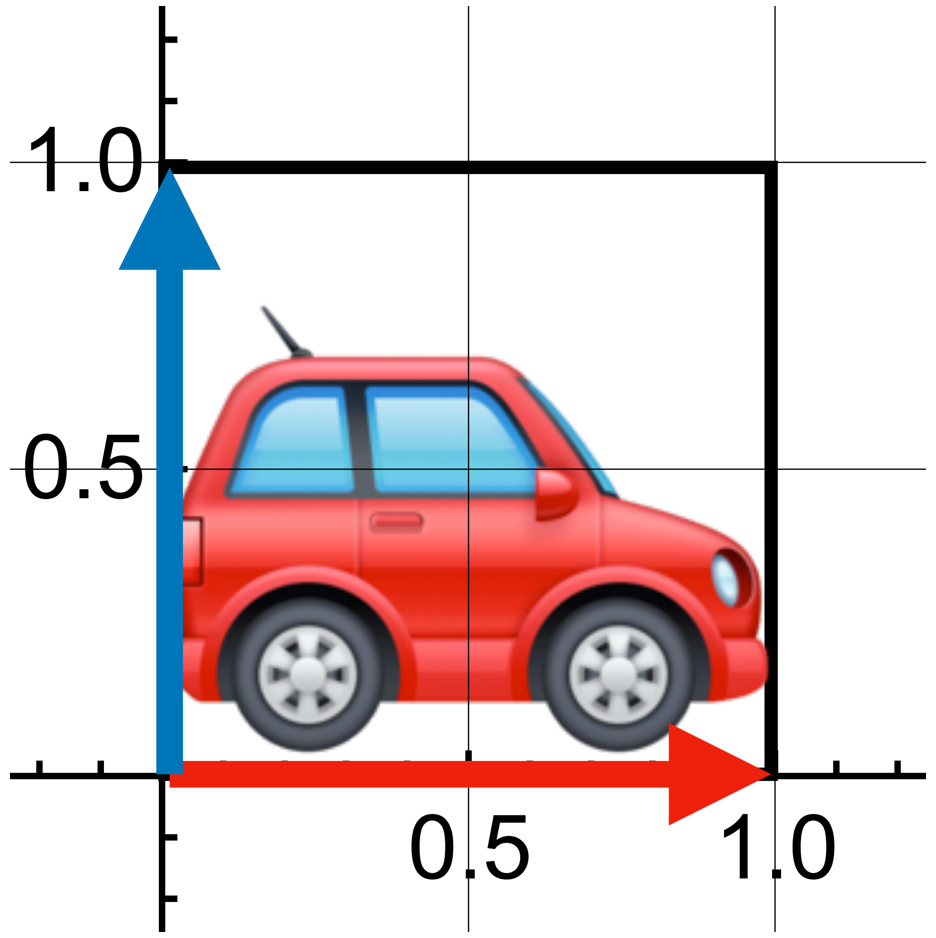
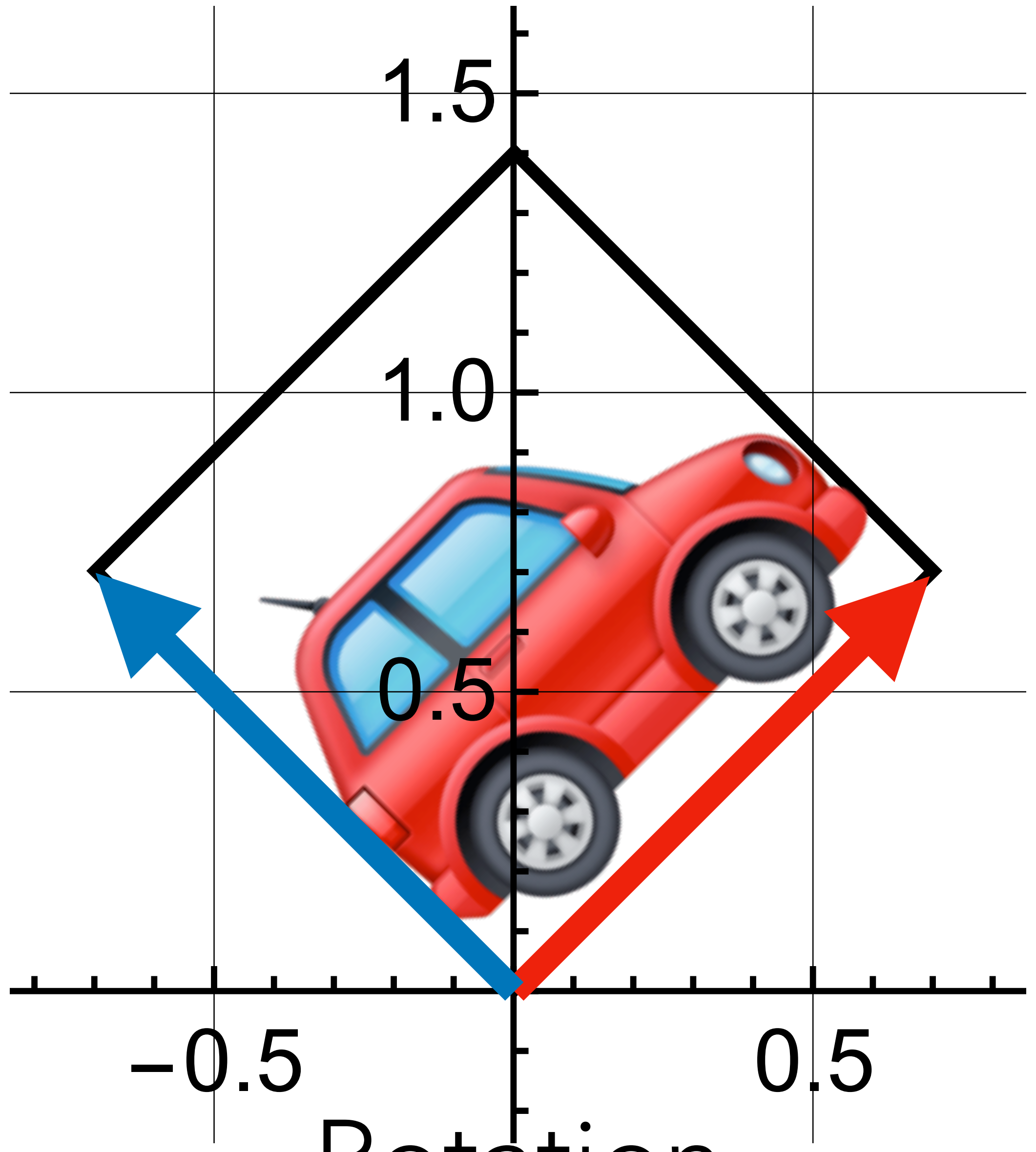
COL781: Computer Graphics

5. Affine Transformations

Blender manual

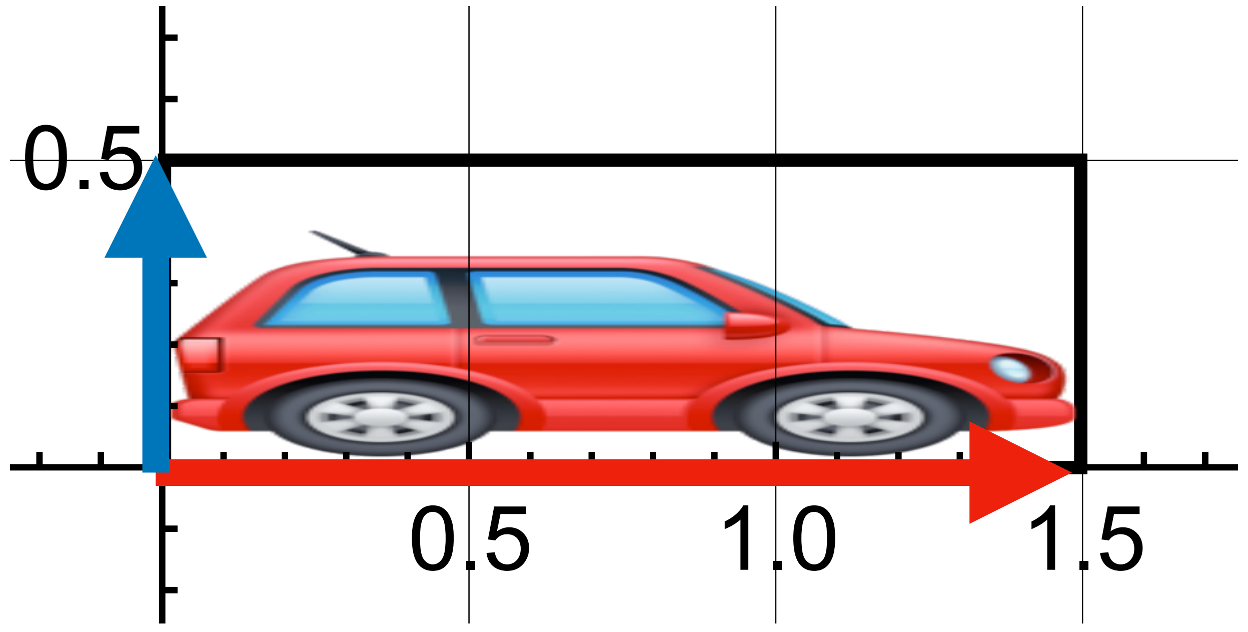
Continuing from last class…





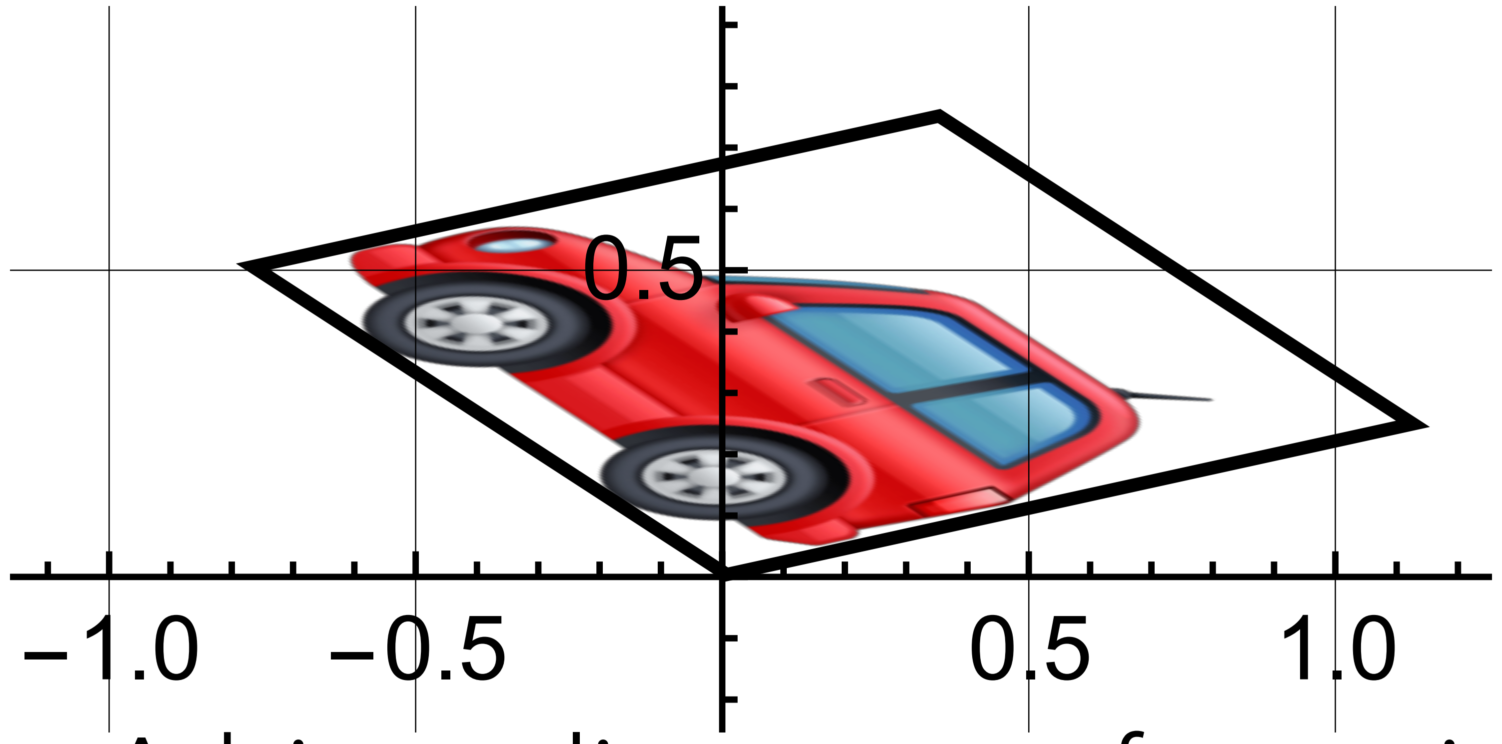
Rotation

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$



Nonuniform scaling

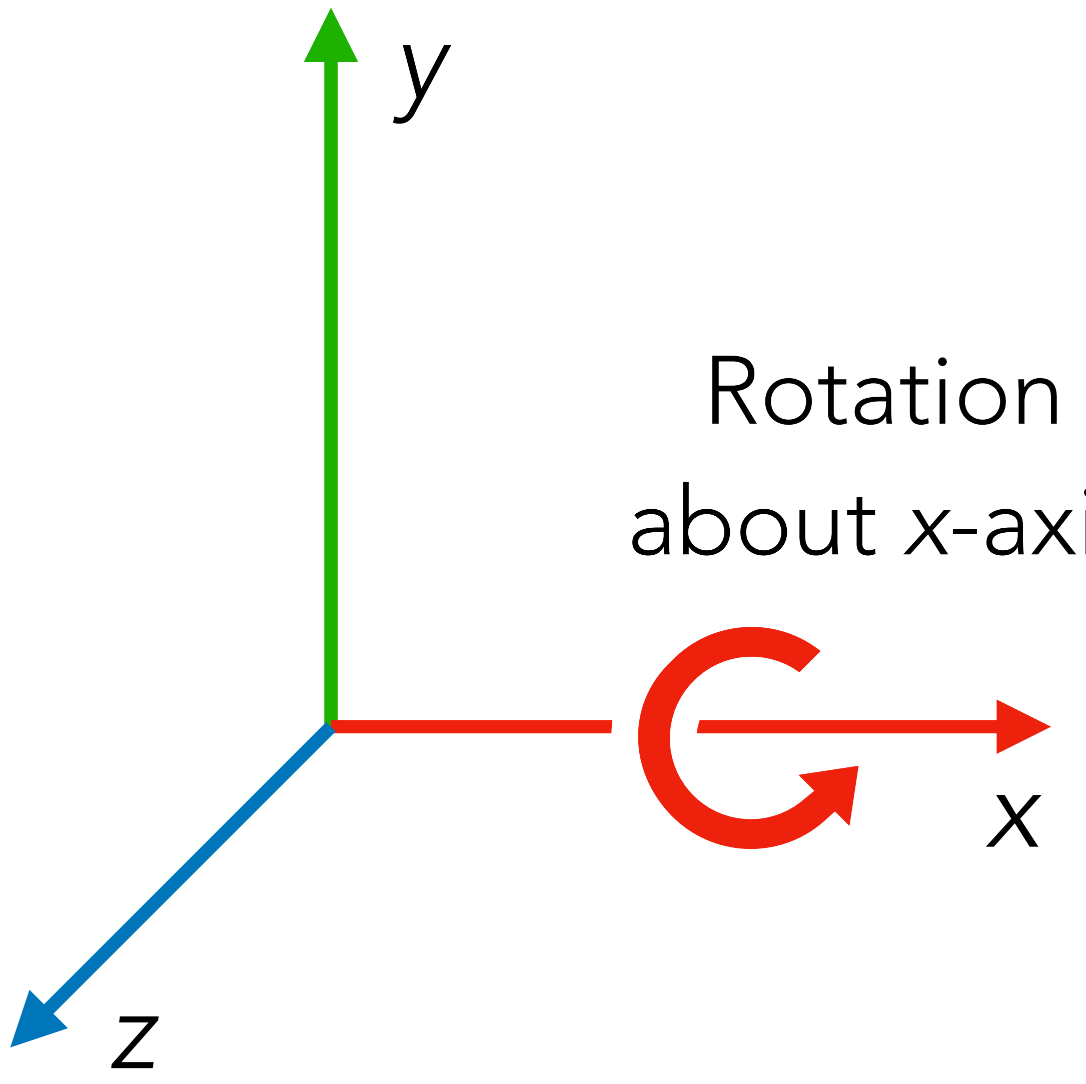
$$\begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix}$$



Arbitrary linear transformation

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

Rotations in 3D



Rotations about the coordinate axes:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

Rotation about *x*-axis
= Rotation in *yz*-plane

$$\begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

Rotation about *y*-axis
= Rotation in *zx*-plane

$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Rotation about *z*-axis
= Rotation in *xy*-plane

Are these all the possible rotations?

Rotations in 3D

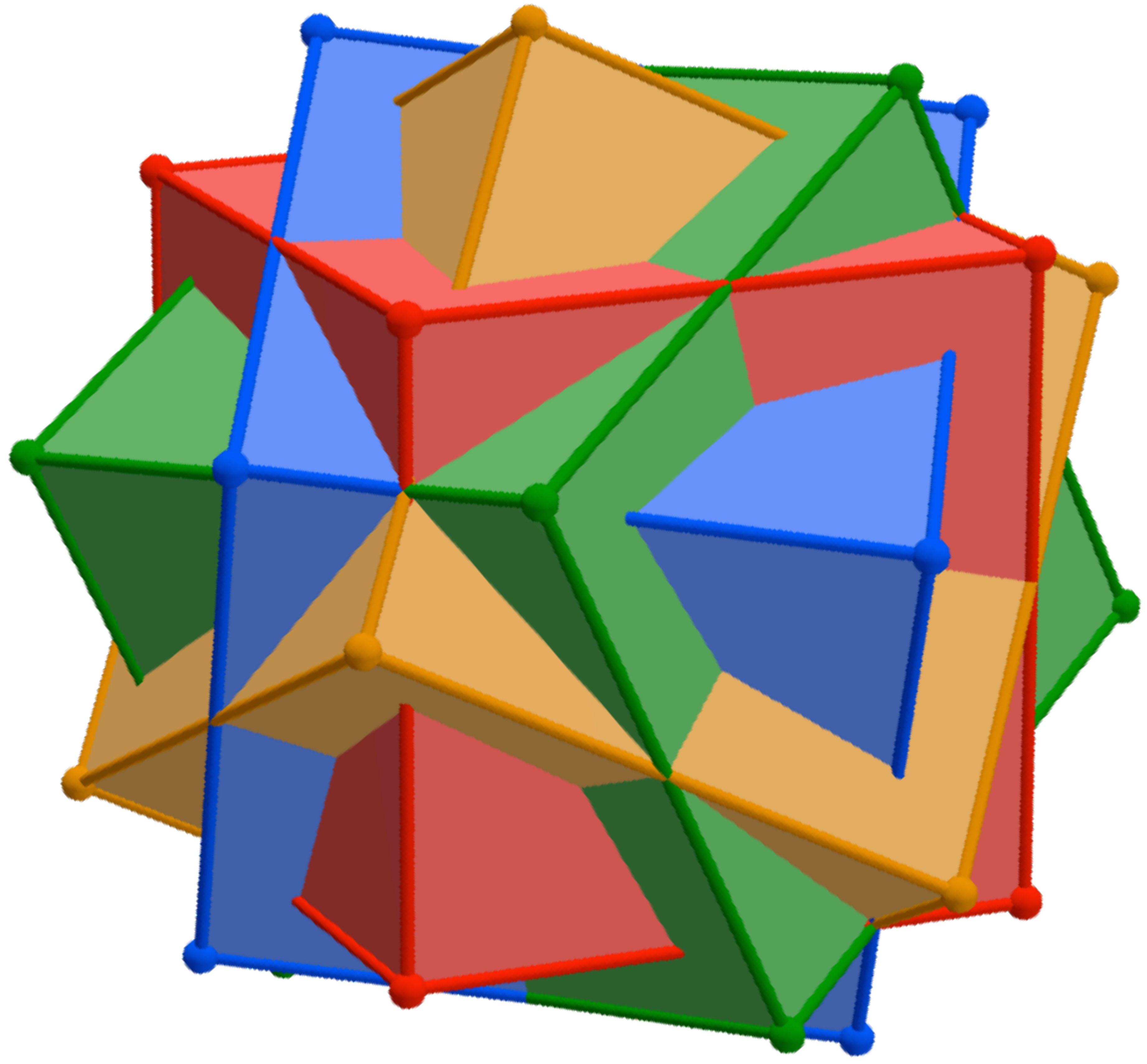
Are these all possible rotations?

Not at all!

A rotation is any transformation which:

- preserves distances and angles
- preserves orientation

Equivalently, $\mathbf{R}^T\mathbf{R} = \mathbf{I}$, and $\det \mathbf{R} = 1$



Euler angles

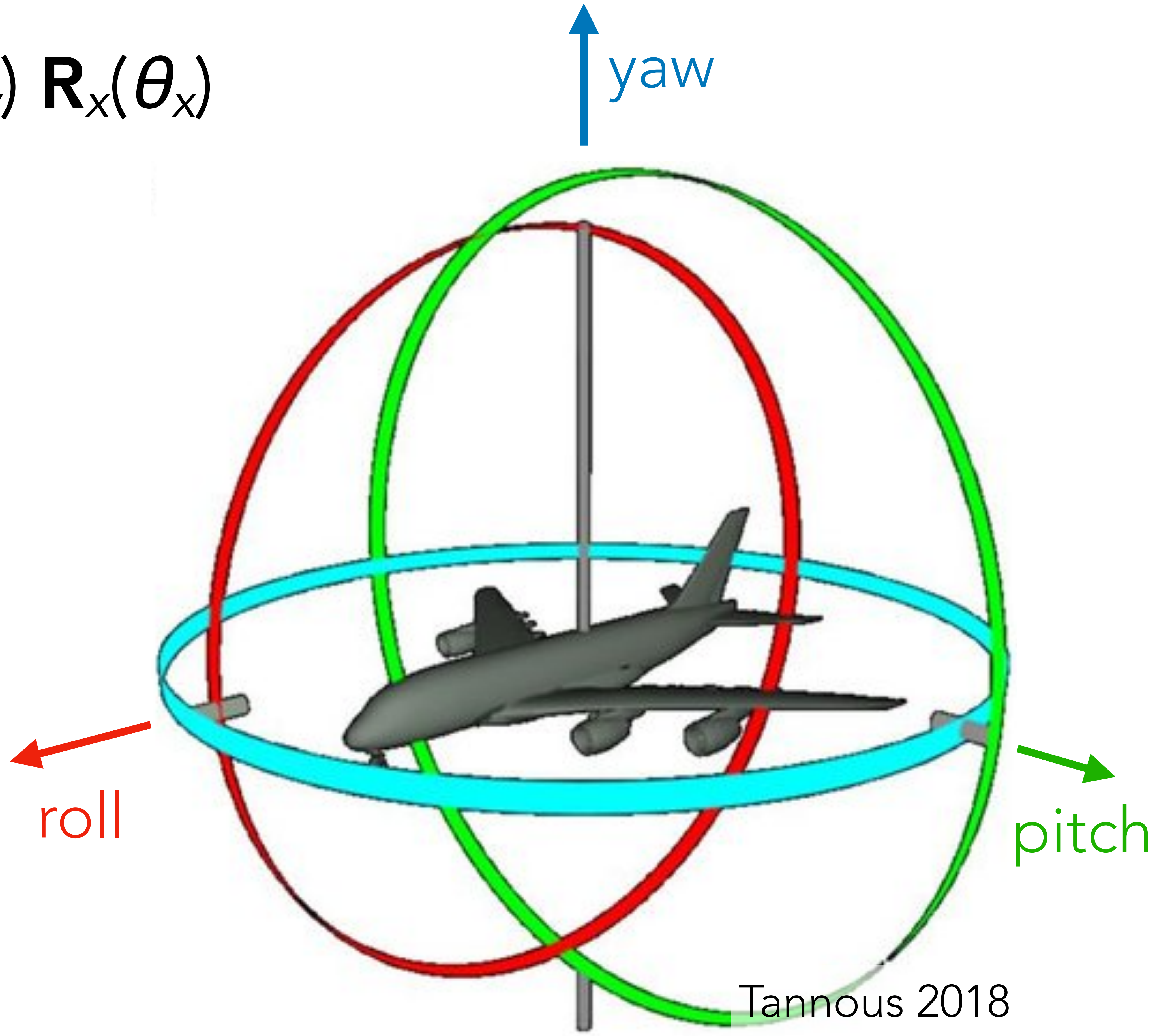
Any rotation in 3D can be expressed using 3 rotations about coordinate axes!

$$\text{e.g. } \mathbf{R} = \mathbf{R}_z(\theta_z)\, \mathbf{R}_y(\theta_y)\, \mathbf{R}_x(\theta_x)$$

θ_x, θ_y, θ_z are called Euler angles

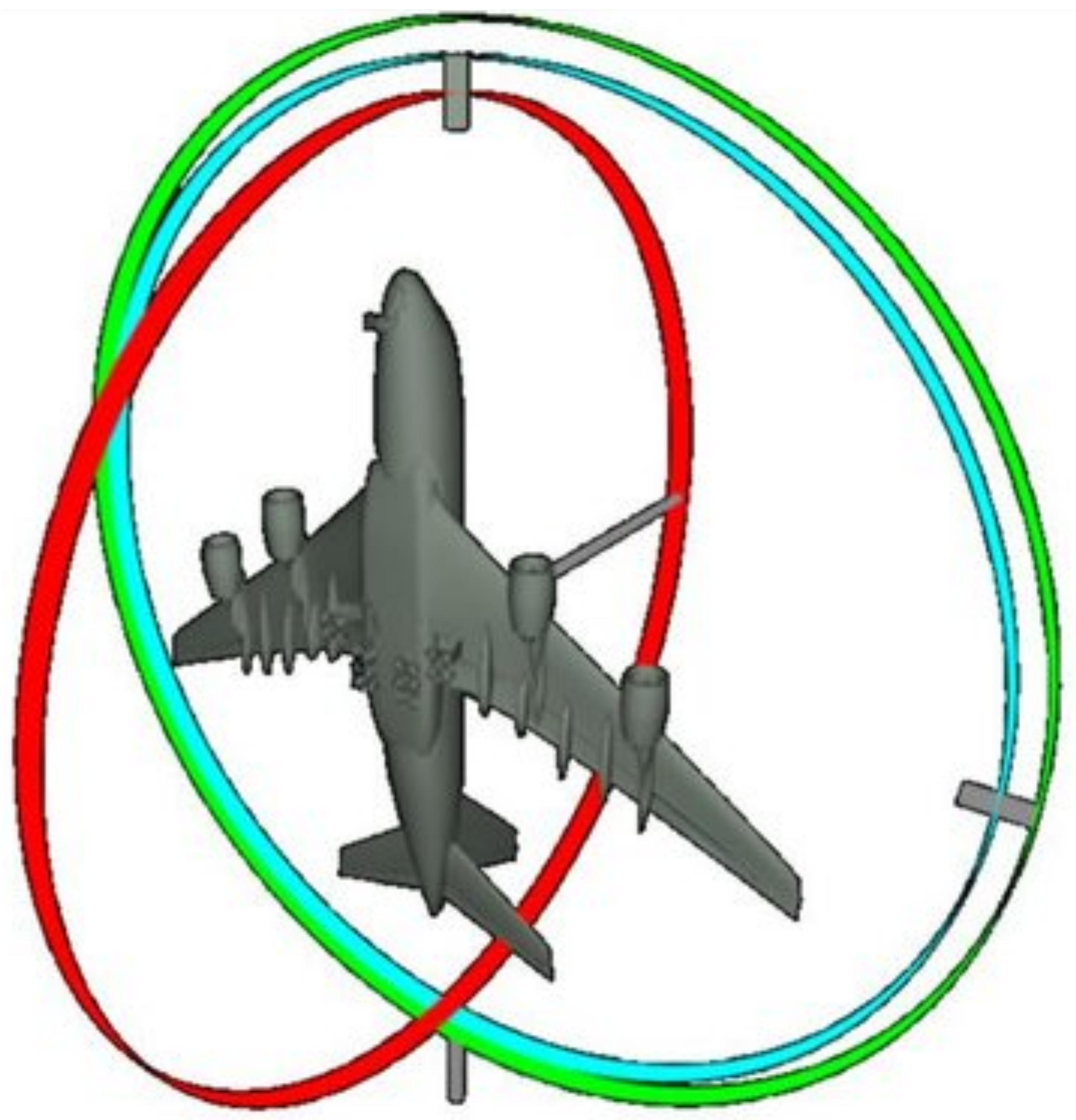
Also called "roll, pitch, yaw" in aircraft

Note: Order of rotation matters! Need to know which angle for which axis, and also which order to multiply them.



In some configurations, Euler angles lose one degree of freedom!

This is called **gimbal lock**



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Rodrigues' rotation formula

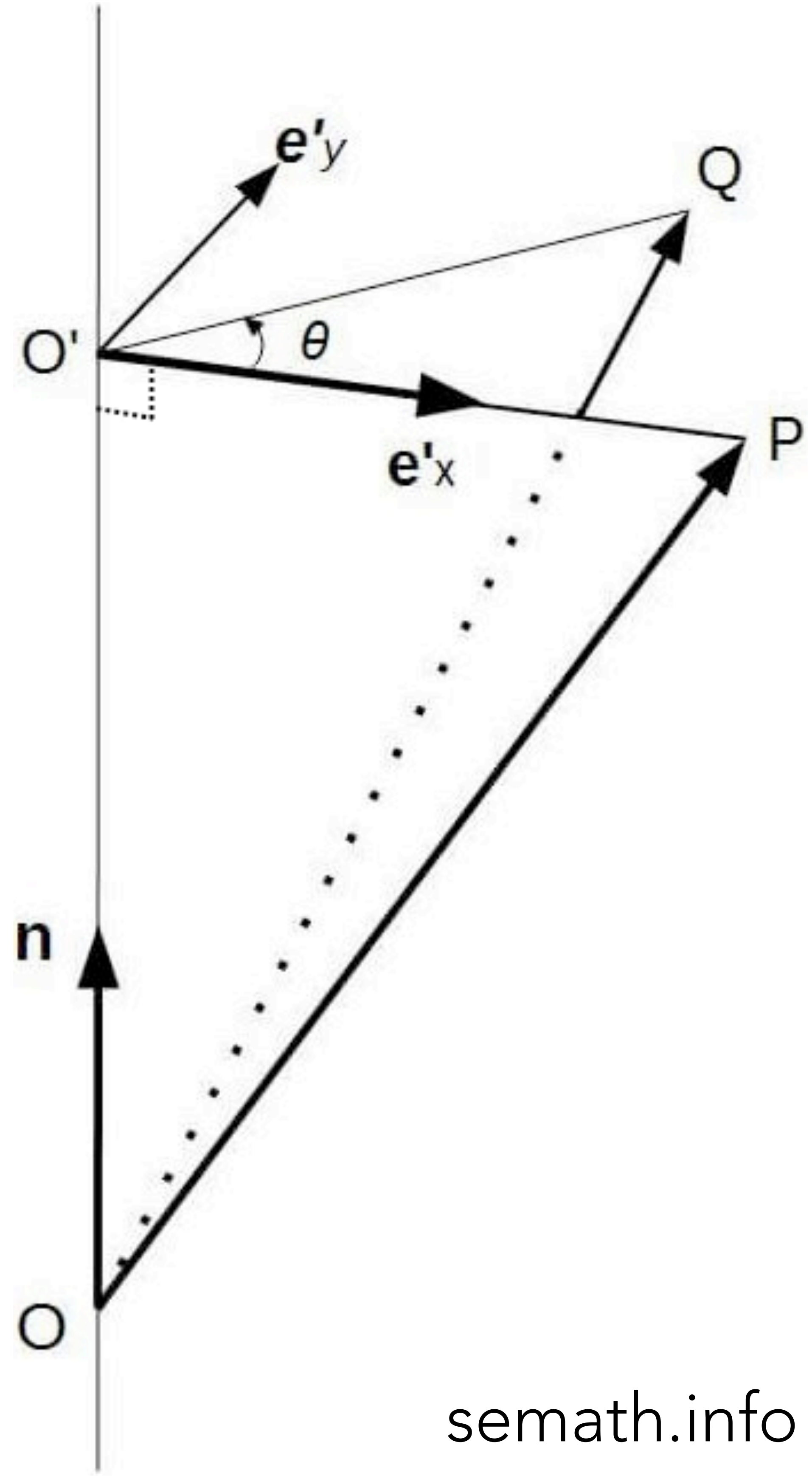
Rotation around an axis **n** by angle θ:

$$\mathbf{R} = \mathbf{I}\cos\theta + [\mathbf{n}]_\times \sin\theta + \mathbf{n}\,\mathbf{n}^\mathsf{T}(1 - \cos\theta)$$

$$\text{where } [\mathbf{n}]_\times = \begin{bmatrix} 0 & -n_z & n_y \\ n_z & 0 & -n_x \\ -n_y & n_x & 0 \end{bmatrix}$$

How? Hints:

- $[\mathbf{n}]_\times$ is the "cross-product matrix": $[\mathbf{n}]_\times \mathbf{v} = \mathbf{n} \times \mathbf{v}$
- Assume an orthogonal basis $\mathbf{n}$, $\mathbf{e}_1$, $\mathbf{e}_2$ and see what **R** does to it



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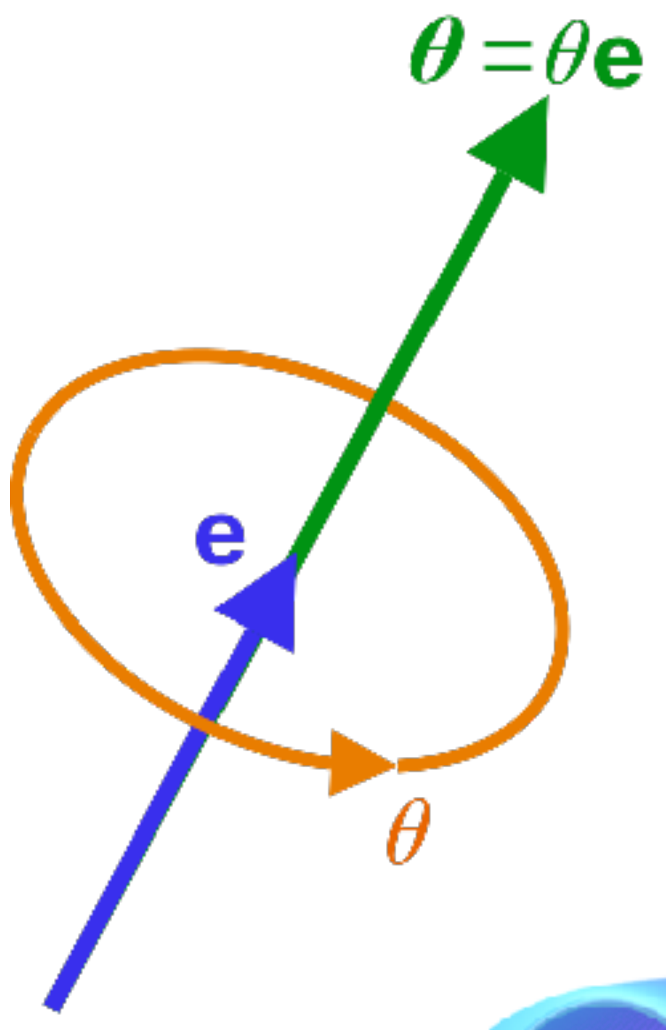
Other rotation representations we won't cover:

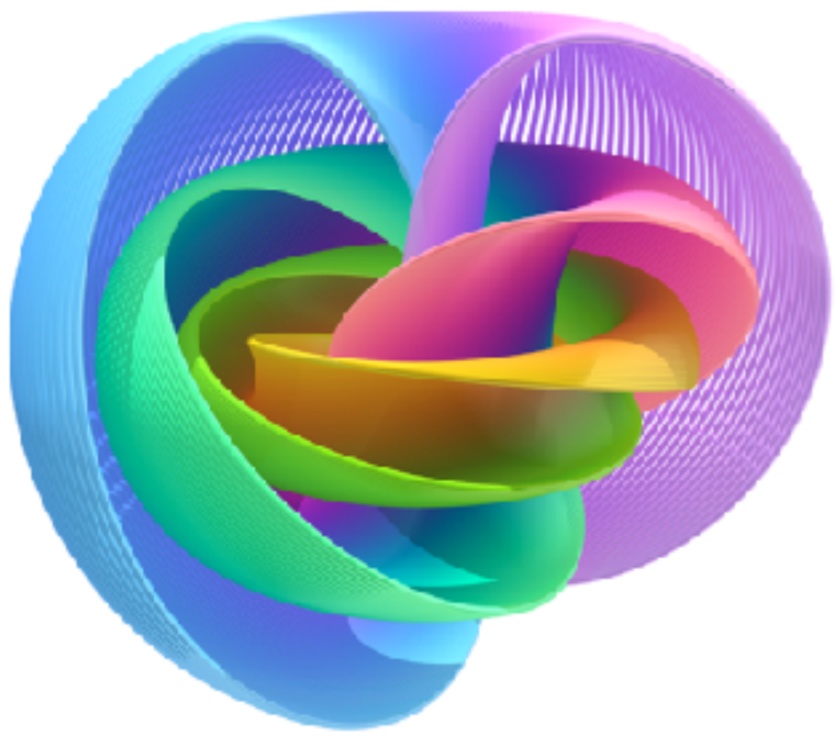
- Angle vector / exponential map

$$\boldsymbol{\theta} = \theta \mathbf{e}$$



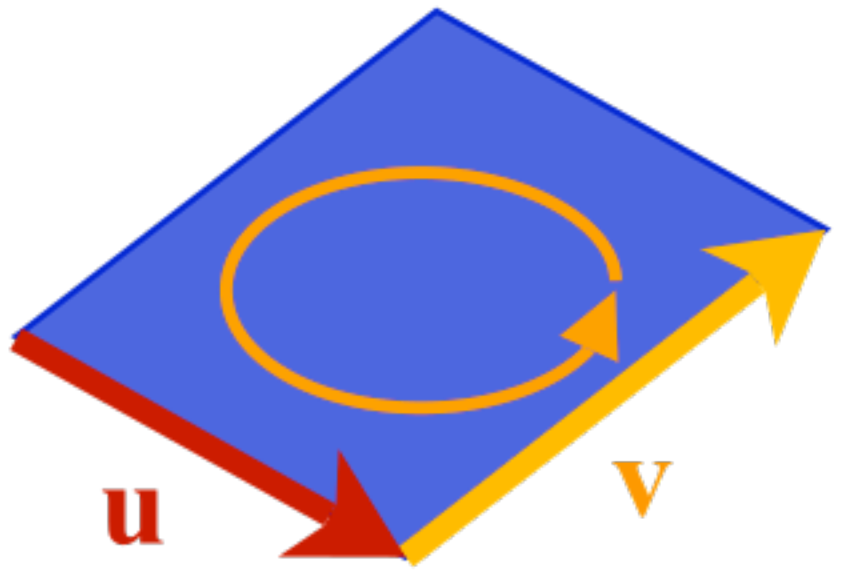
- Quaternions

$$\mathbf{q} = s + ix + jy + kz$$



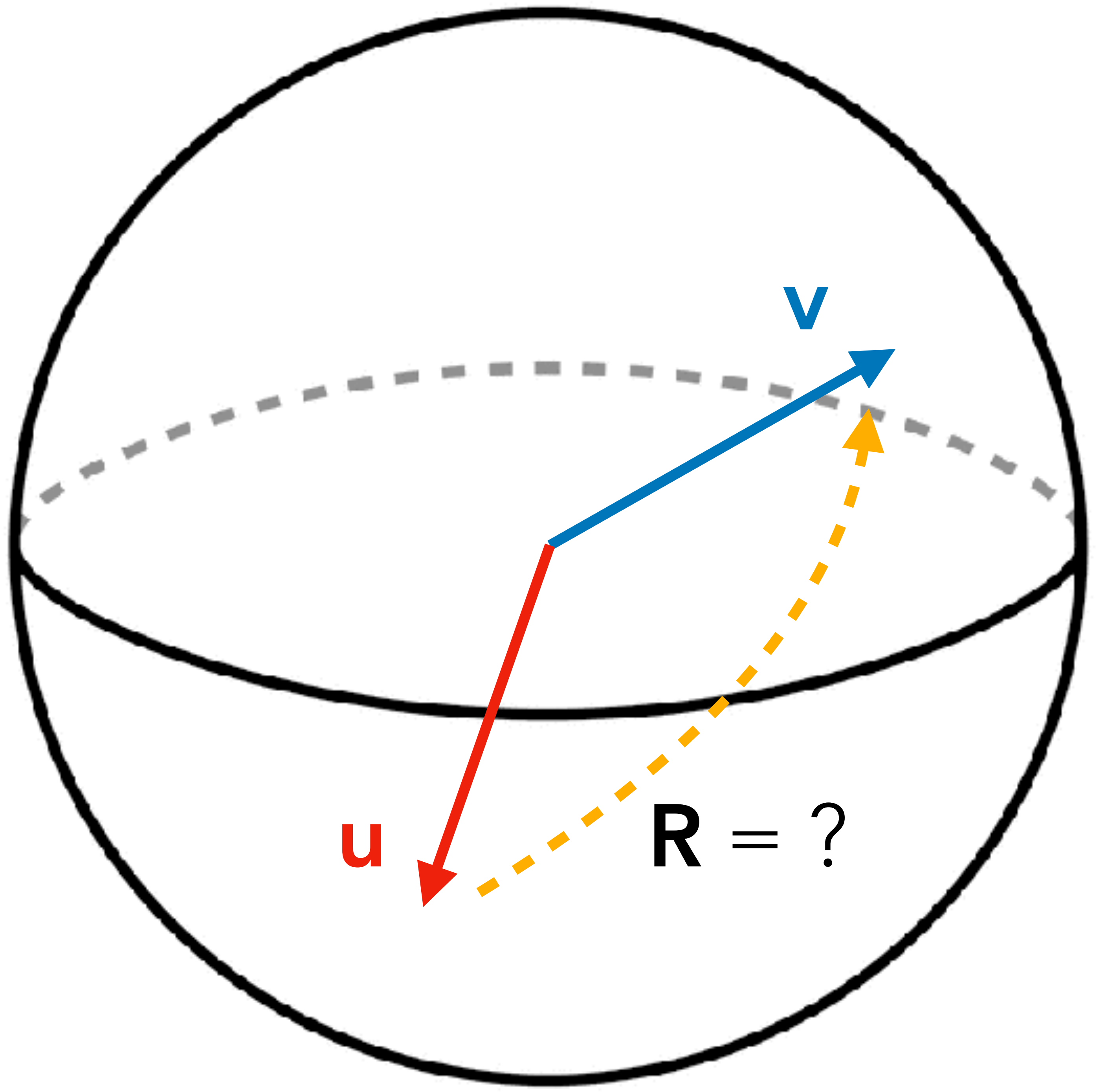
- Rotors

$$\mathbf{uv} = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \wedge \mathbf{v}$$



Homework exercise

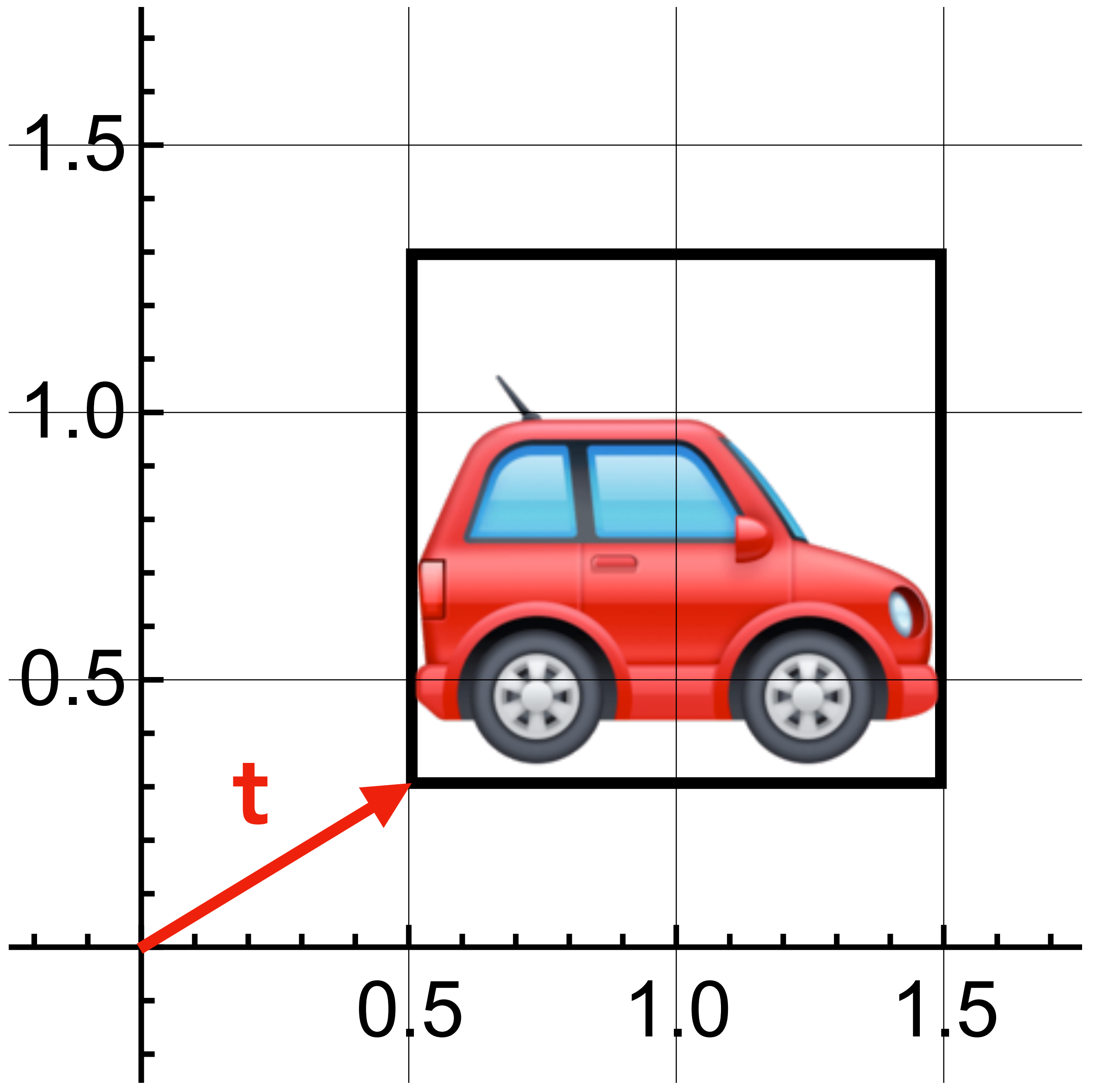
Given unit vectors **u** and **v**, find a way to construct a rotation matrix **R** which maps **u** to **v**, i.e. **Ru** = **v**. Is it unique, or are there many different such rotations?



Translations

Move all points by a constant displacement

$$T(\mathbf{p}) = \mathbf{p} + \mathbf{t}$$

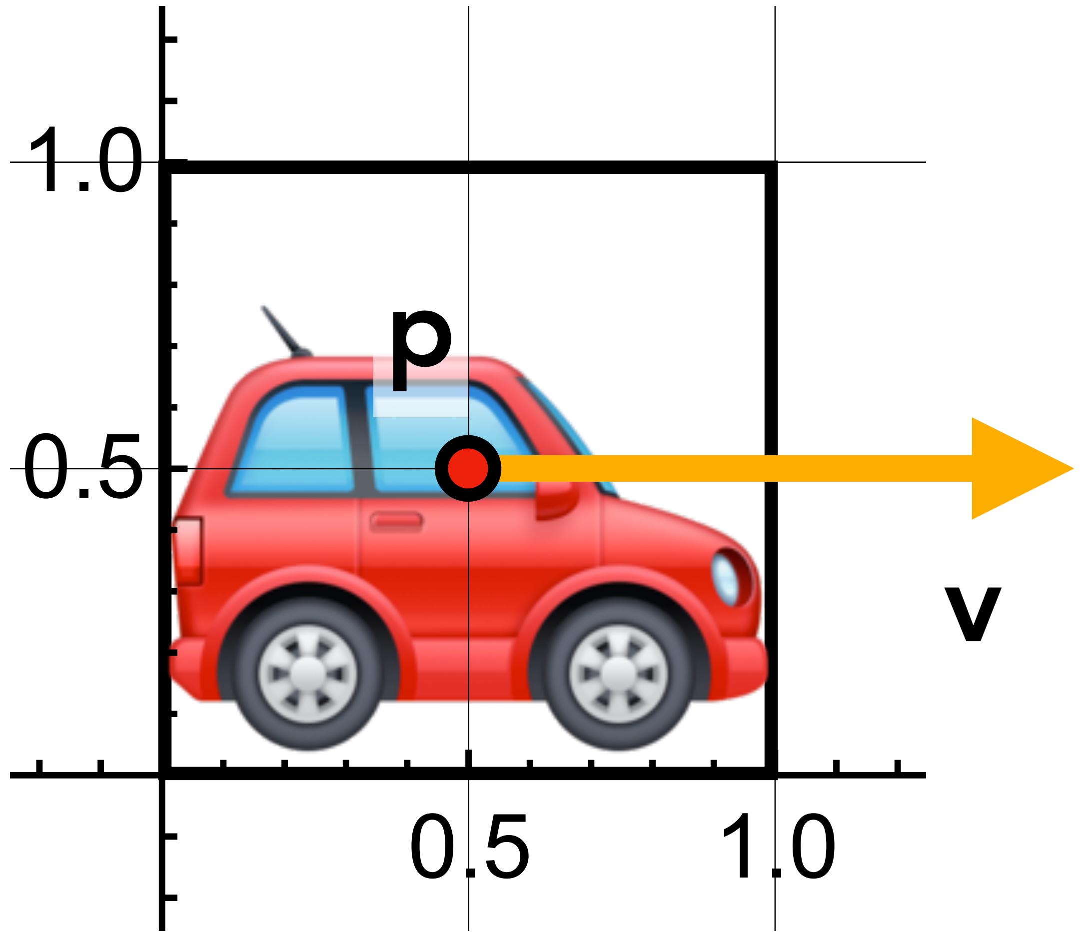


So a linear transformation followed by a translation will be of the form $T(\mathbf{p}) = \mathbf{A}\mathbf{p} + \mathbf{b}$

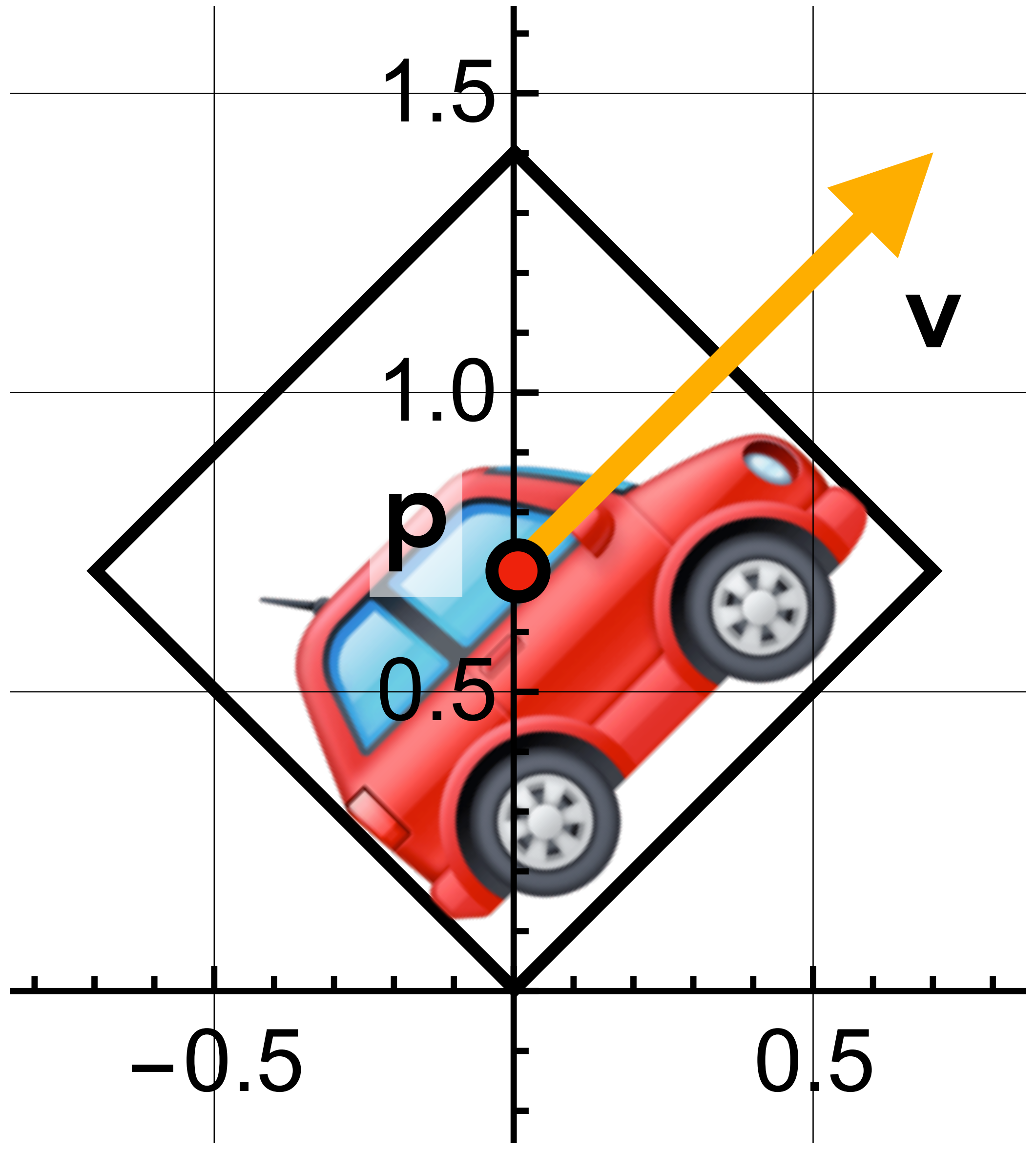
A bit tedious to compose:

$$T_2(T_1(\mathbf{p})) = \mathbf{A}_2(\mathbf{A}_1\mathbf{p} + \mathbf{b}_1) + \mathbf{b}_2 = (\mathbf{A}_2\mathbf{A}_1)\mathbf{p} + (\mathbf{A}_2\mathbf{b}_1 + \mathbf{b}_2)$$

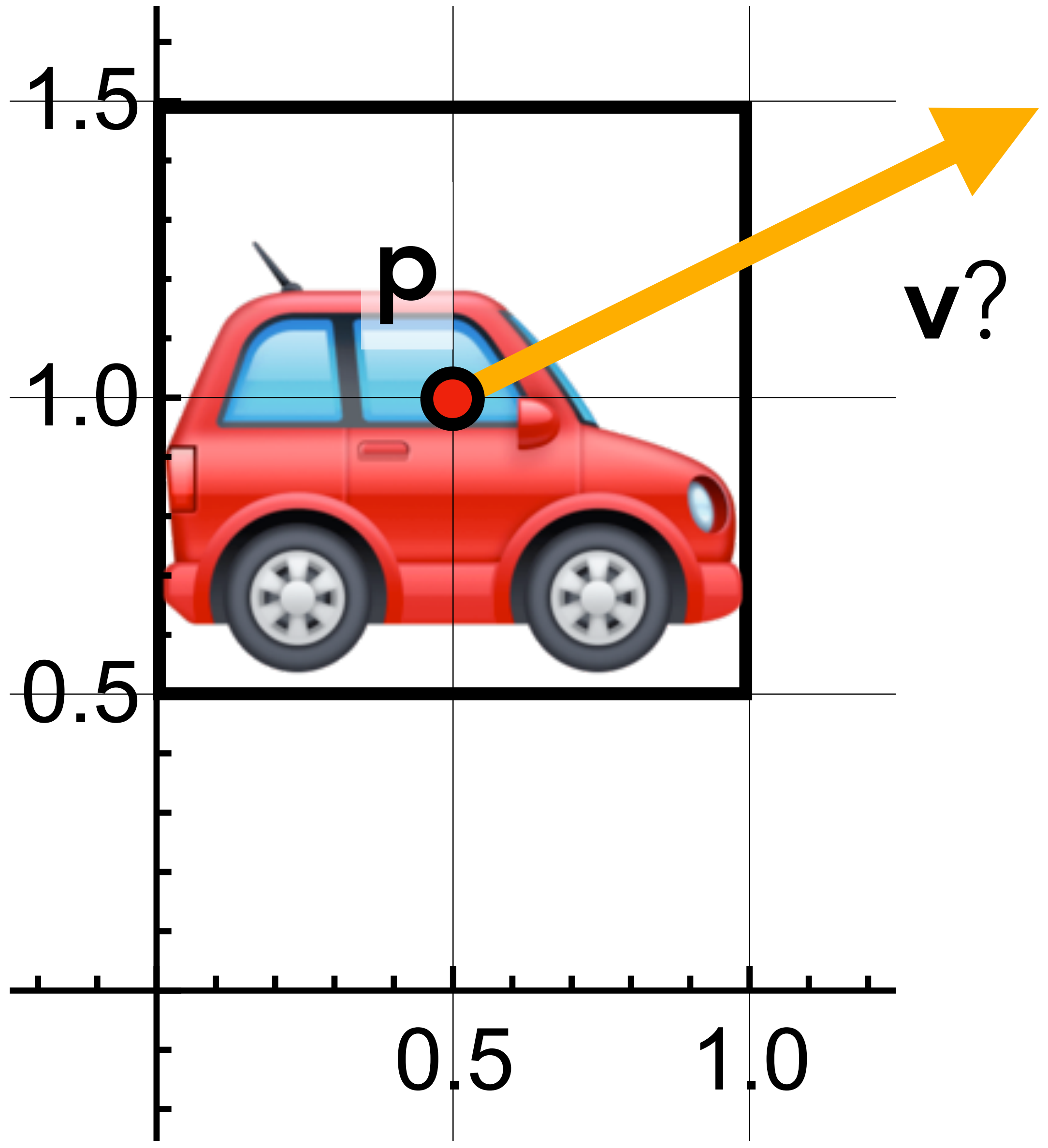
Suppose I have both points and directions/velocities/etc. to transform.



Original:
p = (0.5, 0.5)
v = (1, 0)



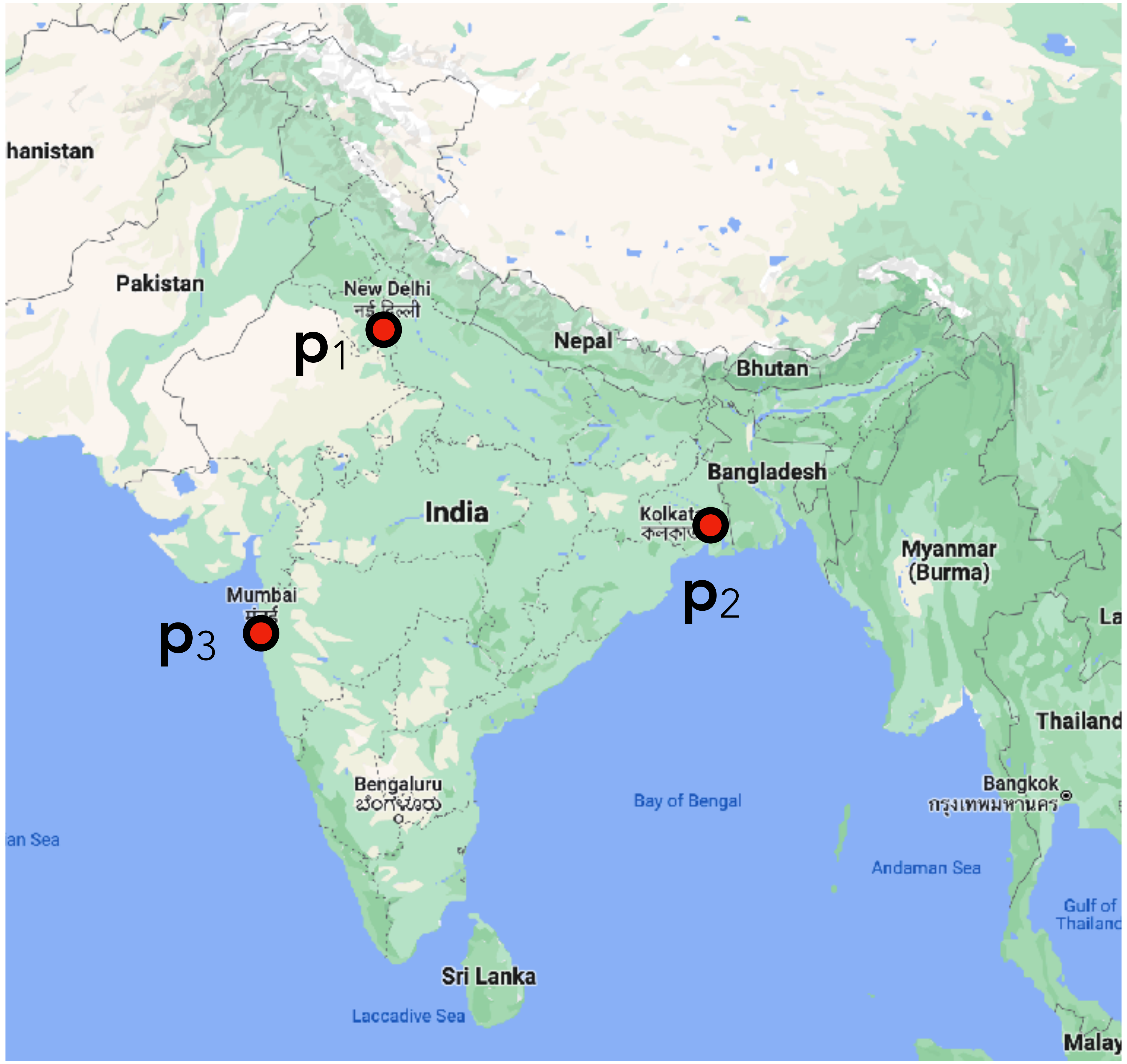
Rotation by 45°:
p = (0, 0.7)
v = (0.7, 0.7)



Translation by (0, 0.5):
p = (0.5, 1)
v = (1, 0.5)?

It seems translation should only affect some things, not others. But why?

Are points really vectors?



$\mathbf{p}_1 + \mathbf{p}_2 = ?$

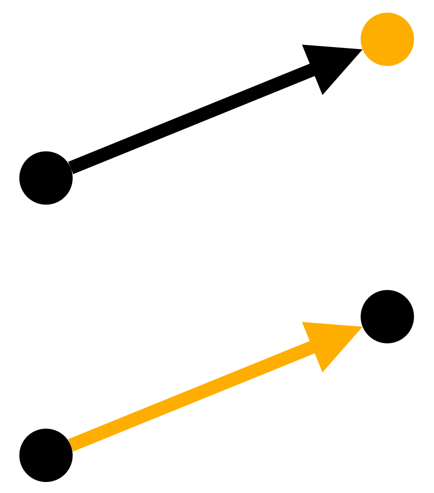
$5\mathbf{p}_3 = ?$

How about I just choose an origin and then add the displacement vectors?

Points vs. vectors

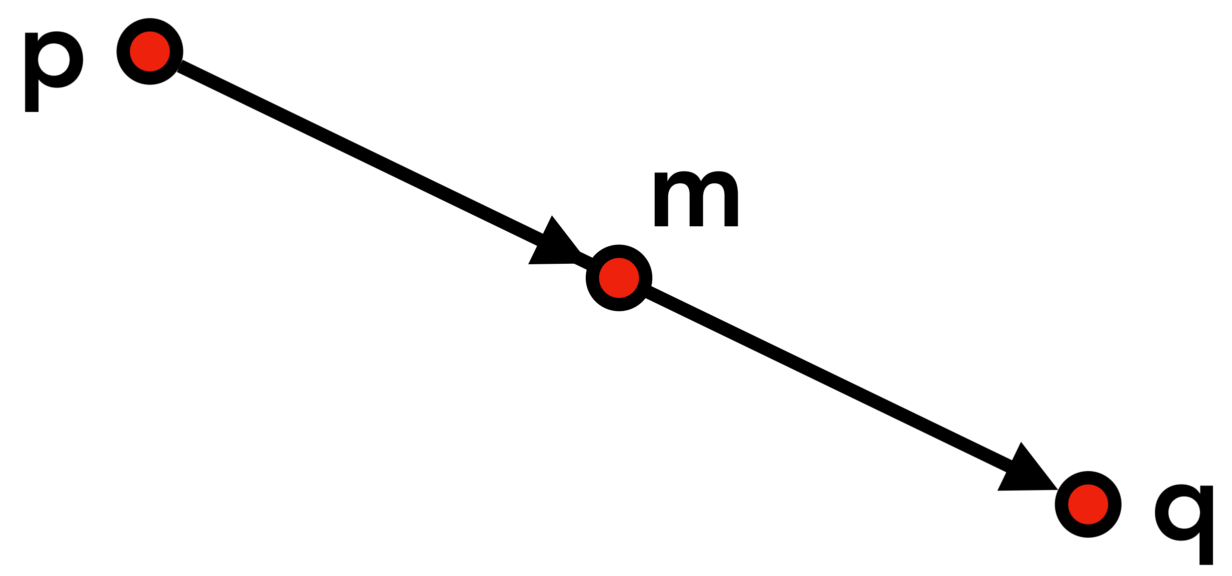
Points form an **affine space** A over the vector space V.

- Point-vector addition: $A \times V \to A$
- Point subtraction: $A \times A \to V$



with the obvious properties e.g. $(\mathbf{p} + \mathbf{u}) + \mathbf{v} = \mathbf{p} + (\mathbf{u} + \mathbf{v})$, $\mathbf{p} + (\mathbf{q} - \mathbf{p}) = \mathbf{q}$, etc.

Example: midpoint of two points $\mathbf{p}$ and $\mathbf{q}$



$$\mathbf{m} = \tfrac{1}{2}(\mathbf{p} + \mathbf{q})?$$

Not allowed! But can rewrite as

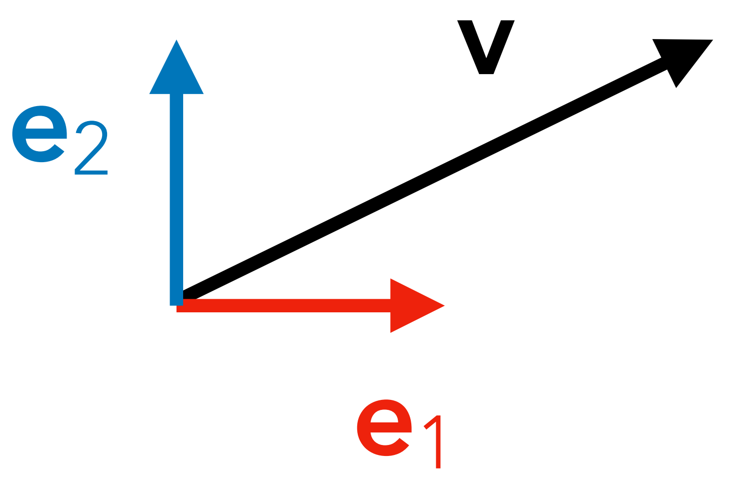
$$\mathbf{m} = \mathbf{p} + \tfrac{1}{2}(\mathbf{q} - \mathbf{p}) = \mathbf{q} + \tfrac{1}{2}(\mathbf{p} - \mathbf{q})$$

In fact it's valid to take any **affine combination** $w_1\mathbf{p}_1 + w_2\mathbf{p}_2 + \cdots + w_n\mathbf{p}_n$ as long as $w_1 + w_2 + \cdots + w_n = 1$.

(**Exercise:** Check that this can be done using only the allowed operations)

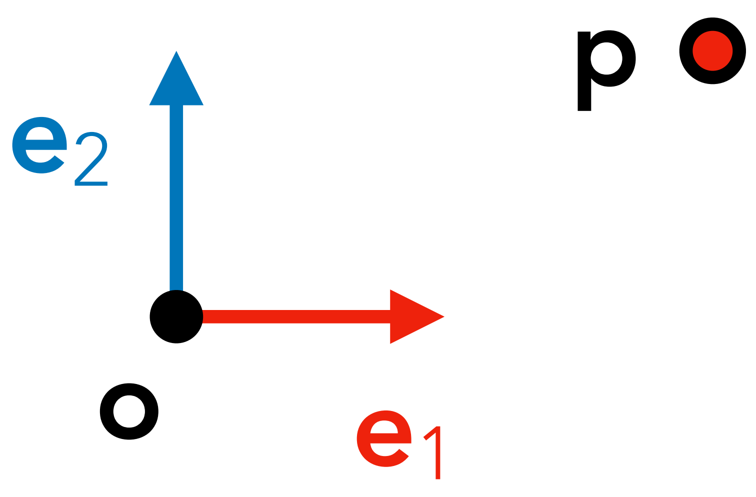
Coordinate frames

To specify a vector numerically, we need a basis



$$\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + \cdots \quad \Leftrightarrow \quad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \end{bmatrix} \text{ in the basis}$$

To specify a point numerically, we need a **coordinate frame**: origin and basis



$$\mathbf{p} = p_1\mathbf{e}_1 + p_2\mathbf{e}_2 + \cdots + \mathbf{o} \quad \text{so maybe} \quad \mathbf{p} = \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ 1 \end{bmatrix}?$$

Write a point as an $(n+1)$-tuple $\mathbf{p} = \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ 1 \end{bmatrix}$ to mean $\mathbf{p} = p_1\mathbf{e}_1 + p_2\mathbf{e}_2 + \cdots + \mathbf{o}$.

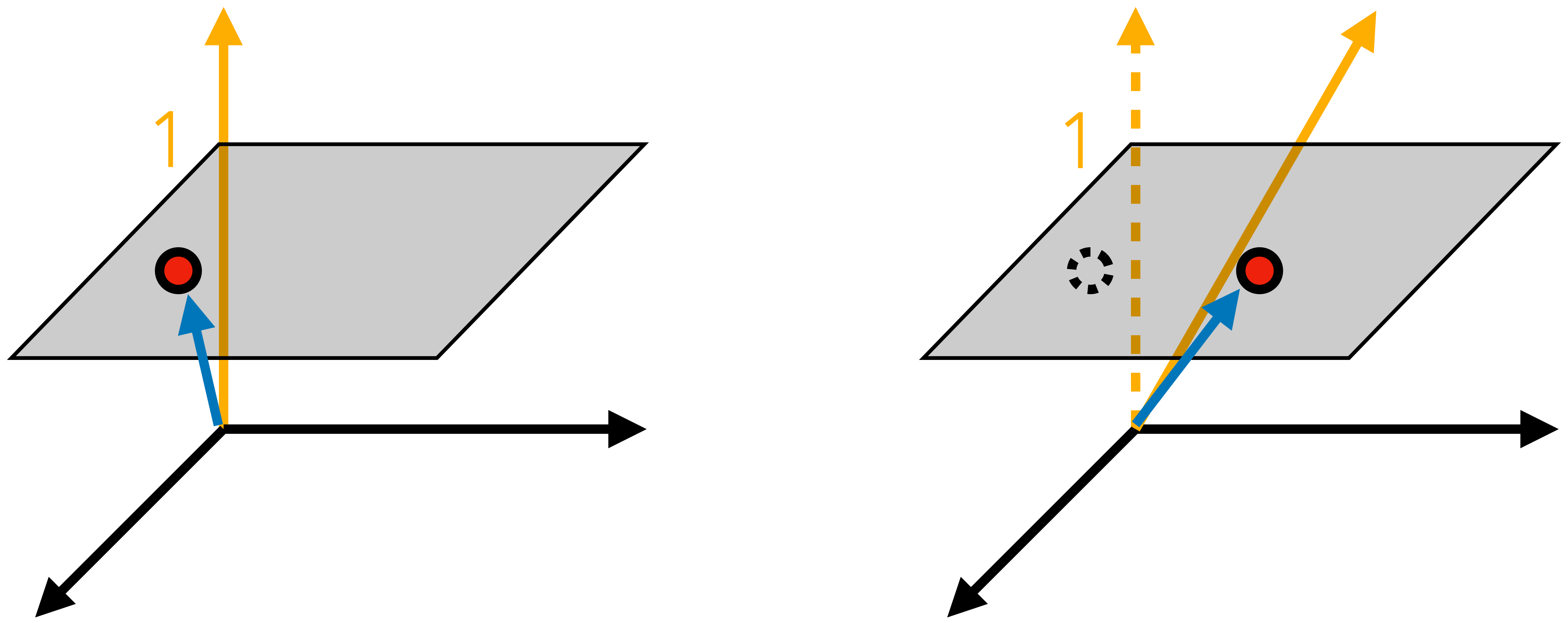
Linear transformations are now $\begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix}$, mapping $\mathbf{e}_i \rightarrow \mathbf{A}\mathbf{e}_i$ and $\mathbf{o} \rightarrow \mathbf{o}$

e.g. $\begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ 1 \end{bmatrix} = \begin{bmatrix} s_x p_x \\ s_y p_y \\ 1 \end{bmatrix}$

Translation by a vector **t**: $\begin{bmatrix} \mathbf{I} & \mathbf{t} \\ \mathbf{0} & 1 \end{bmatrix}$, mapping $\mathbf{e}_i \rightarrow \mathbf{e}_i$ but $\mathbf{o} \rightarrow \mathbf{o} + \mathbf{t}$

$$\text{e.g.} \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ 1 \end{bmatrix} = \begin{bmatrix} p_x + t_x \\ p_y + t_y \\ 1 \end{bmatrix}$$

If we plot the extra coordinate as well: it's a shear transformation in (n+1) dimensions!



What about vectors?

$$\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + \cdots + 0\mathbf{o} \quad \Leftrightarrow \quad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ 0 \end{bmatrix}$$

Apply a translation:

$$\begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_x \\ v_y \\ 0 \end{bmatrix} = \begin{bmatrix} v_x \\ v_y \\ 0 \end{bmatrix}$$

Homogeneous coordinates

Add an extra coordinate *w* at the end.

- Points: $w = 1$
- Vectors: $w = 0$

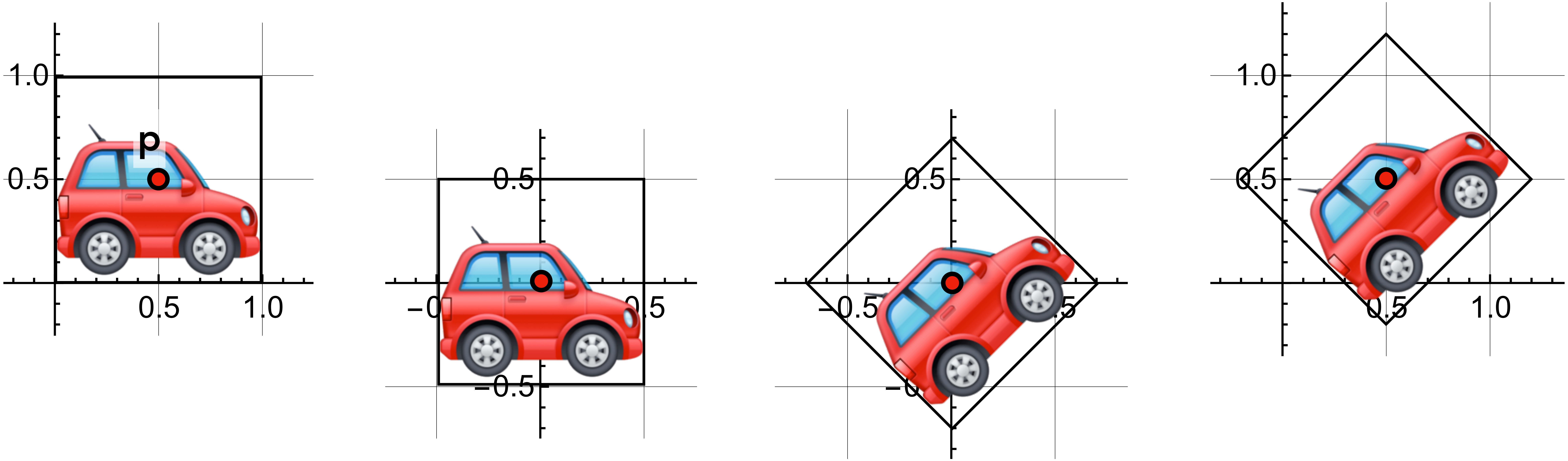
Transformations become $(n+1)\times(n+1)$ matrices

- Linear transformations: $\begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix}$
- Translations: $\begin{bmatrix} \mathbf{I} & \mathbf{t} \\ \mathbf{0} & 1 \end{bmatrix}$

General **affine transformation**: $\begin{bmatrix} \mathbf{A} & \mathbf{t} \\ \mathbf{0} & 1 \end{bmatrix}$

- Corresponds to linearly transforming basis vectors $\mathbf{e}_i \to \mathbf{A}\mathbf{e}_i$ and translating origin $\mathbf{o} \to \mathbf{o} + \mathbf{t}$
- Composition: just matrix multiplication again.

Example: Rotate by given angle θ about given point $\mathbf{p}$ (instead of about origin)



Translate by $-\mathbf{p}$

Rotate by θ about origin

Translate by $\mathbf{p}$

$$\mathbf{M} = \mathbf{T}(\mathbf{p})\ \mathbf{R}(\theta)\ \mathbf{T}(-\mathbf{p})$$

Given coordinates of **p** in frame 1, what are its coordinates in frame 2?

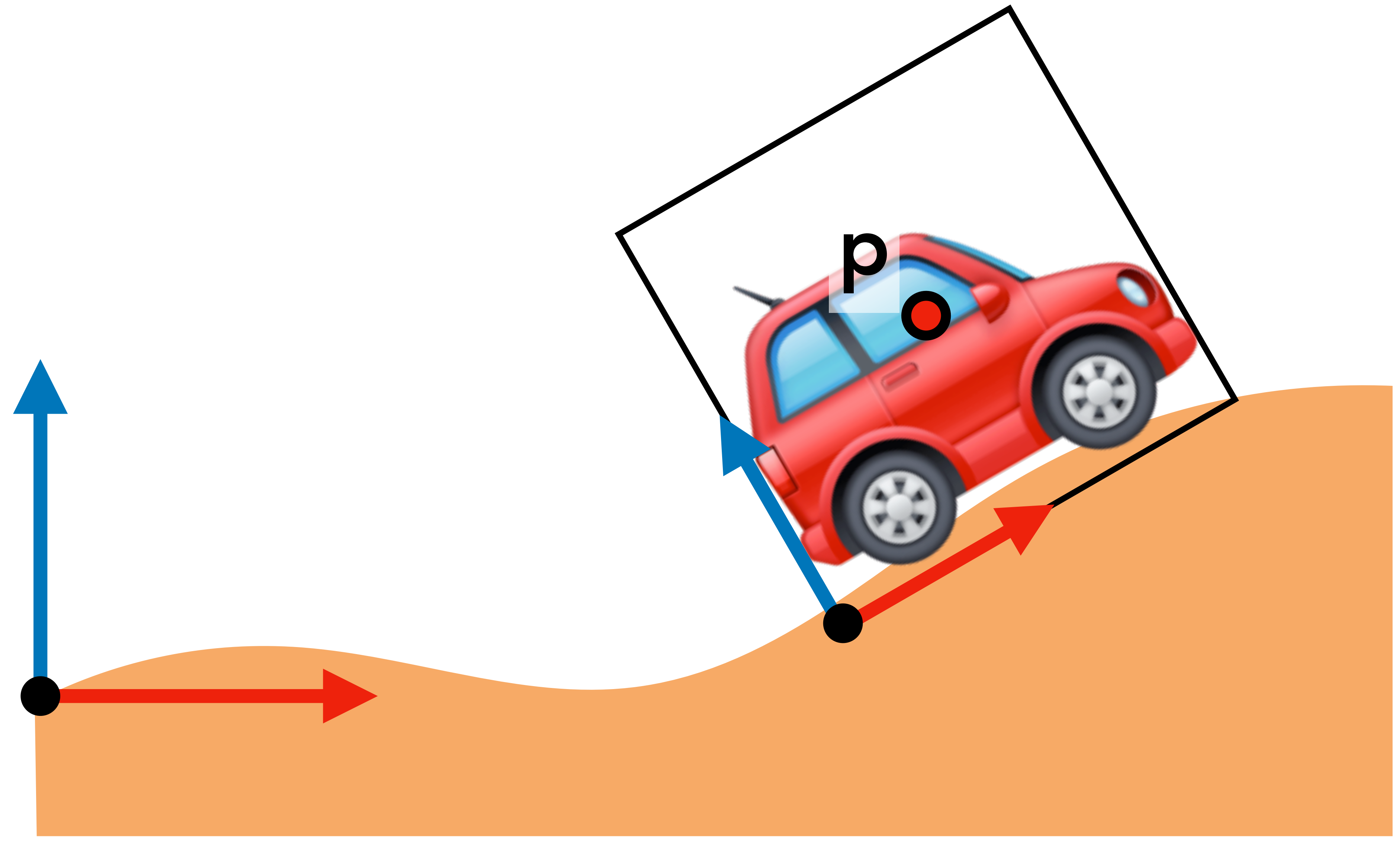
$$\mathbf{p} = p_1\mathbf{e}_1 + p_2\mathbf{e}_2 + \cdots + \mathbf{o}$$

Write coords of $\mathbf{e}_1$, $\mathbf{e}_2$, … and $\mathbf{o}$ in frame 2:

$$\mathbf{e}_i = \begin{bmatrix} \bullet \\ \bullet \\ \vdots \\ 0 \end{bmatrix}, \quad \mathbf{o} = \begin{bmatrix} \bullet \\ \bullet \\ \vdots \\ 1 \end{bmatrix}$$

Then $\mathbf{p} = \begin{bmatrix} \bullet & \bullet & \cdots & \bullet \\ \bullet & \bullet & \cdots & \bullet \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ 1 \end{bmatrix}$

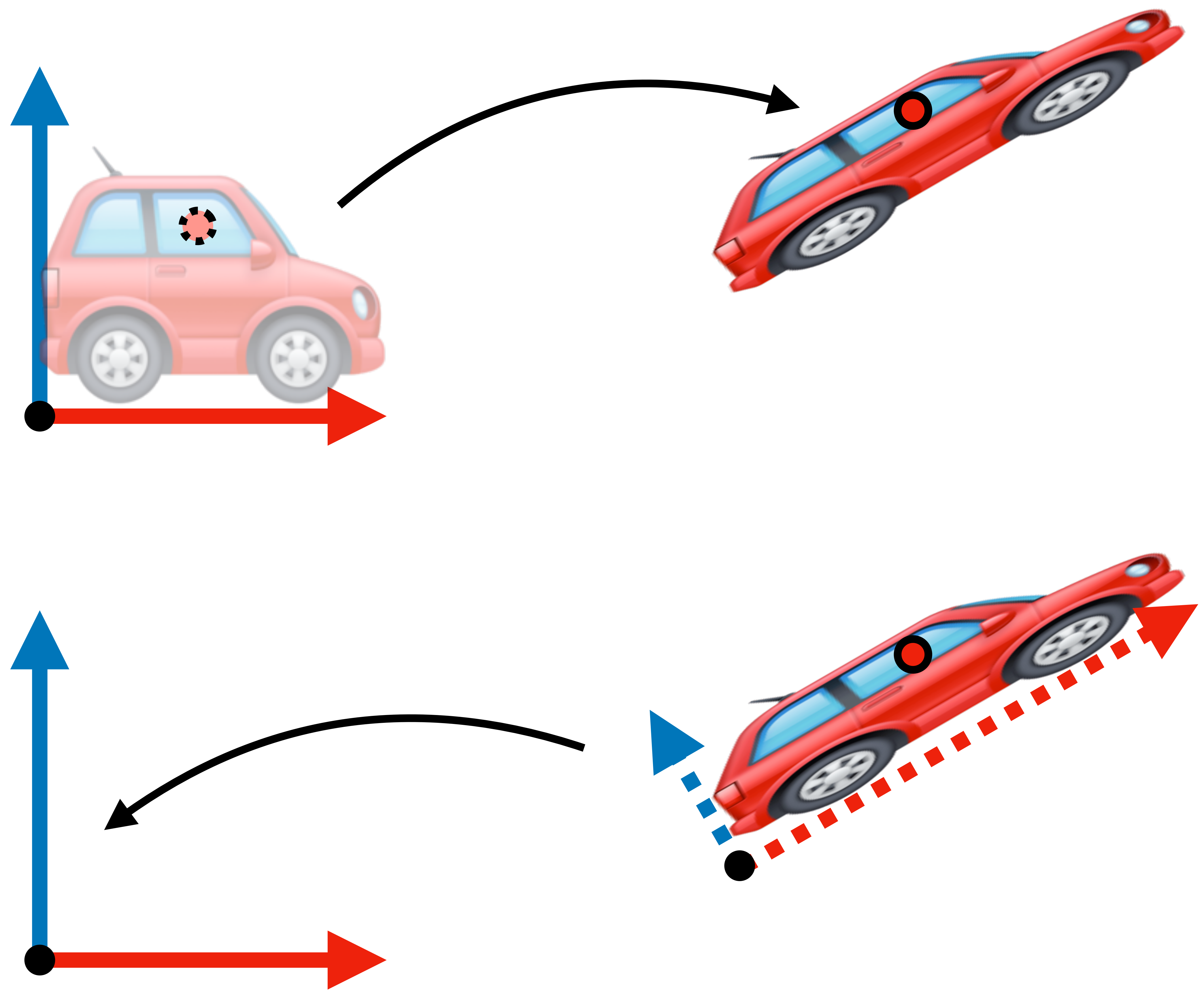
$\mathbf{e}_1$ $\mathbf{e}_2$ $\mathbf{o}$



Change of coordinates looks exactly like a transformation matrix!

Active transformation: Moves points to new locations in the same frame

Change of coordinates (passive transformation): Gives coordinates of the same point in a different frame



Matrices are the same but the meaning is different! You have to keep track.

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e.g. world_driver = world_from_car * car_driver
          Vec3           Mat3x3            Vec3
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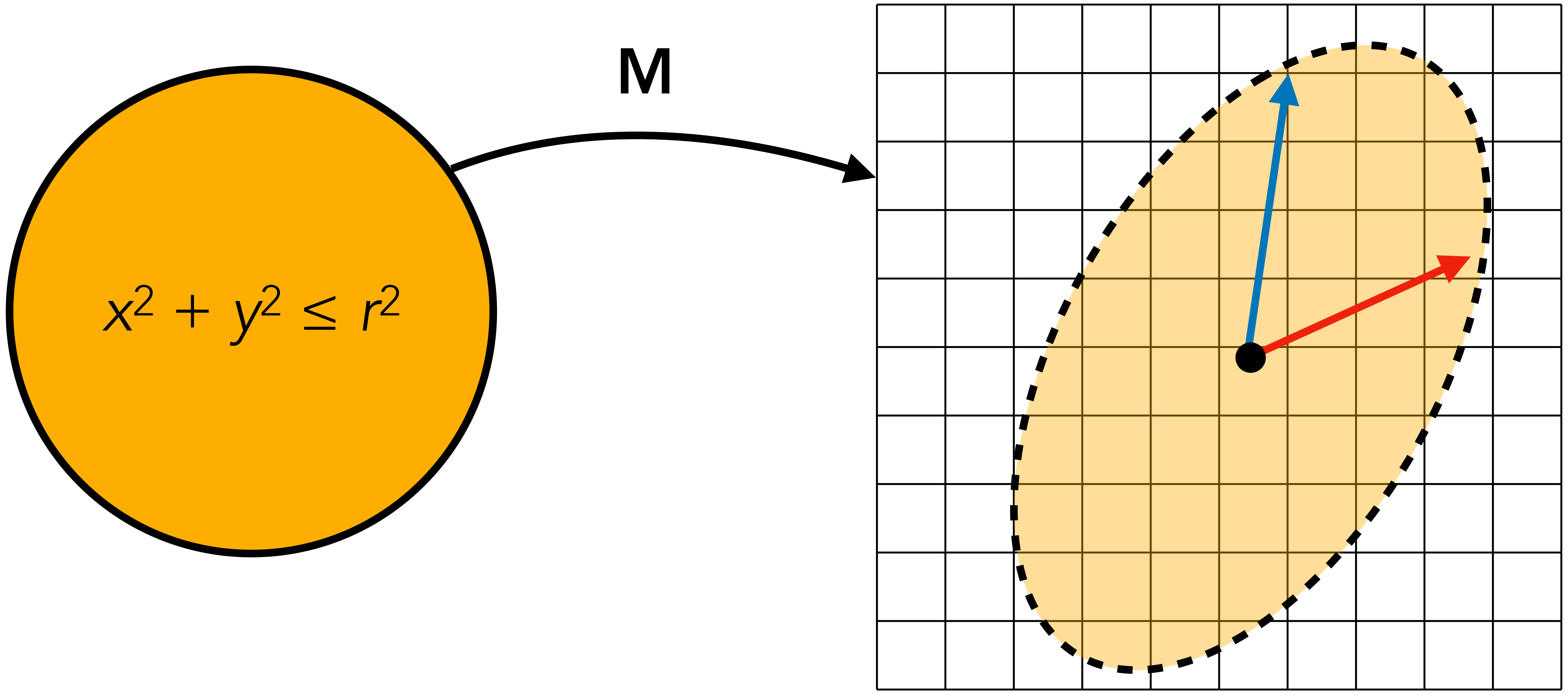
Puzzle:

To draw a transformed polygon, I can just transform the vertices.



If something is instead specified by a function $f(x,y)$ (e.g. a circle or an image), can I still draw its transformed version?