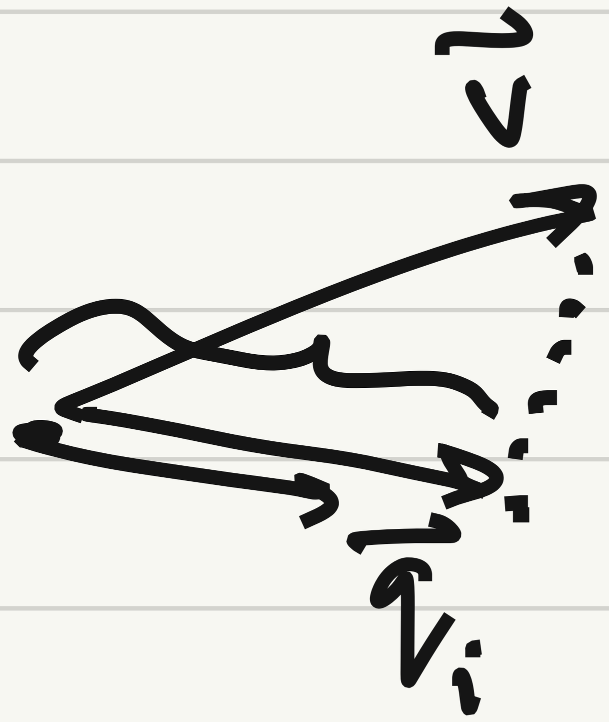
Norms and SVD

$$\vec{v} = \sum \overbrace{(\vec{q}_i^{\,*} \vec{v})}^{\text{scalar mult.}} \vec{q}_i$$

$$= \sum \vec{q}_i (\vec{q}_i^{\,*} \vec{v})$$

$$= \sum \underbrace{(\vec{q}_i \vec{q}_i^{\,*})}_{\text{matrix}} \vec{v}$$



Correction from prev. lecture

A matrix Q: square, orthonormal cols is $\boxed{\text{orthogonal matrix}}$ if Q is real

If complex, $\boxed{\text{unitary matrix}}$

$$Q^*Q = I = QQ^*$$

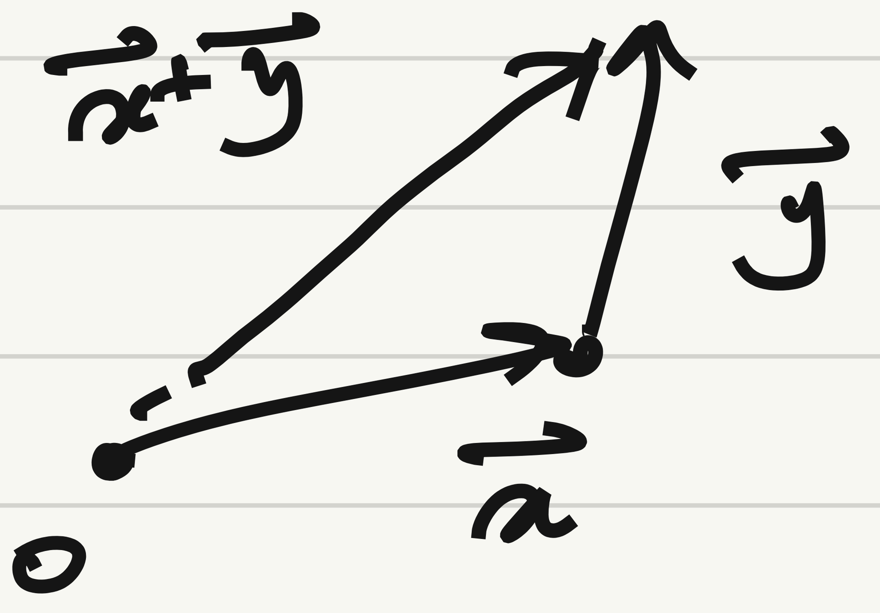
A norm is function: $\mathbb{C}^m \to \mathbb{R}$

which gives "size" of a vector (ie. how far it is from $\vec{0}$)

$$\|\vec{x}\|_2 = \sqrt{\sum_i |x_i|^2} \qquad \|\vec{x}\|_\infty = \max_i |x_i|$$

Def. A function $\|\cdot\| : V \to \mathbb{R}$ is called a $\boxed{\text{norm}}$ if:

1. $\|\vec{x}\| \geq 0$, and $\|\vec{x}\| = 0 \iff \vec{x} = \vec{0}$
2. $\| s\vec{x} \| = |s| \, \|\vec{x}\|$
3. $\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$, triangle inequality



$\boxed{\text{p-norm}}$ or $\boxed{L^p\text{-norm}}$

$$\|\vec{x}\|_2 = \left(\sum |x_i|^2\right)^{1/2}$$

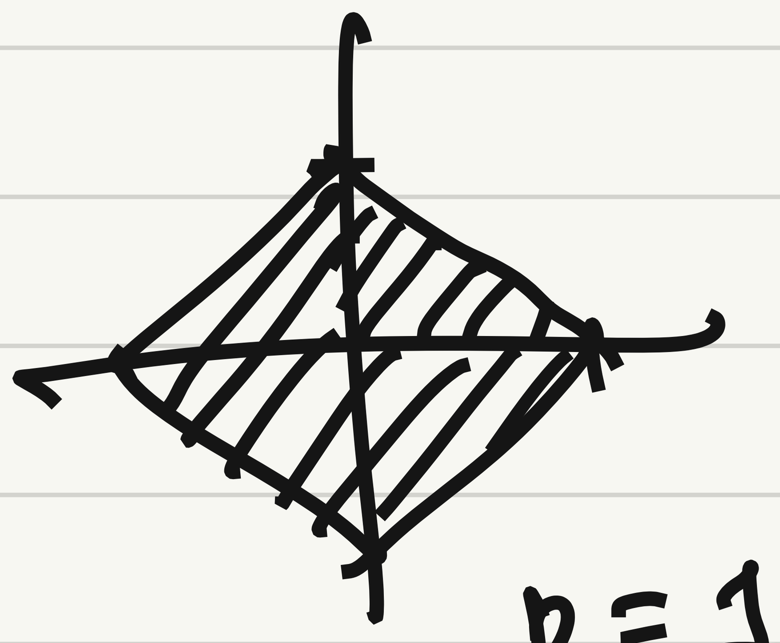
for any $p \geq 1$, $\|\vec{x}\|_p = \underbrace{\left(\sum |x_i|^p\right)^{1/p}}$

unit ball: $\{\vec{x} : \|\vec{x}\| \leq 1\}$

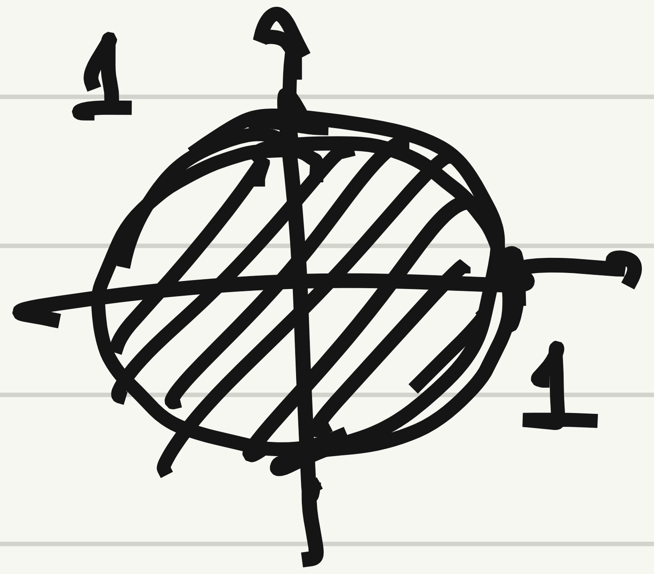
eg. $\|\vec{x}\|_1 = \sum |x_i|$

$\|\vec{x}\|_2 = \left(\sum |x_i|^2\right)^{1/2}$

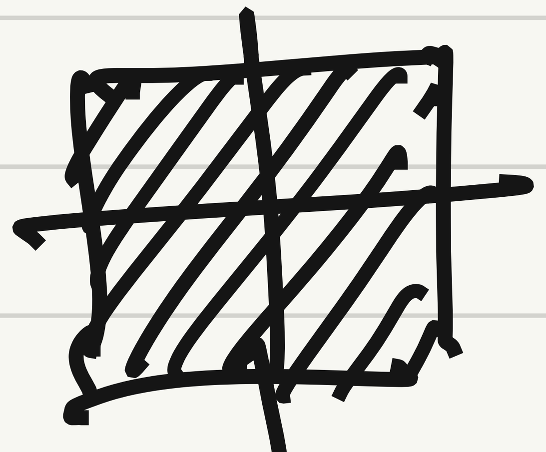
$\|\vec{x}\|_\infty = \max_i |x_i| = \lim_{p \to \infty} \|\vec{x}\|_p$



$p = 1$



$p = 2$



$p = \infty$

weighted norm : $W = \begin{bmatrix} w_1 & & & \\ & w_2 & & \\ & & \ddots & \\ & & & w_m \end{bmatrix}$. $w_i > 0$

$$\|\vec{x}\|' = \|W\vec{x}\| = \left\| \begin{bmatrix} w_1 x_1 \\ w_2 x_2 \\ \vdots \\ w_m x_m \end{bmatrix} \right\|$$

(Prove that this is also a norm)

1. Cauchy-Schwarz inequality

$$|\vec{x}^* \vec{y}| \leq \|\vec{x}\|_2 \|\vec{y}\|_2$$

2. Hölder inequality : $|\vec{x}^* \vec{y}| \leq \|\vec{x}\|_p \cdot \|\vec{y}\|_q$ when $\frac{1}{p} + \frac{1}{q} = 1$

Norms on matrices.

sA , $A + B$

$\|\cdot\| : \mathbb{C}^{m \times n} \to \mathbb{R}$

$$\left.\begin{array}{l} 1.\ \|A\| \geq 0 , \quad \|A\| = 0 \iff A = 0 \\ 2.\ \|sA\| = |s| \, \|A\| \\ 3.\ \|A + B\| \leq \|A\| + \|B\| \end{array}\right\}$$

$$\begin{array}{ccccc} \vec{x} & \longrightarrow & A\vec{x} & = & \vec{y} \\ \downarrow & & \downarrow & & \downarrow \\ \|\vec{x}\| & & \underbrace{\|A\|} & & \|\vec{y}\| \ ? \end{array}$$

$$A : \mathbb{C}^{m \times n}, \quad \vec{x} \in \mathbb{C}^{n}, \quad \vec{y} \in \mathbb{C}^{m}$$

$$\begin{array}{ccc} \vec{x} \in \mathbb{C}^{n} & & \vec{y} \in \mathbb{C}^{m} \\ \downarrow & & \downarrow \\ \|\cdot\|_{(n)} & & \|\cdot\|_{(m)} \end{array}$$

change in norm : $\dfrac{\|\vec{y}\|_{(m)}}{\|\vec{x}\|_{(n)}} = \dfrac{\|A\vec{x}\|_{(m)}}{\|\vec{x}\|_{(n)}}$

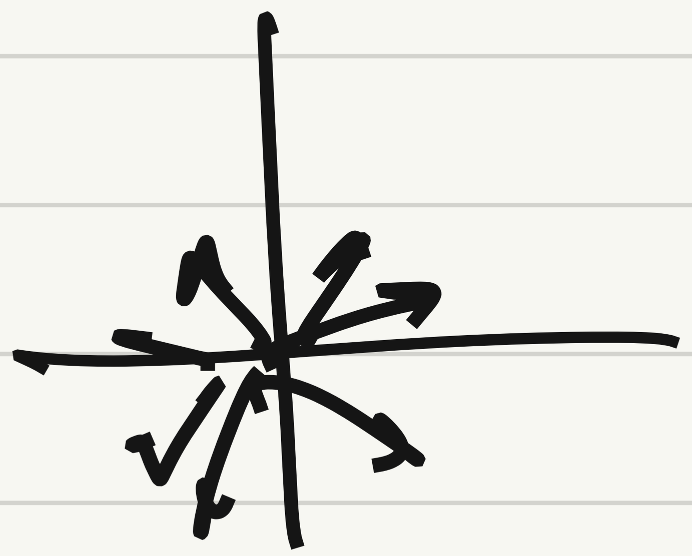
$$\|A\|_{(m,n)} = \sup_{\vec{x} \neq 0} \frac{\|A\vec{x}\|_{(m)}}{\|\vec{x}\|_{(n)}}$$

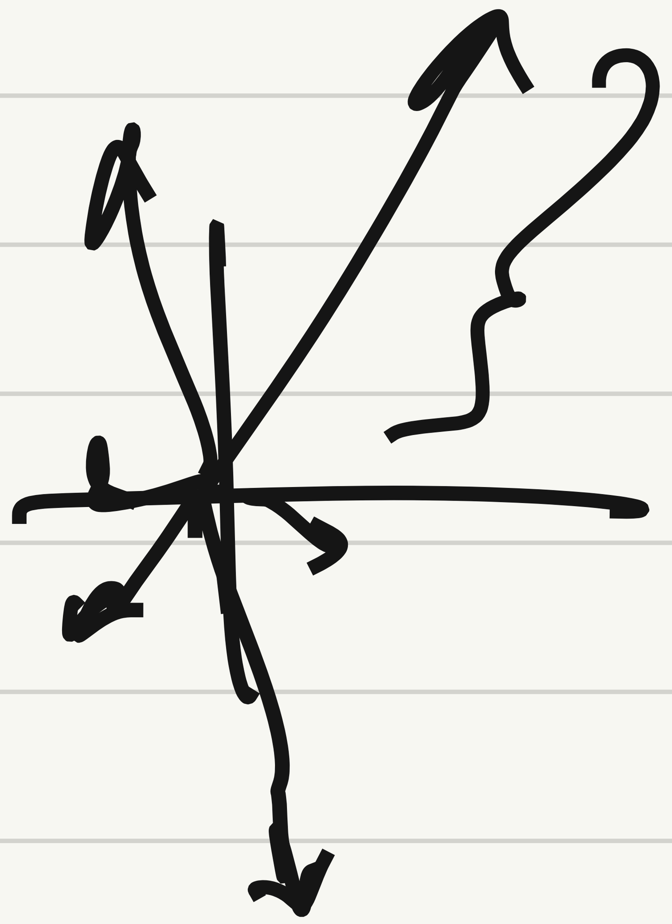
: $\boxed{\text{induced norm}}$ or $\boxed{\text{operator norm}}$ of A

$$= \sup_{\|\vec{x}\|_{(n)} = 1} \|A\vec{x}\|_{(m)}$$

Is this a norm?

Exercise.





$$\boxed{\|A\vec{x}\| \leq \|A\| \cdot \|\vec{x}\|}$$

only for induced norms

$$\|A\|_{(m,n)} = \sup_{\|\vec{x}\|_{(n)}=1} \|A\vec{x}\|_{(m)}$$

$\|\vec{x}\|_1 = 1$

$\left\|\begin{bmatrix}2\\0\end{bmatrix}\right\|_\infty = 2$

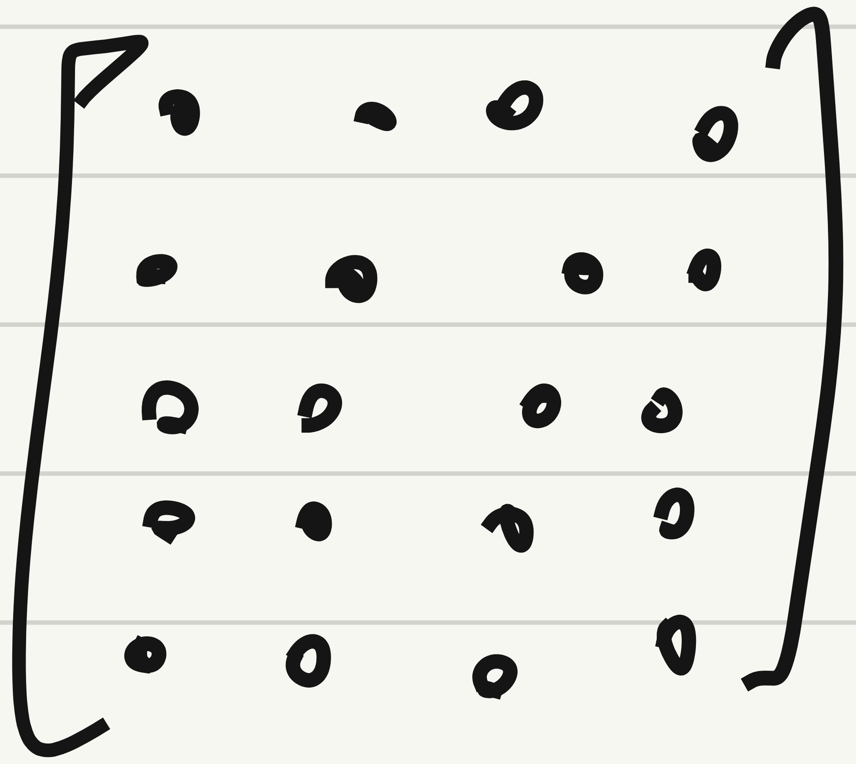
$$A = \begin{bmatrix}2 & 1\\ 0 & 1\end{bmatrix}$$

$$\|\vec{x}\|_{(n)} = \|\vec{x}\|_1$$

$$\|\vec{y}\|_{(m)} = \|\vec{y}\|_\infty$$

$$\|A\| = 2$$

$$\|A\|_p = \sup_{\|\vec{x}\|_p=1} \|A\vec{x}\|_p$$



How to compute $\|A\|_p$?

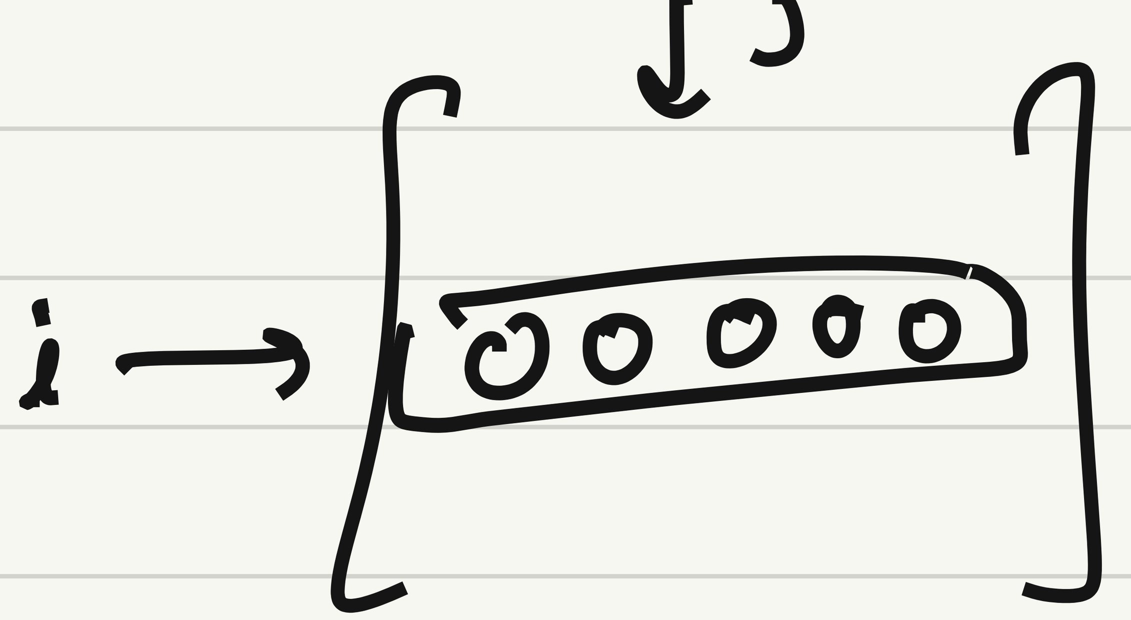
$$\|A\|_1 = \underbrace{\text{max abs. col. sum}} = \max_j \sum_i |a_{ij}| = \underbrace{\boxed{\max_j \|\vec{a}_j\|_1}}$$

$$\sup_{\|\vec{x}\|_1=1} \|A\vec{x}\|_1 = \ldots$$

① for all $\|\vec{x}\|_1 = 1$, $\|A\vec{x}\|_1 \le$

② $\exists\, \vec{x} : \|\vec{x}\|_1 = 1$, $\|A\vec{x}\|_1 =$

$\|A\|_\infty$ = max. abs. row sum = $\max_i \sum_j |a_{ij}|$



$\|A\|_2 = ?$ need singular value decomposition (SVD)

for induced norms:

4. $\|AB\| \leq \|A\|\,\|B\|$ $\boxed{\text{submultiplicativity}}$ (cf. $\|A+B\| \leq \|A\| + \|B\|$)

induced $\Rightarrow$ submult

$\not\Uparrow$

$\boxed{\text{frobenius norm}}$

or $\boxed{\text{Hilbert-Schmidt norm}}$

$$\|A\|_F = \left(\sum_i \sum_j |a_{ij}|^2\right)^{1/2}$$

$$= \|\mathrm{vec}(A)\|_2 = \left(\sum_j \|\vec{a}_j\|_2^2\right)^{1/2}$$

$$= \left(\mathrm{tr}(A^*A)\right)^{1/2} = \left(\mathrm{tr}(AA^*)\right)^{1/2}$$

$$\mathrm{vec}(A) = \begin{bmatrix} \vec{a}_1 \\ a_2 \\ \vdots \end{bmatrix} \in \mathbb{C}^{mn}$$

$\mathrm{tr}(A) = \sum_i a_{ii}$

$\|AB\|_F \leq \|A\|_F \|B\|_F$ Proof via $AB = \left[A\vec{b}_1 \,\middle|\, A\vec{b}_2 \,\middle|\, \cdots \,\middle|\, A\vec{b}_m \right]$

and $\|A\|_F = \sqrt{\sum_j \|\vec{a}_j\|_2^2}$

How to show $\|\cdot\|_F$ is not induced?

<u>Hint</u>: Consider $\|I\|_F$

$\boxed{\text{Unitary invariance}}$: $\underbrace{A \in \mathbb{C}^{m \times n}}$, $\left. \begin{array}{l} Q \in \mathbb{C}^{m \times m}, \\ R \in \mathbb{C}^{n \times n} \end{array} \right\}$ unitary

$\|QA\|_F = \|A\|_F = \|AR\|_F$

$\|QA\|_2 = \|A\|_2 = \|AR\|_2$

not true for $\|\cdot\|_1$, $\|\cdot\|_\infty$

	1	2	∞	F
Induced	✓	✓	✓	✗
Easy to compute	✓	✗	✓	✓
Unitary invariance	✗	✓	✗	✓
Submult.	✓	✓	✓	✓

$$\underbrace{f(\vec{x}) = A\vec{x}}$$

$$\kappa(\vec{x}) = \sup_{\delta\vec{x}} \frac{\|\delta f\| / \|f\|}{\|\delta\vec{x}\| / \|\vec{x}\|} = \sup_{\delta\vec{x}} \frac{\|A\,\delta\vec{x}\| / \|A\vec{x}\|}{\|\delta\vec{x}\| / \|\vec{x}\|}$$

$$= \frac{\|\vec{x}\|}{\|A\vec{x}\|} \underbrace{\sup_{\delta\vec{x}} \frac{\|A\vec{x}\|}{\|\delta\vec{x}\|}}_{\|A\|} = \frac{\|A\|\,\|\vec{x}\|}{\|A\vec{x}\|}$$

Condition number of a <u>matrix</u>

$$\kappa(A) = \sup_{\vec{x},\,\delta\vec{x}} \frac{\|A\,\delta\vec{x}\| \,/\, \|A\vec{x}\|}{\|\delta\vec{x}\| \,/\, \|\vec{x}\|} = \underbrace{\sup_{\vec{x}\neq 0} \frac{\|\vec{x}\|}{\|A\vec{x}\|}} \cdot \underbrace{\sup_{\delta\vec{x}} \frac{\|A\,\delta\vec{x}\|}{\|\delta\vec{x}\|}}_{\|A\|}$$

If A is singular, $\exists\,\vec{x} : A\vec{x} = \vec{0}$

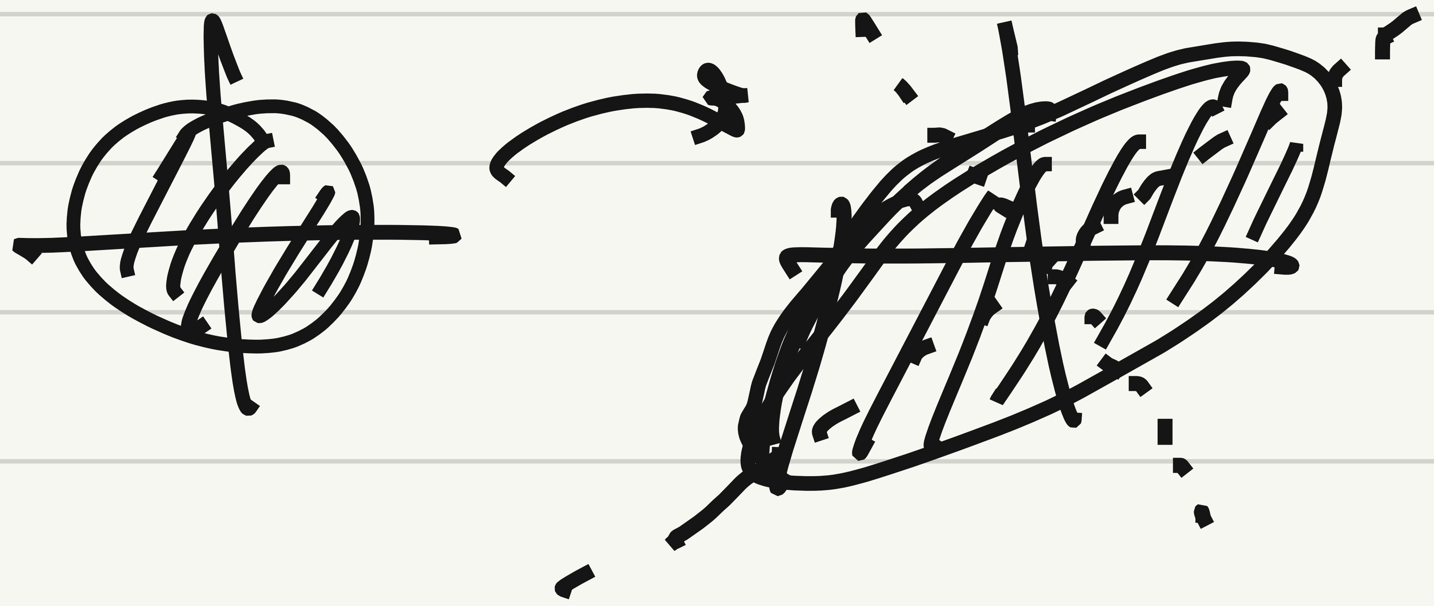
so $\kappa(A) = \infty$

otherwise $A\vec{x} = \vec{y} \iff \vec{x} = A^{-1}\vec{y}$: $\kappa(A) = \underbrace{\sup_{\vec{y}\neq 0} \frac{\|A^{-1}\vec{y}\|}{\|\vec{y}\|}}_{\|A^{-1}\|} \cdot \|A\|$

$\underbrace{\kappa(A) = \|A\| \cdot \|A^{-1}\|}$: worst-case conditioning of $f(\vec{x}) = A\vec{x}$, and lots of other problems

In this section $\|\cdot\| = \|\cdot\|_2$

Singular Value Decomposition (SVD)



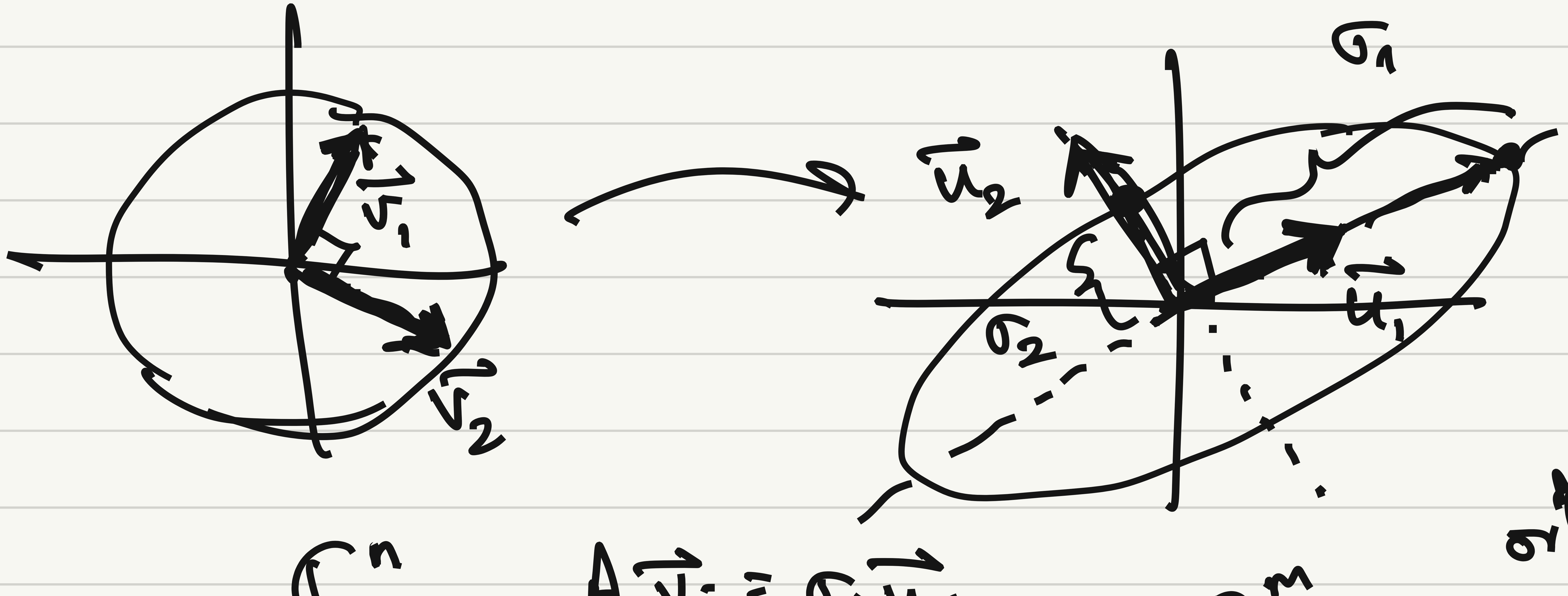
$A \in \mathbb{C}^{m \times n}$

Image of $\{\vec{x} : \|\vec{x}\|_2 \leq 1\}$ is an ellipsoid

ball

if $\operatorname{rank}(A) = r$ then r-dim. ellipsoid in $\mathbb{R}^m$

for now, let's assume A is "tall" ($m \geq n$) and full rank ($r = n$)



$A\vec{v}_j = \sigma_j \vec{u}_j$

$\vec{u}_1, \ldots, \vec{u}_n$: orthonormal $\boxed{\text{left singular vectors}}$

$\sigma_1, \ldots, \sigma_n$: ≥ 0 $\boxed{\text{singular values}}$

orthonormal

$\hookrightarrow \vec{v}_1, \ldots, \vec{v}_n$: $\boxed{\text{right singular vectors}}$

$\vec{u}_1, \dots, \vec{u}_n \in \mathbb{C}^m$ orthonormal

$\sigma_1, \dots, \sigma_n > 0$

$\vec{v}_1, \dots, \vec{v}_n \in \mathbb{C}^n$ orthonormal

$A\vec{v}_j = \sigma_j \vec{u}_j \quad \forall j = 1, \dots, n$

$$V = \left[\begin{array}{c|c|c} \vec{v}_1 & \cdots & \vec{v}_n \end{array}\right]$$

square, unitary

$$\hat{U} = \left[\begin{array}{c|c|c} \vec{u}_1 & \cdots & \vec{u}_n \end{array}\right]$$

rectangular

$$AV = \left[\begin{array}{c|c|c} A\vec{v}_1 & \cdots & A\vec{v}_n \end{array}\right]$$

$$\left[\begin{array}{c|c|c} \sigma_1\vec{u}_1 & \cdots & \sigma_n\vec{u}_n \end{array}\right] = ?$$

$$\underbrace{\begin{bmatrix} \vec{u}_1 & \cdots & \vec{u}_n \end{bmatrix}}_{\hat{U}}$$

$$\underbrace{\begin{bmatrix} \sigma_1\vec{u}_1 & \cdots & \sigma_n\vec{u}_n \end{bmatrix}} = \underbrace{\begin{bmatrix} \vec{u}_1 & \cdots & \vec{u}_n \end{bmatrix}}_{\hat{U}} \underbrace{\begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_n \end{bmatrix}}_{\hat{\Sigma}}$$

diagonal.
real

left singular vectors

right singular vectors

$$AV = \hat{U}\hat{\Sigma}$$

$$\iff A = \hat{U}\hat{\Sigma}V^{-1} = \hat{U}\hat{\Sigma}V^{*} \quad : \quad \boxed{\text{reduced SVD}} \text{ of } A$$

$$\begin{bmatrix} A \end{bmatrix} = \begin{bmatrix} \hat{U} \end{bmatrix} \underbrace{\begin{bmatrix} \diagdown \end{bmatrix}}_{\hat{\Sigma}} \begin{bmatrix} V \end{bmatrix}$$

$\hat{U} \in \mathbb{C}^{m\times n}$, orthonormal cols

$\hat{\Sigma} \in \mathbb{R}^{n\times n}$, diagonal.

$V \in \mathbb{C}^{n\times n}$, unitary

$$A = \hat{U}\hat{\Sigma}V^*$$

$$A\vec{x} = \hat{U}\,\overbrace{\hat{\Sigma}\,\underbrace{V^* x}_{\text{orthonormal change of basis}}}$$

$$\hat{\Sigma}\vec{w} = \begin{bmatrix}\sigma_1 w_1\\ \sigma_2 w_2\\ \vdots\\ \sigma_n w_n\end{bmatrix}$$

$\hat{U}$

$$\underbrace{\begin{bmatrix}\sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_n\end{bmatrix}}_{\hat{\Sigma}}\begin{bmatrix}w_1\\ w_2\\ \vdots\\ w_n\end{bmatrix} =$$