Linear Algebra

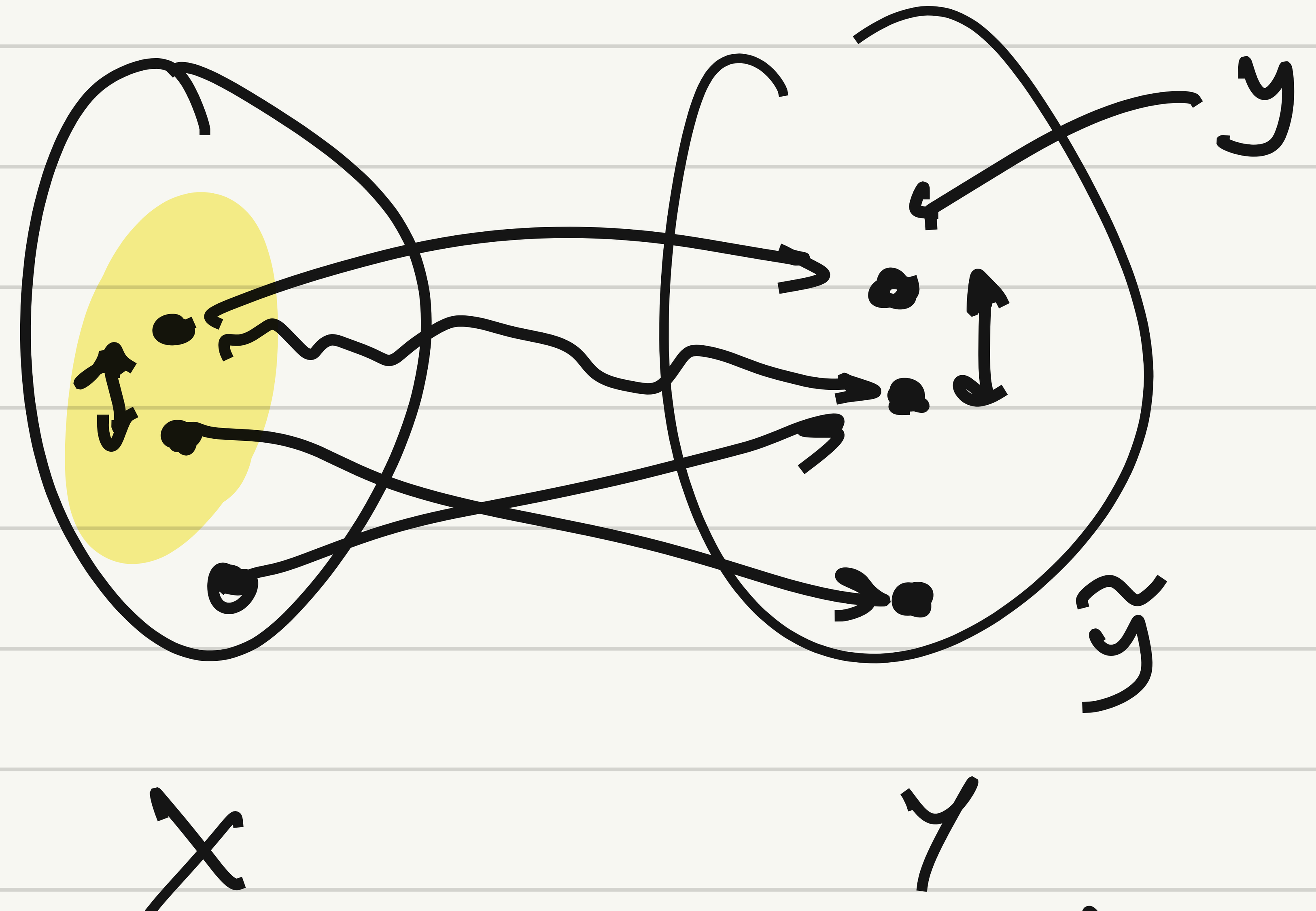
Assignment 1 will be posted today, due in 2 weeks

- Moodle → not really
- course webpage → slides, assignment PDFs
- GradeScope → assignment submission, exams
- Piazza → discussion

$x \pm \varepsilon$

$f(x)$ $\quad x =$ weight of vehicles bridge can support

$f(x) =$ (length, thickness) of main span



requirements (safe load, wind resistance)

design (

fundamental operation : lin. comb.

$$\left.\begin{array}{l} \vec{v}_1, \dots, \vec{v}_n \in V \\ s_1, \dots, s_n \in \mathbb{K} \\ s_1\vec{v}_1 + \dots + s_n\vec{v}_n = \vec{v} \in V \end{array}\right\}$$

→ span $\{v_1, \dots, v_n\} = \langle v_1, \dots, v_n \rangle$

$= \{ s_1 v_1 + \dots : s_1, \dots \in \mathbb{K} \}$

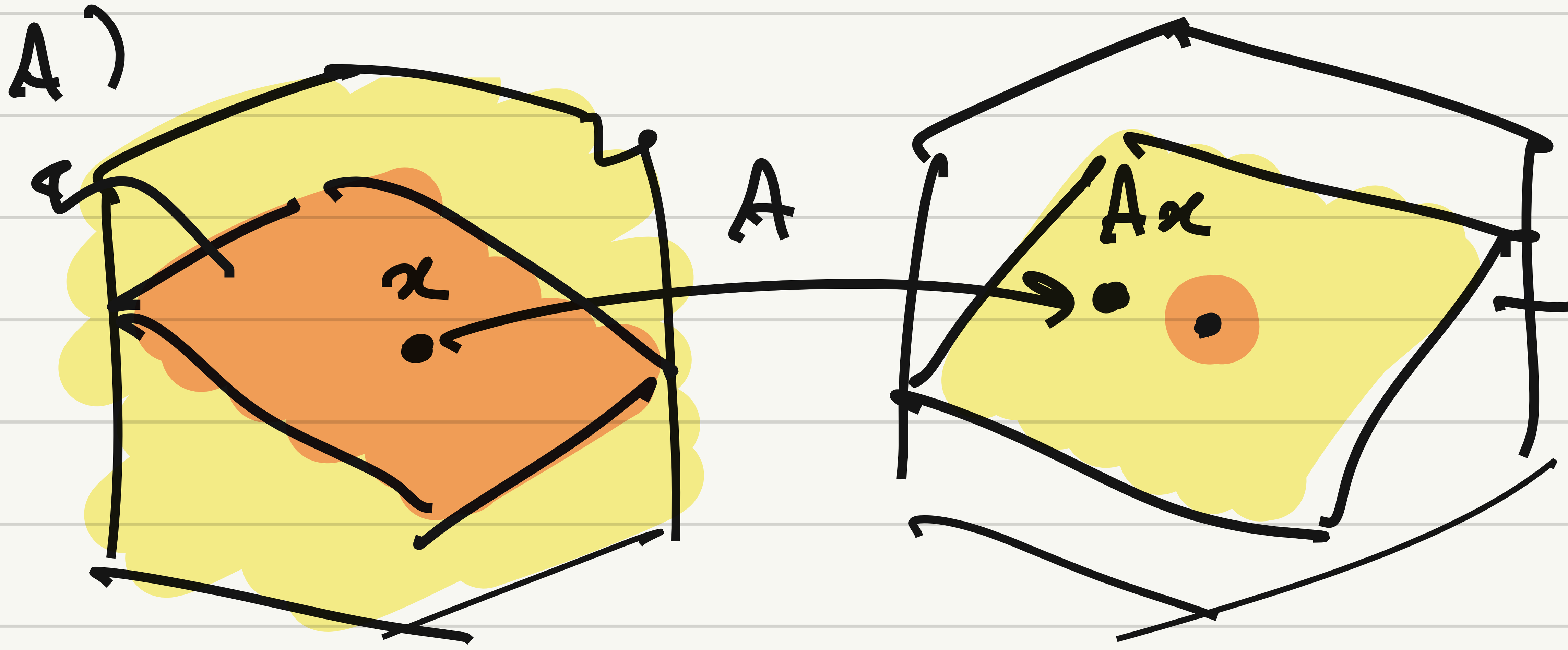
→ lin. indep.

→ basis = lin. indep. set that spans V.

$A\vec{x} = T(\vec{v})$

$(AB)x = (S \circ T)(x)$

null(A)



range(A) = col(A)

$A \in \mathbb{K}^{m \times n}$ $\mathbb{R}^n$ $\mathbb{R}^m$

solve $\underline{A\vec{x} = \vec{b}}$

Solution exists iff. $\vec{b} \in \text{range}(A)$

If $\vec{x}$ is sol. then every other sol. $= x + z$ where $z \in \text{null}(A)$

$A\vec{x} = \vec{b}$, $A\vec{y} = \vec{b}$ then $A(\vec{y} - \vec{x}) = \vec{0} \Leftrightarrow \vec{y} - \vec{x} \in \text{null}(A)$

$\boxed{\cancel{\text{column}}\ \text{rank}}$ of A = # of lin. indep. cols of A. $= \dim(\text{range}(A))$

row rank " " = " " " rows " " equal!

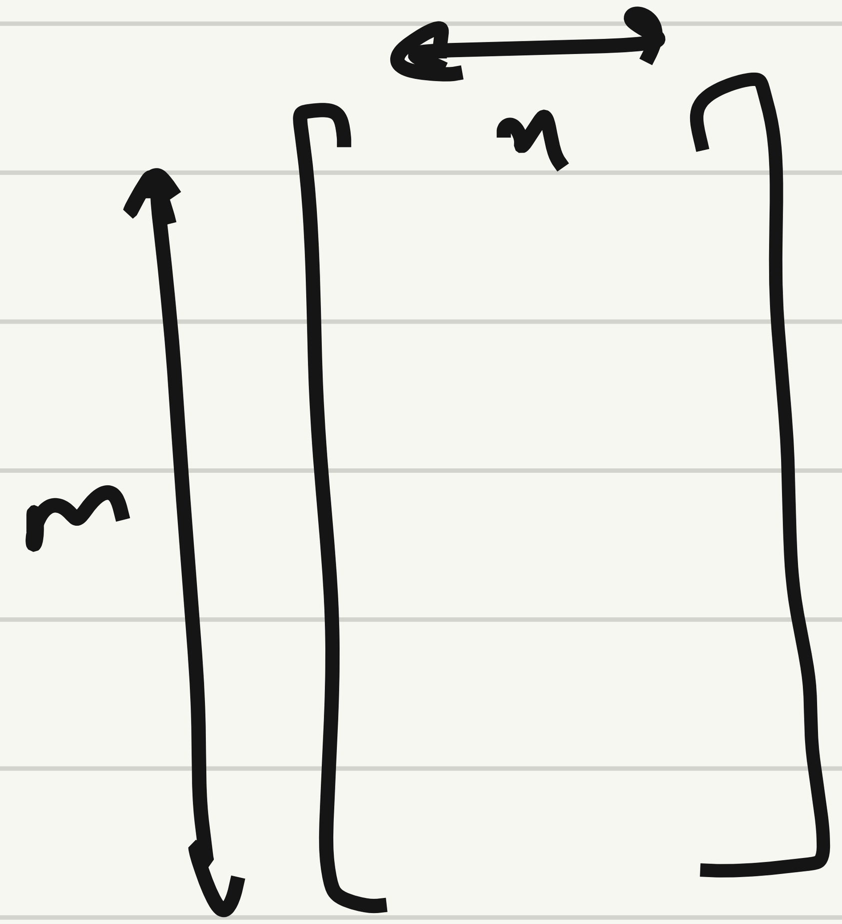
rank $\leq \min(m, n)$

If $=$, $\boxed{\text{full rank}}$

"tall matrix"

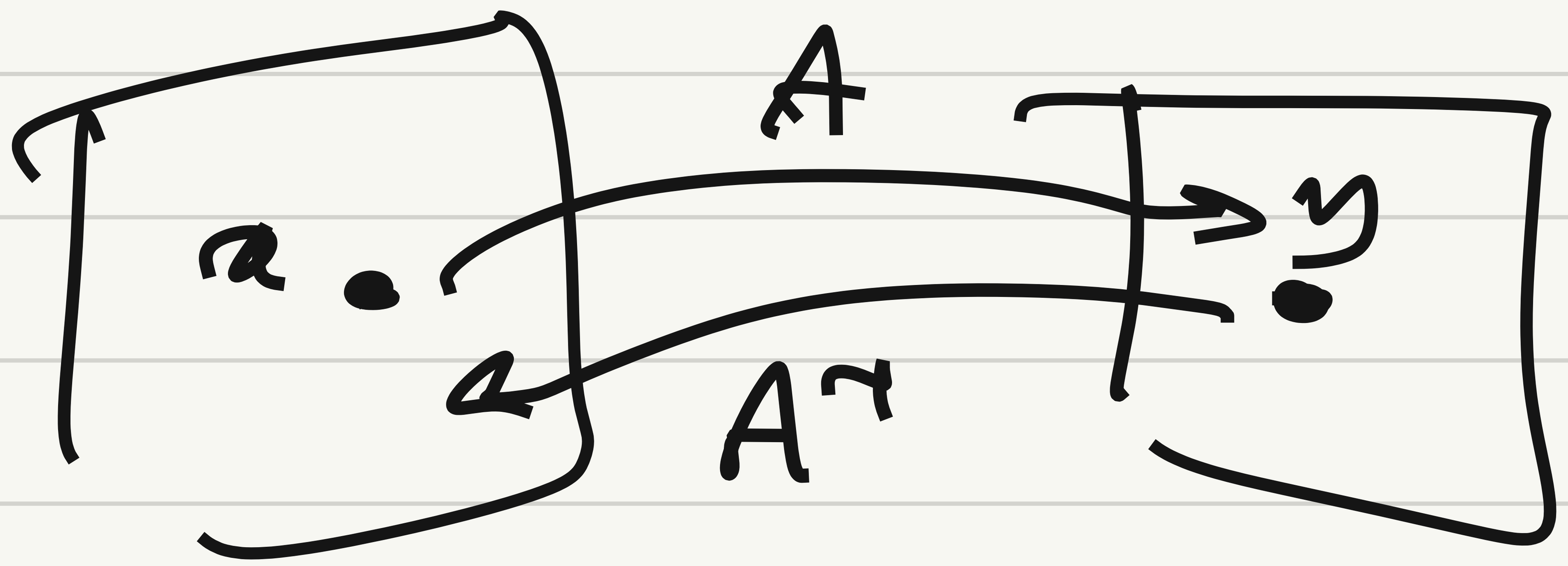
$\mathbb{R}^n \to \mathbb{R}^m$

<u>Thm</u>: A matrix $A \in \mathbb{K}^{m \times n}$, $n \leq m$, is full rank <u>iff</u> $T(x) = Ax$ is injective (one-to-one)



$\vec{x}_1 \neq \vec{x}_2 \Rightarrow A\vec{x}_1 \neq A\vec{x}_2$.

$\boxed{\text{Inverse}}$ of a matrix

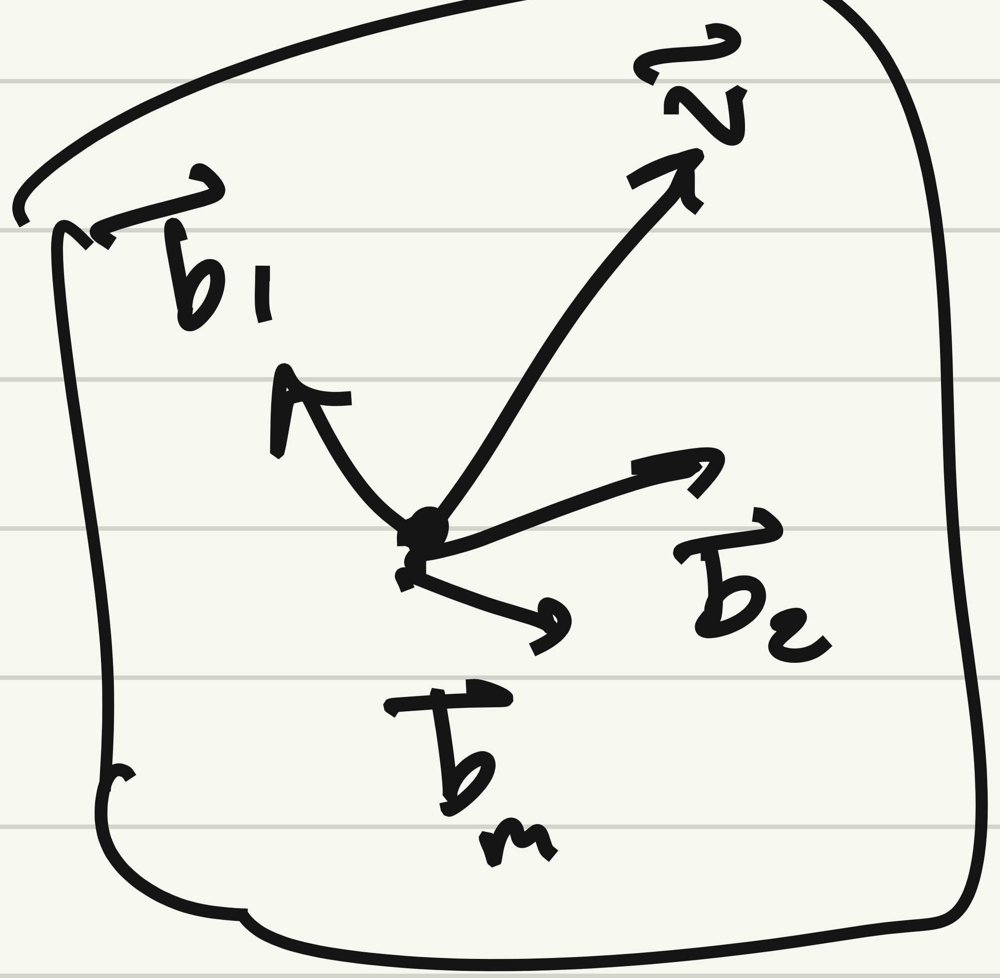


$$A\vec{x} = \vec{y} \Leftrightarrow A^{-1}\vec{y} = \vec{x}$$

Inverse only exists if A is square and full rank

If A is square: inverse exists $\Leftrightarrow$ A is $\boxed{\text{invertible}}$

does not exist $\Leftrightarrow$ A is $\boxed{\text{Singular}}$



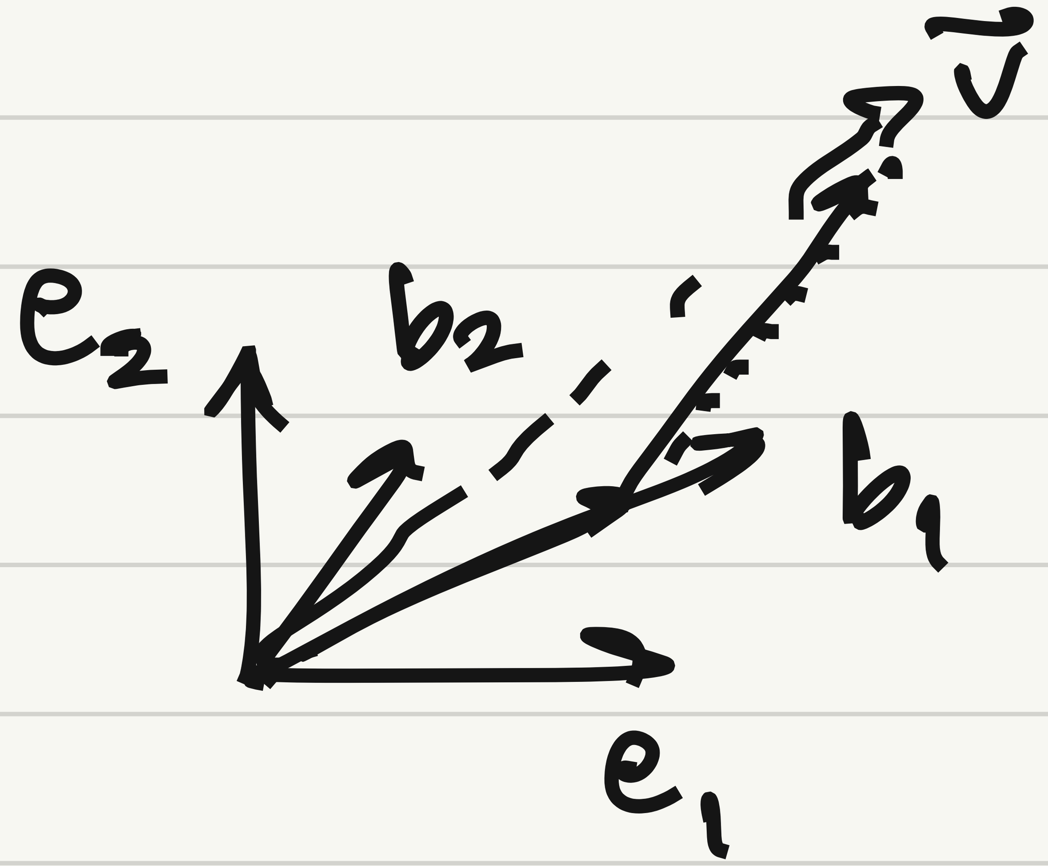
$\vec v \in \mathbb{K}^m$, basis $\beta = (\vec b_1, \dots, \vec b_m)$

$\cancel{\langle \vec x, \vec y \rangle = \vec x^* \vec y}$

$\vec v = \underbrace{x_1 \vec b_1 + \dots + x_m \vec b_m}$, what are $x_1, \dots, x_m$?

$$= \underbrace{\begin{bmatrix} \vec b_1 & \Big| & \cdots & \Big| & \vec b_m \end{bmatrix}}_{B} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} = B\vec x$$

coord. vector of $\vec v$ in basis β



B: square & full rank $\Rightarrow$ invertible

$$\Rightarrow \quad \vec x = \underbrace{B^{-1} \vec v}_{\text{solve } B\vec x = \vec v \text{ for } \vec x}$$

<u>change of basis</u>

never actually compute B^{-1} then multiply!

Orthogonality

$\boxed{\text{Transpose}}$ of $A = A^T$

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots \\ a_{21} & a_{22} & a_{23} & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix}}_{A} \longrightarrow \underbrace{\begin{bmatrix} a_{11} & a_{21} & \cdots \\ a_{12} & a_{22} & \cdots \\ a_{13} & a_{23} & \cdots \\ \vdots & \vdots & \end{bmatrix}}_{A^T}$$

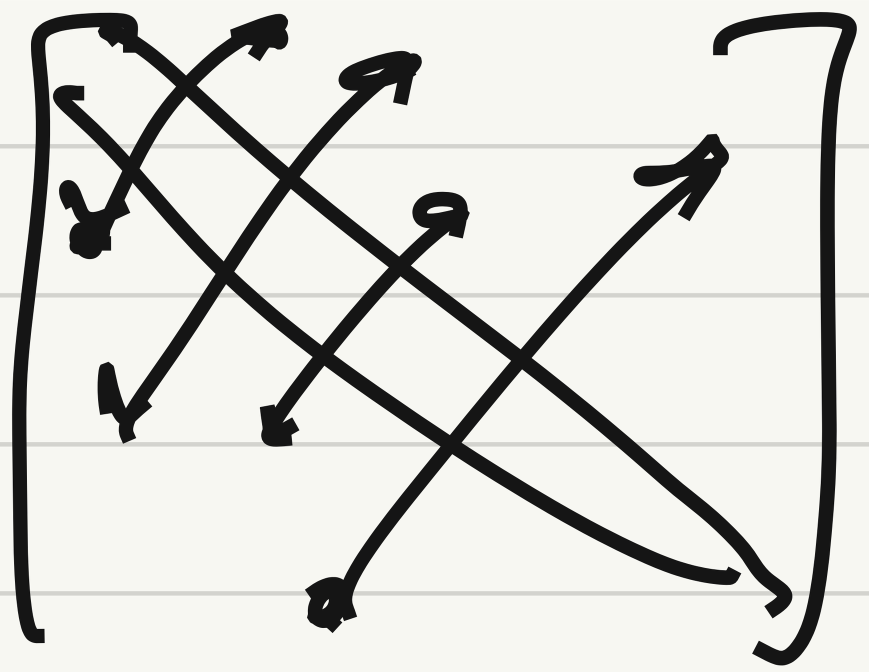
Complex matrices

$\boxed{\text{Adjoint}}$ $A^* =$ conjugate transpose

$$\overline{x + iy} = x - iy$$

$$\underbrace{\begin{bmatrix} \bar{a}_{11} & \bar{a}_{21} & \cdots \\ \bar{a}_{12} & \bar{a}_{22} & \cdots \\ \bar{a}_{13} & \bar{a}_{23} & \cdots \end{bmatrix}}_{A^*}$$

$A = A^T$ $\boxed{\text{symmetric}}$ matrix



$A = A^*$: $\boxed{\text{Hermitian}}$ matrix

$$\underbrace{(AB)^{-1} = B^{-1}A^{-1}}$$

$$(A+B)^* = A^* + B^*, \quad (sA)^* = \bar{s}A^*, \quad (AB)^* = B^*A^*$$

$\boxed{(A^*)^{-1} = (A^{-1})^* = A^{-*}}$ $\quad$ (real case : $(A^T)^{-1} = (A^{-1})^T = A^{-T}$)

Dot product : $\vec{x} \cdot \vec{y} = \sum x_i y_i$ for real, in general $\sum \bar{x}_i y_i$

$\vec{x} \cdot \vec{x} = \sum \bar{x}_i x_i = \sum |x_i|^2$

$$\vec{x}^* \vec{y} = \begin{bmatrix} \bar{x}_1 & \bar{x}_2 & \cdots & \bar{x}_m \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}$$

$\boxed{\text{inner product}}$

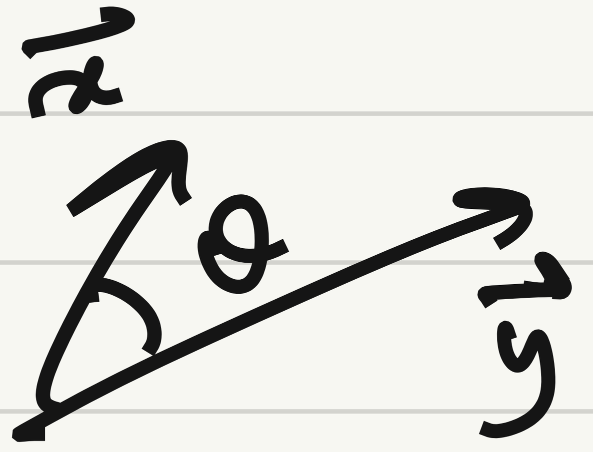
① $\vec{x}^* \vec{y} = \overline{\vec{y}^* \vec{x}}$, ② $\vec{x}^*(s_1 \vec{y}_1 + s_2 \vec{y}_2) = s_1 \vec{x}^* \vec{y}_1 + s_2 \vec{x}^* \vec{y}_2$

③ $\vec{x}^* \vec{x} \geq 0$, $\vec{x}^* \vec{x} \neq 0 \iff \vec{x} \neq \vec{0}$.

$$\|\vec{x}\|_2 = \sqrt{\sum |x_i|^2} = \sqrt{\vec{x}^* \vec{x}}$$

$\boxed{\text{unit vector}}$: $\|\vec{x}\|_2 = 1$

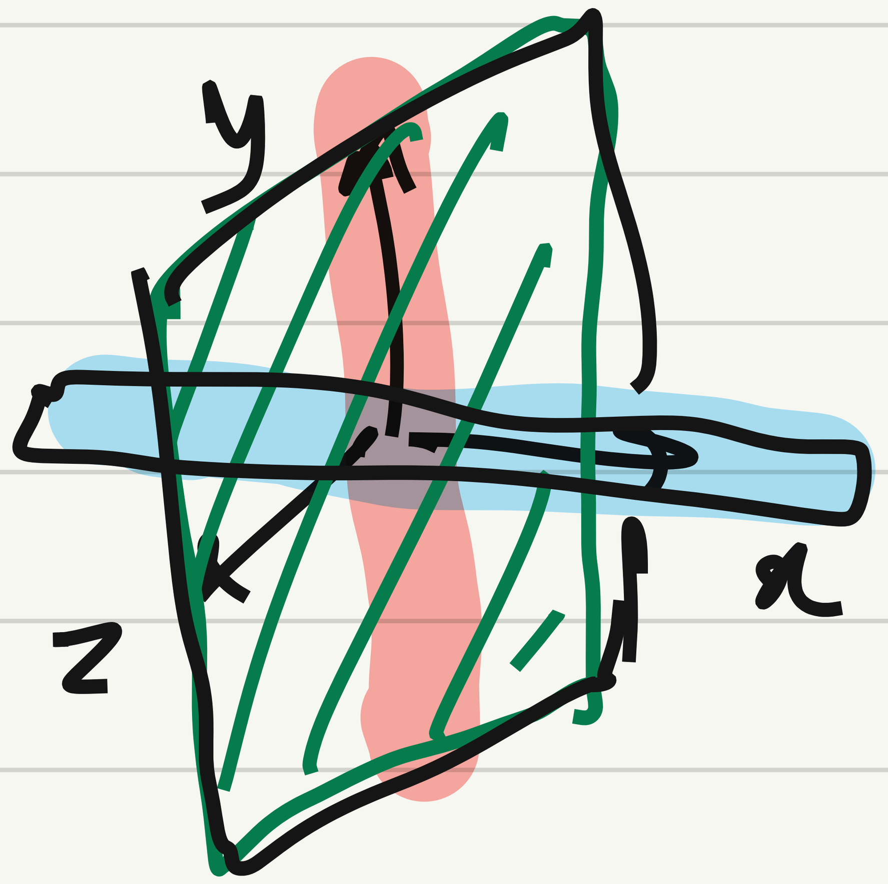
Two vectors $\vec{x}, \vec{y}$ are $\boxed{\text{orthogonal}}$ if $\vec{x}^{*}\vec{y} = 0$.

In $\mathbb{R}^n$ 

, $\cos\theta = \dfrac{\vec{x}^{T}\vec{y}}{\|\vec{x}\|_2 \cdot \|\vec{y}\|_2}$

Two sets X, Y are $\boxed{\text{orthogonal to each other}}$

if $\forall \vec{x} \in X, \vec{y} \in Y$, $\vec{x}^{*}\vec{y} = 0$



A set S is $\boxed{\text{orthogonal}}$

if $\forall \vec{x} \neq \vec{y} \in S$, $\vec{x}^{*}\vec{y} = 0$

and $\forall \vec{x} \in S$, $\vec{x} \neq 0$

$$\begin{tikzcd}
V \arrow[r, "T(v)"] \arrow[d, "\text{basis}"'] & W \arrow[d, "\text{basis}"] \\
\mathbb{R}^n \arrow[r, "Ax"] & \mathbb{R}^m
\end{tikzcd}$$

$S = \{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$

If S is orthogonal, it is also lin. indep. (If S is ortho in $\mathbb{K}^m$, $|S| \le m$)

$\Rightarrow$ if $|S| = m$ then it is a basis of $\mathbb{R}^m$ (or $\mathbb{K}^m$)

S is $\boxed{\text{orthonormal}}$ if it is orthogonal and all $\vec{x} \in S$ are unit vectors.

unity = 1

If $(\vec{a}_1, \vec{a}_2, \dots, \vec{a}_m)$ are a basis, to get coeffs of $\vec{b} \in \mathbb{K}^m$

Solve $A\vec{x} = \vec{b}$. $\Longleftrightarrow$ $\vec{b} = x_1\vec{a}_1 + \cdots + x_m\vec{a}_m$

But if $(\vec{q}_1, \vec{q}_2, \dots, \vec{q}_m)$ are orthonormal basis, then...

$$\vec{b} = x_1\vec{q}_1 + \cdots + x_m\vec{q}_m$$

Take inner prod. with $\vec{q}_1$:

$$\vec{q}_1^{\,*}\vec{b} = x_1\underbrace{(\vec{q}_1^{\,*}\vec{q}_1)}_{\|\vec{q}_1\|^2 = 1} + x_2\underbrace{(\vec{q}_1^{\,*}\vec{q}_2)}_{0} + \cdots + x_m\underbrace{(\vec{q}_1^{\,*}\vec{q}_m)}_{0}$$

$$\vec{q}_1^{\,*}b = x_1$$

$$x_i = \vec{q}_i^* \vec{b}$$

$$\Leftrightarrow \vec{b} = (\vec{q}_1^* \vec{b})\vec{q}_1 + \cdots + (\vec{q}_m^* \vec{b})\vec{q}_m$$

. If Q is orthogonal then Q^* is also orthogonal.

$$Q = \left[\vec{q}_1 \,\middle|\, \cdots \,\middle|\, \vec{q}_m \right] \qquad \vec{x} = \begin{bmatrix} \vec{q}_1^* \vec{b} \\ \vdots \\ \vec{q}_m^* \vec{b} \end{bmatrix} = Q^* \vec{b}$$

$$\boxed{Q^* Q = I = Q Q^*}$$

$$Q^* = \begin{bmatrix} \vec{q}_1^* \\ \hline \vdots \\ \hline \vec{q}_m^* \end{bmatrix} \qquad \vec{x} = Q^{-1} \vec{b}$$

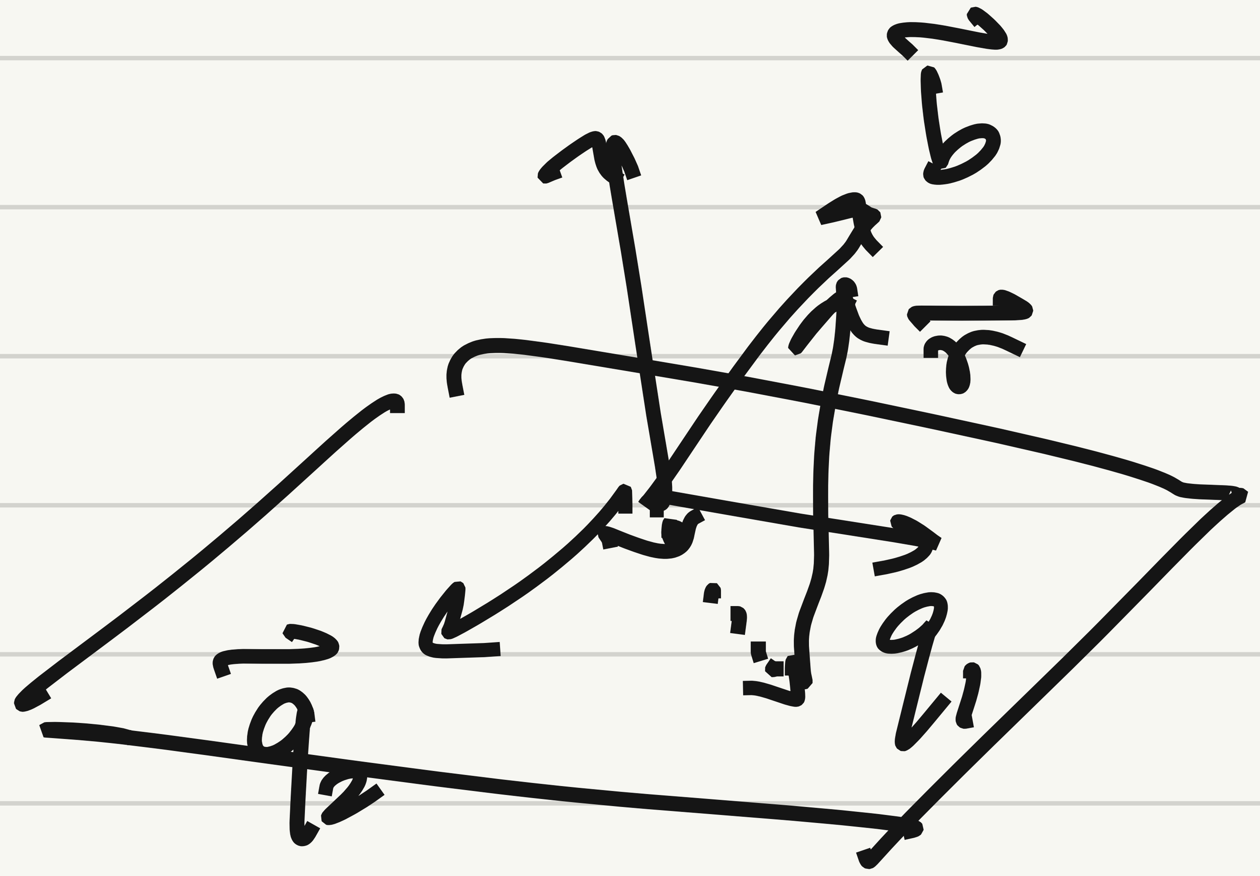
$$\boxed{Q^{-1} = Q^*}$$

Q is an $\boxed{\text{orthogonal}}$ matrix

Q is square and columns of Q are orthonormal

Orthogonal matrix Q : $Q^*Q = I = QQ^*$

Orthonormal set $\vec{q}_1, \dots, \vec{q}_n \in \mathbb{K}^m$, $n < m$



$$\vec{b} = \underbrace{\underbrace{(\vec{q}_1^{\,*}\vec{b})\vec{q}_1} + \dots + \underbrace{(\vec{q}_n^{\,*}\vec{b})\vec{q}_n}} + \underbrace{\vec{r}}$$

$$\vec{r} = \vec{b} - (\dots)$$

$\vec{r}$ is orthogonal to $\vec{q}_1, \dots, \vec{q}_n$ (prove it)

$\Rightarrow$ we have $\boxed{\text{orthogonally decomposed}}$ $\vec{b}$ into $n+1$ vectors

$$\vec{x}, \vec{y} \longrightarrow Q\vec{x}, Q\vec{y}$$

$$(Q\vec{x})^*(Q\vec{y}) = \vec{x}^*\underbrace{Q^*Q}_{I}\vec{y} = \vec{x}^*\vec{y}$$

similarly $\|Q\vec{x}\|_2 = \|\vec{x}\|_2$

$\Rightarrow$ inner prod, 2-norm are $\boxed{\text{invariant}}$ to orthogonal matrices

Orthogonal matrices preserve lengths & angles

Geometrically, (in $\mathbb{R}^n$) orthogonal matrices = rotations & reflections

$\Longrightarrow$ Orthonormal changes of basis don't affect inner product.