathrm{d}x$
- But $E\left[\frac{f(X)}{p(X)}\right] = \int_a^b \frac{f(x)}{p(x)}\,p(x)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x$

Importance sampling

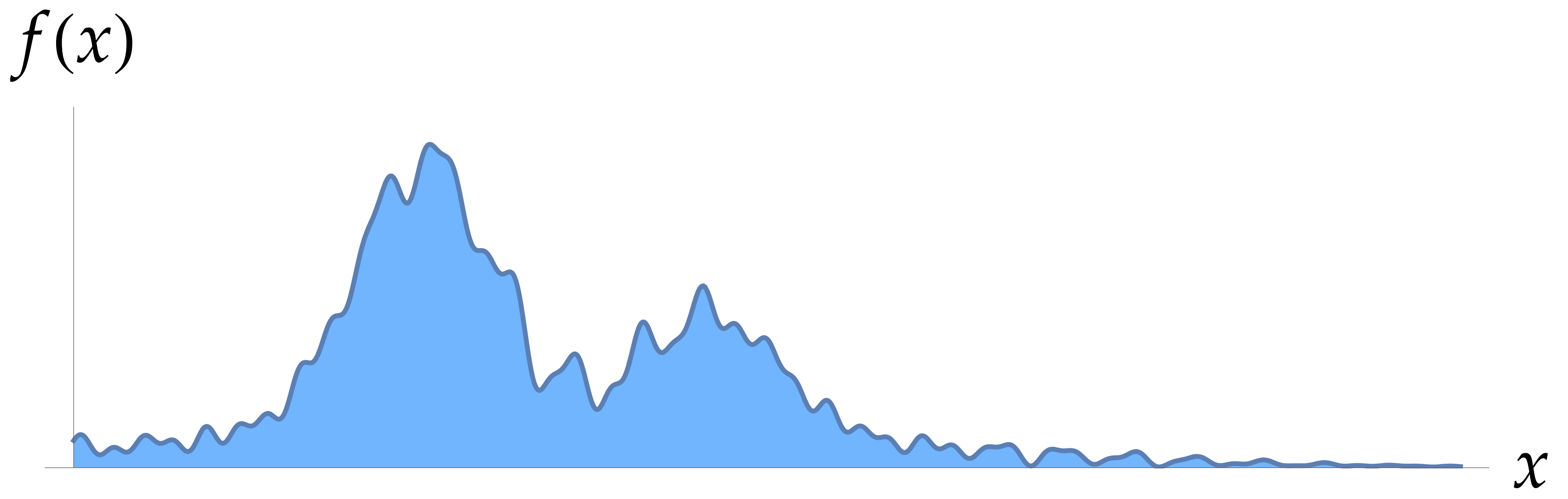
$$E\left[\frac{f(X)}{p(X)}\right] = \int_a^b f(x)\,\mathrm{d}x$$

$$\int_a^b f(x)\,\mathrm{d}x \approx \frac{1}{N}\sum_{i=1}^{N}\frac{f(X_i)}{p(X_i)}$$

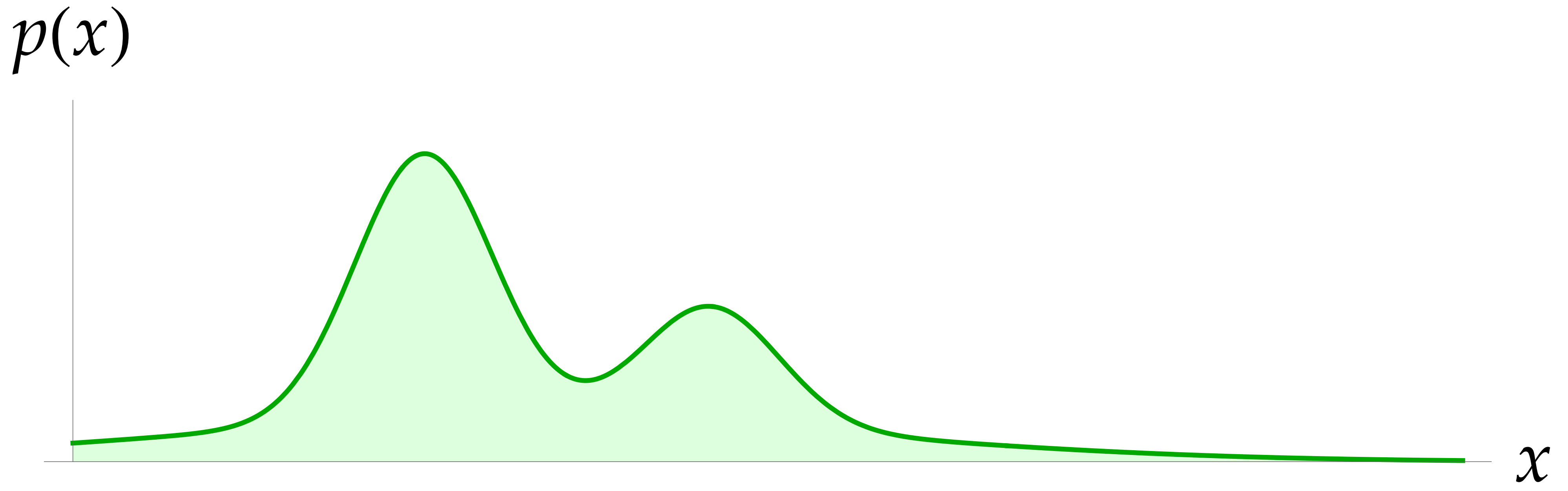
Variance of the estimator now depends on variance of $f(X)/p(X)$, not of $f(X)$

Choose a sampling distribution $p(x)$ which is…

- close to $f(x)$
- easy to sample from



complicated integrand



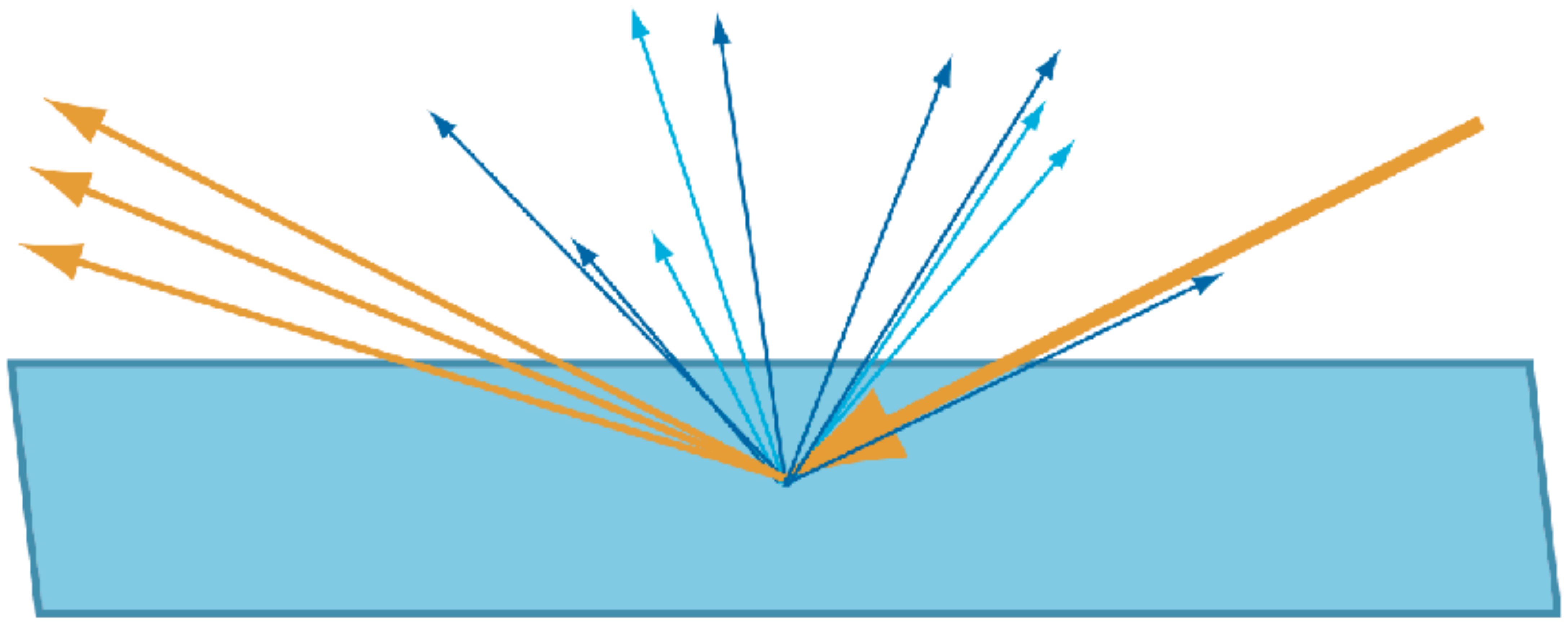
our best guess for where the integrand is "big"

$$L_o(\mathbf{p}, \boldsymbol{\omega}_o) = L_e(\mathbf{p}, \boldsymbol{\omega}_o) + \int_{H^2} f_r(\mathbf{p}, \boldsymbol{\omega}_i \rightarrow \boldsymbol{\omega}_o)\, L_i(\mathbf{p}, \boldsymbol{\omega}_i) \cos(\theta_i)\, d\boldsymbol{\omega}_i$$

In general, we don't know anything about the distribution of L_i.

Only thing we can importance sample is $f_r(\mathbf{p}, \boldsymbol{\omega}_i \rightarrow \boldsymbol{\omega}_o) \cos(\theta_i)$: Shoot more rays in directions where BRDF is large

What if the surface is perfectly specular (BRDF f_r is a delta function)?



Example: Cosine-weighted sampling

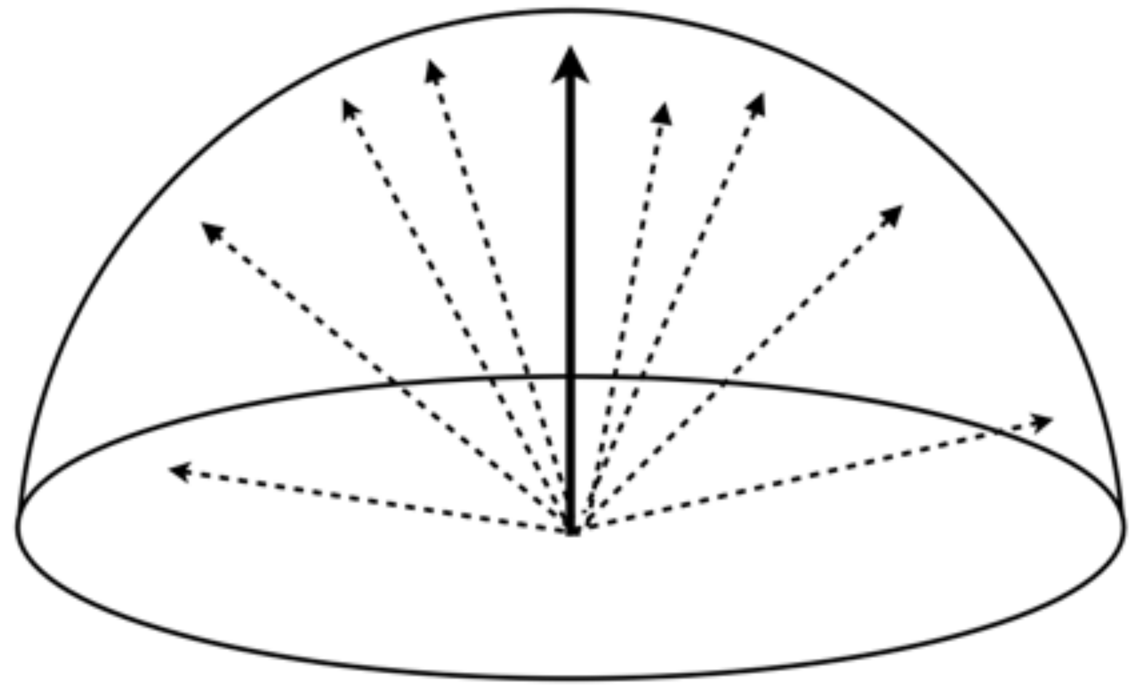
Uniformly sampling the hemisphere is not optimal even for a Lambertian surface!

$$L_o(\mathbf{p}) = f_r \int_{H^2} L_i(\mathbf{p}, \boldsymbol{\omega}) \cos(\theta) \, d\boldsymbol{\omega}$$

Choose $p(\boldsymbol{\omega}) = \cos(\theta)/\pi$. Then

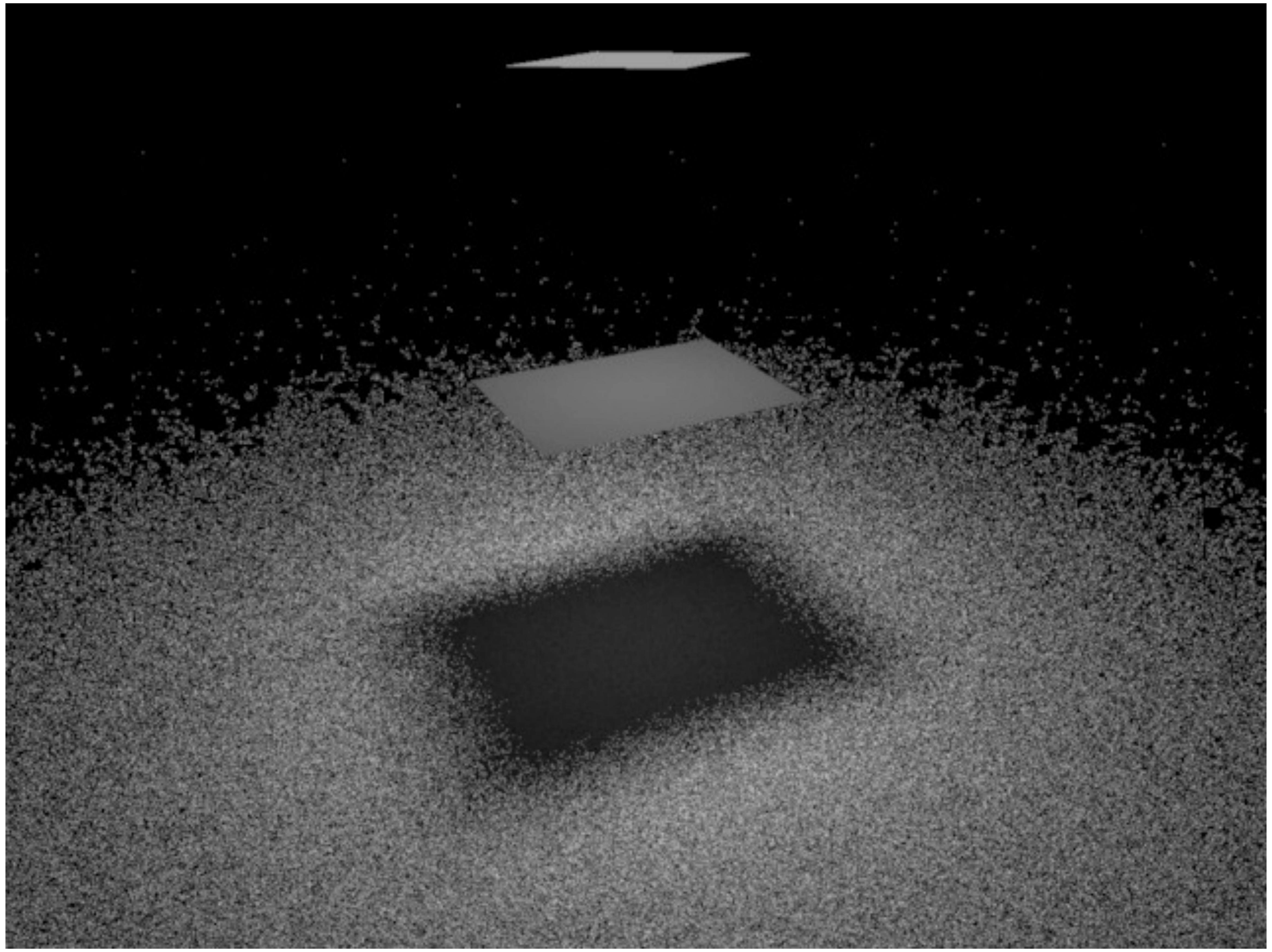
$$\int_{H^2} L_i(\mathbf{p}, \boldsymbol{\omega}) \cos(\theta) \, d\boldsymbol{\omega} \approx \frac{1}{N} \sum_{i=1}^{N} \frac{L_i(\mathbf{p}, \boldsymbol{\omega}_i) \cos(\theta_i)}{\cos(\theta_i)/\pi} = \frac{\pi}{N} \sum_{i=1}^{N} L_i(\mathbf{p}, \boldsymbol{\omega}_i)$$

Exercise: Figure out how to sample directions according to $p(\boldsymbol{\omega})$.
(There is a very nice geometrical approach!)

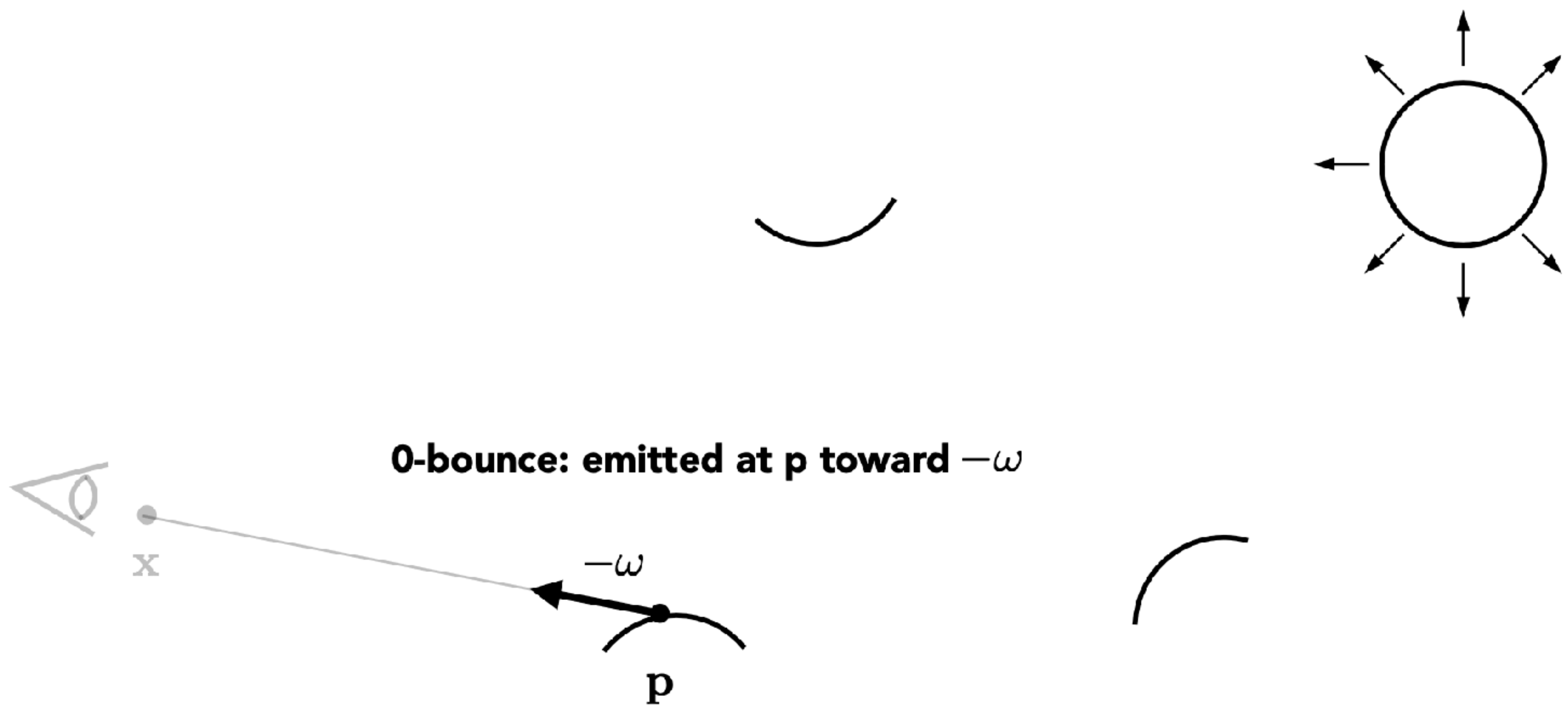


```
incidentRadiance(x, ω):
    p = intersectScene(x, ω)
    L = p.emittedLight(−ω)
    ωi, pdf = p.BRDF.sampleDirection(−ω)
    pc = continuationProbability(p, ωi, −ω)
    if random() < pc:
        L += incidentRadiance(p, ωi) * p.BRDF(ωi, −ω) * cos_theta_i / pdf / pc
    return L
```

Wait, BRDF sampling will take us back to these noisy shadows.



Recall that the radiance is the sum of contributions from many different paths…



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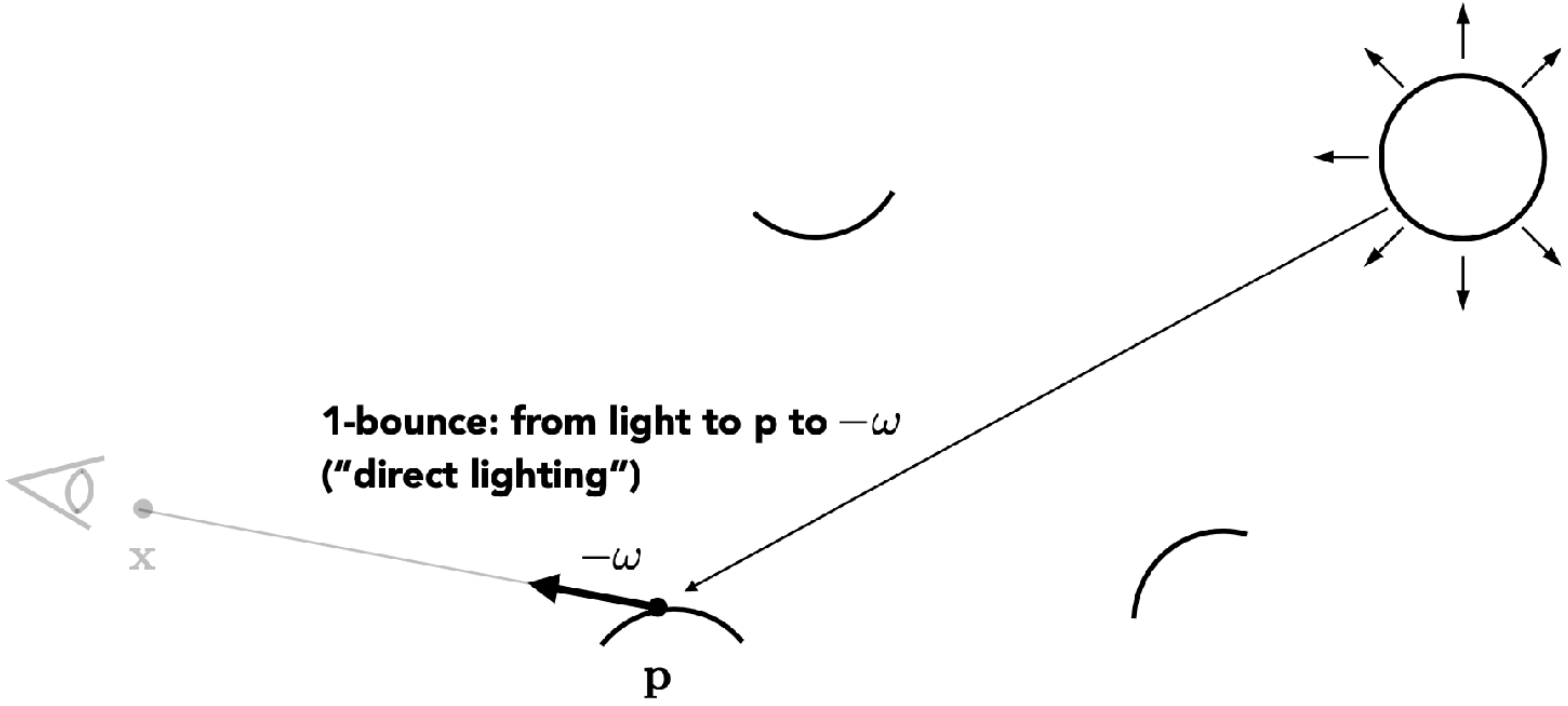

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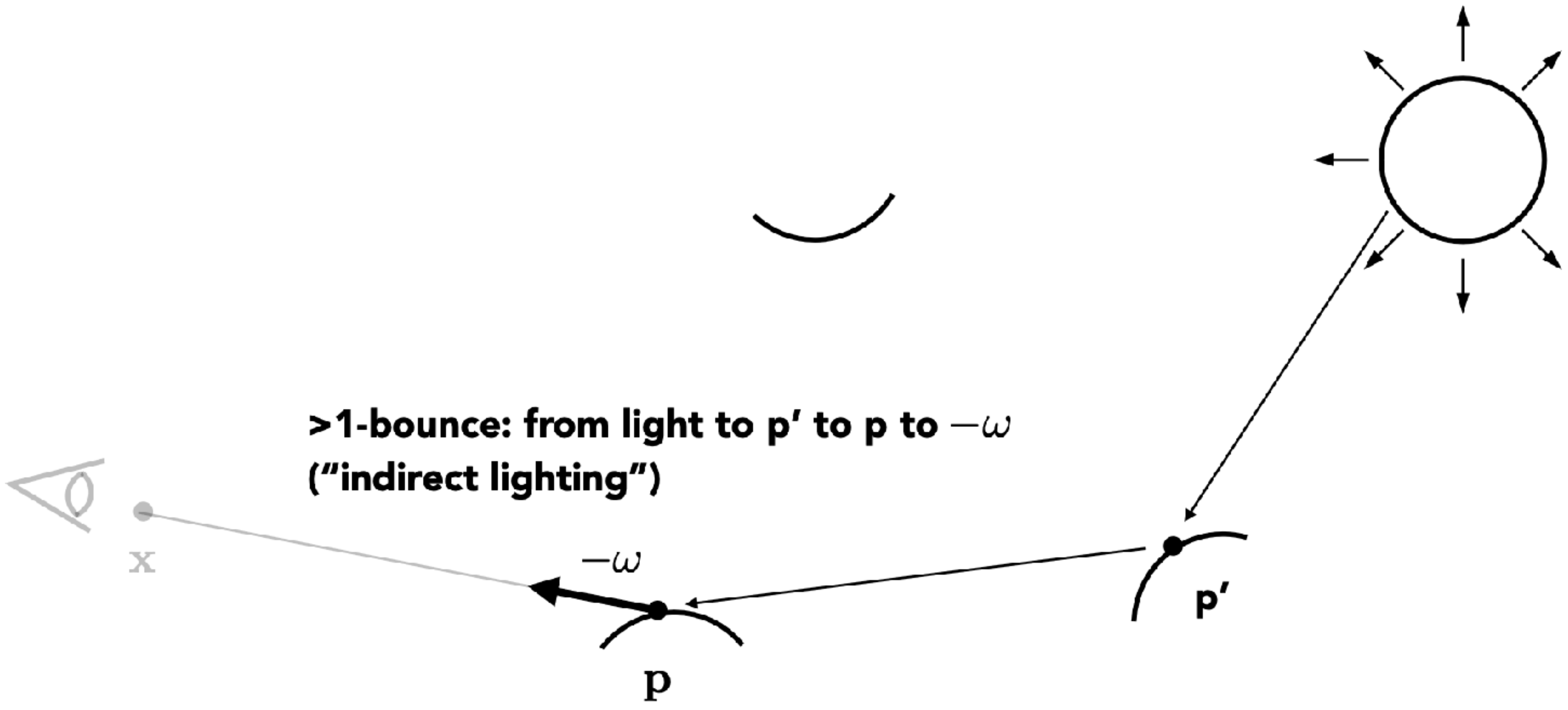


Total radiance
= 0-bounce radiance (emission)
+ 1-bounce radiance (direct illumination)
+ >1-bounce radiance (indirect illumination)

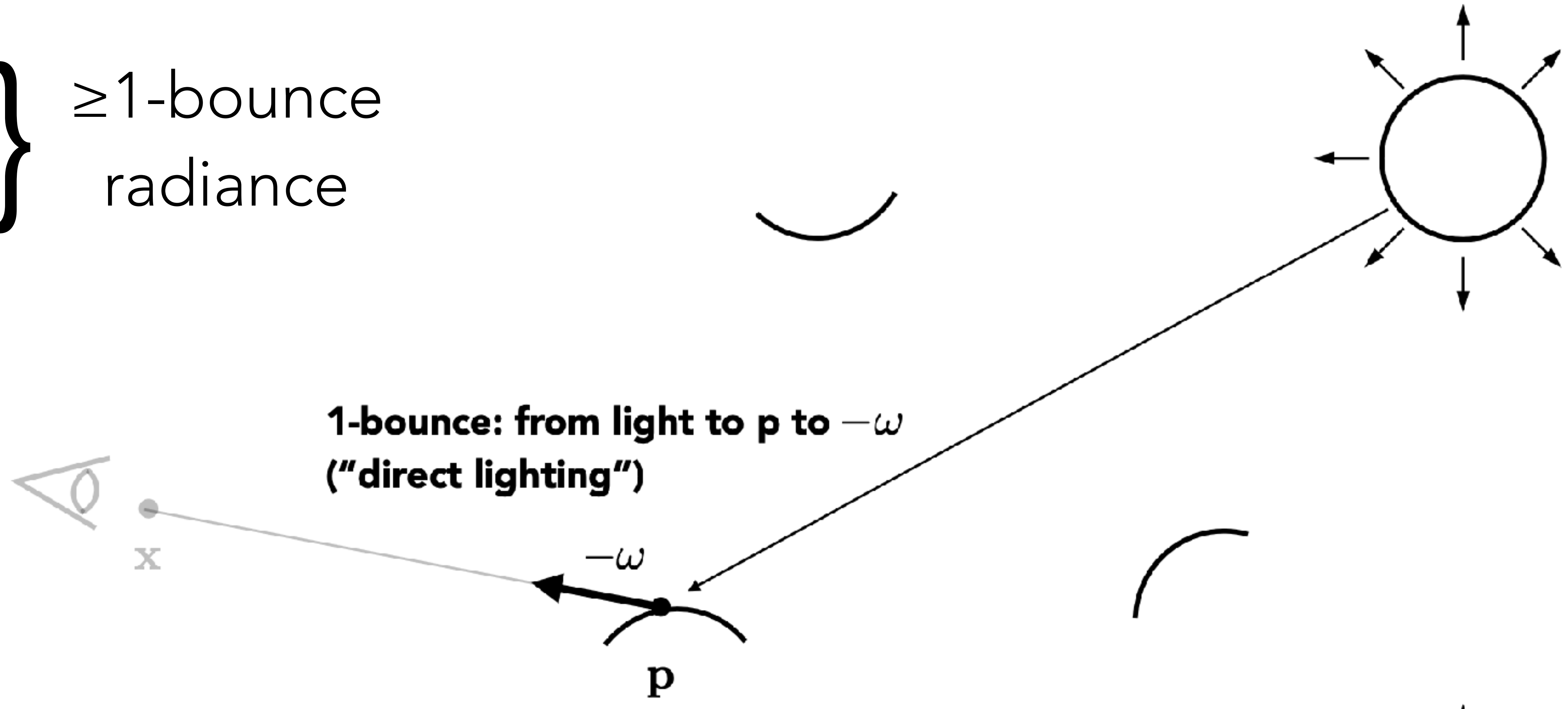
} ≥1-bounce radiance

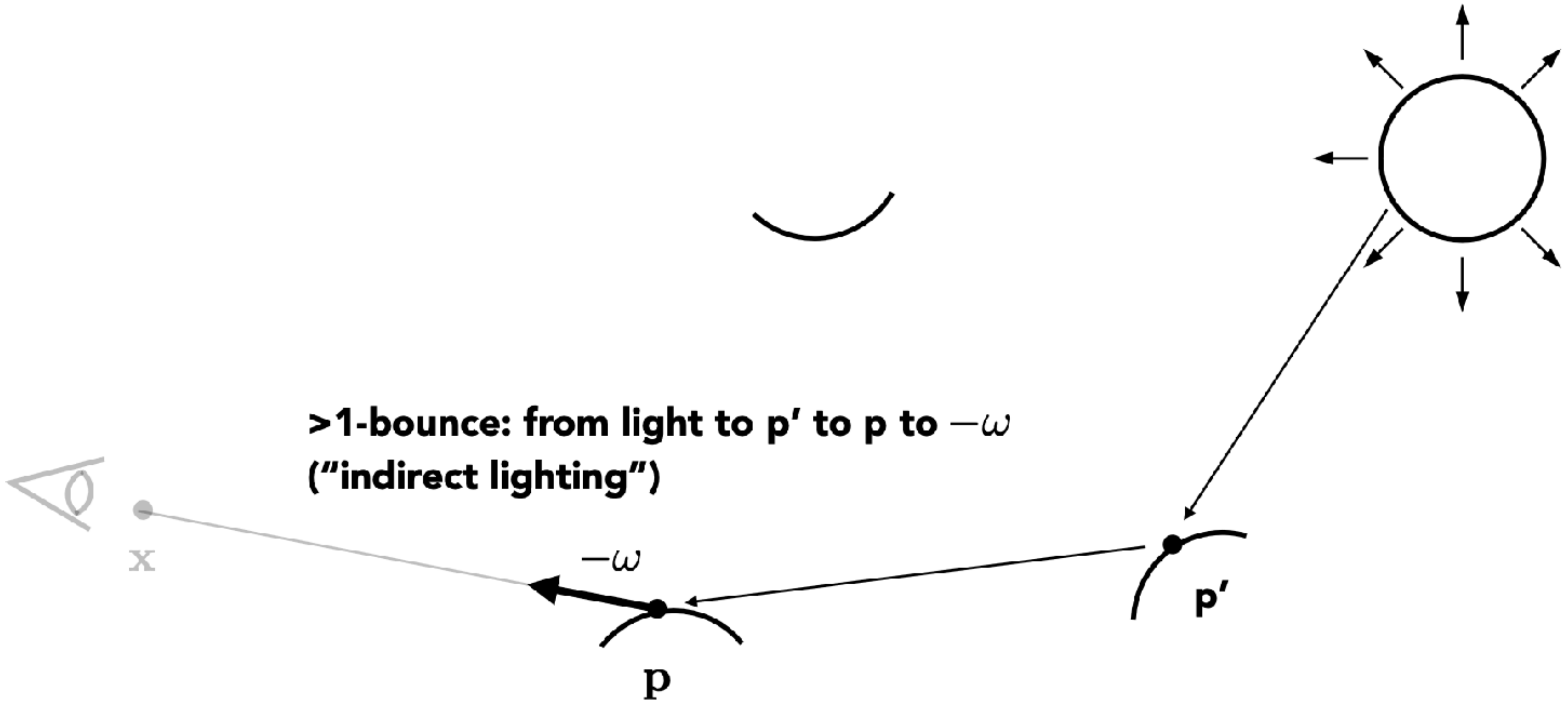
1-bounce radiance:

- Sample point on light source(s)
- Integrate 0-bounce radiance of light source

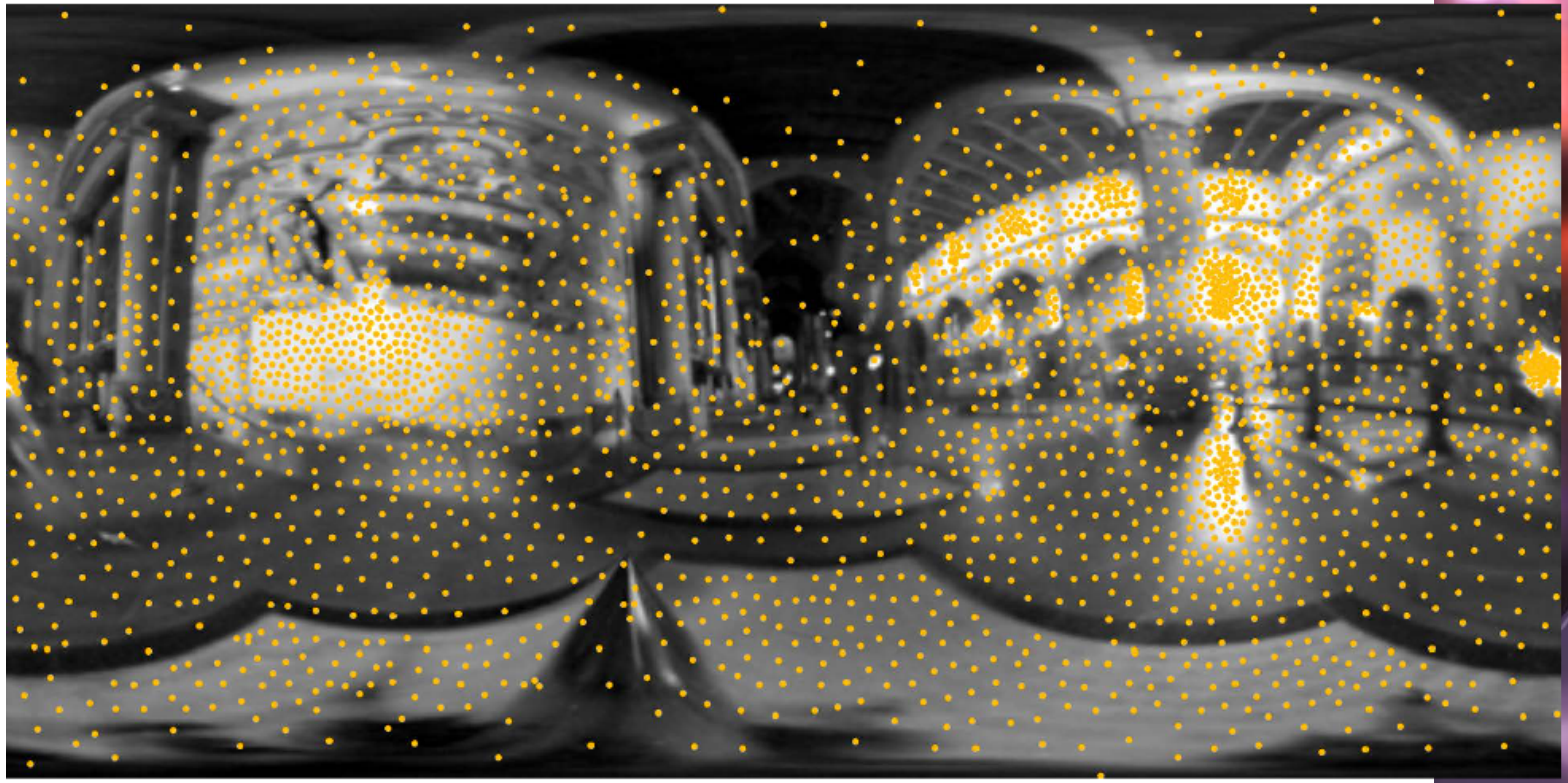
>1-bounce radiance:

- Sample direction from BRDF
- Recursively integrate only ≥1-bounce radiance from other point **p′**





Example: Environment lighting



Now you understand the essentials of path tracing!

Let’s see some examples of its limitations…

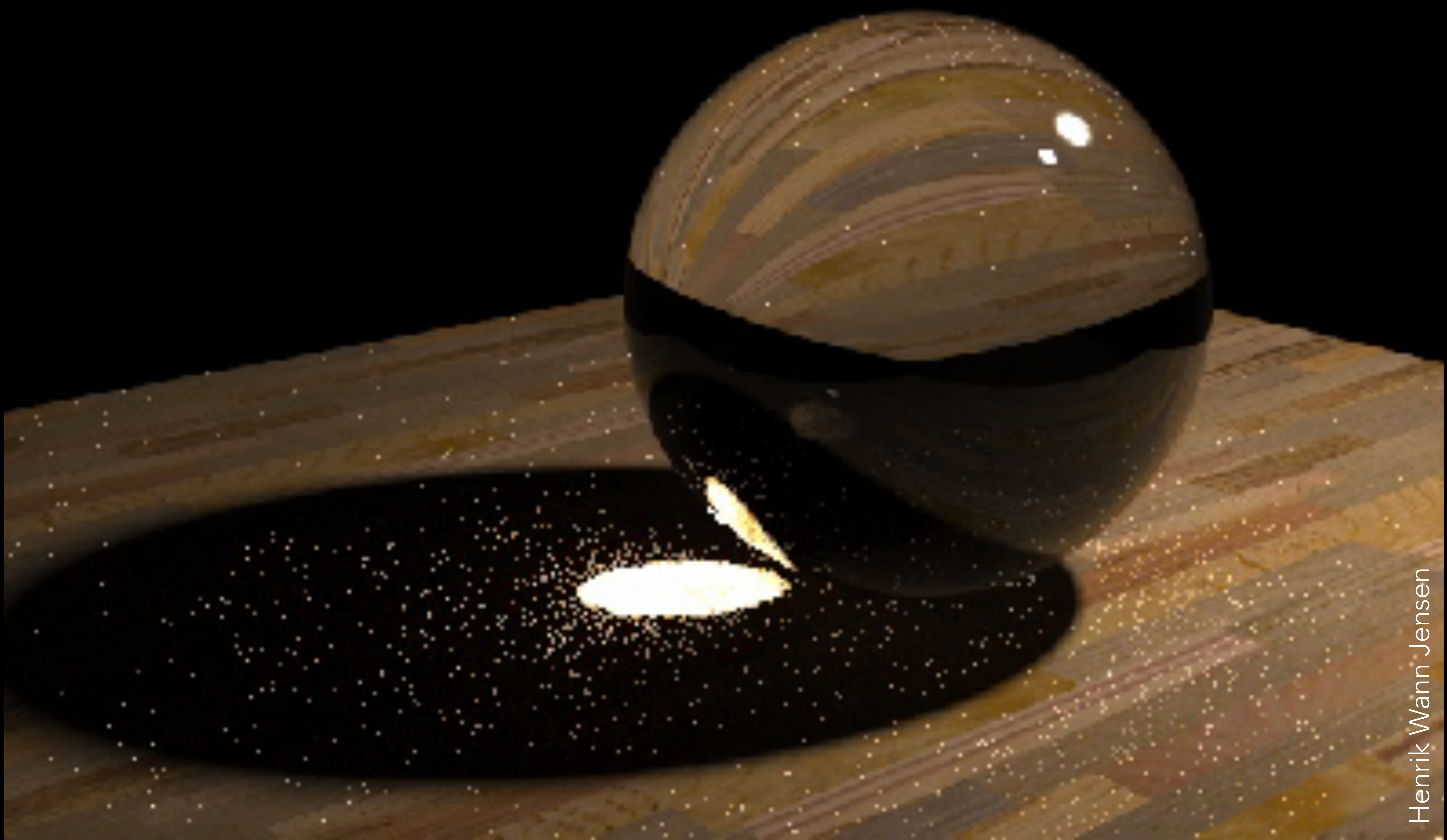
Diffuse vs. glossy surfaces, point vs. area lights

Light sampling

BRDF sampling

Veach and Guibas 1995

Caustics



Hard-to-find light paths

Veach and Guibas 1995

Homework exercise

Find a way to sample directions on the hemisphere according to the cosine-weighted distribution, $p(\boldsymbol{\omega}) = \cos(\theta)/\pi$.

(A very nice geometrical approach exists, but a straightforward application of inversion sampling should also work.)