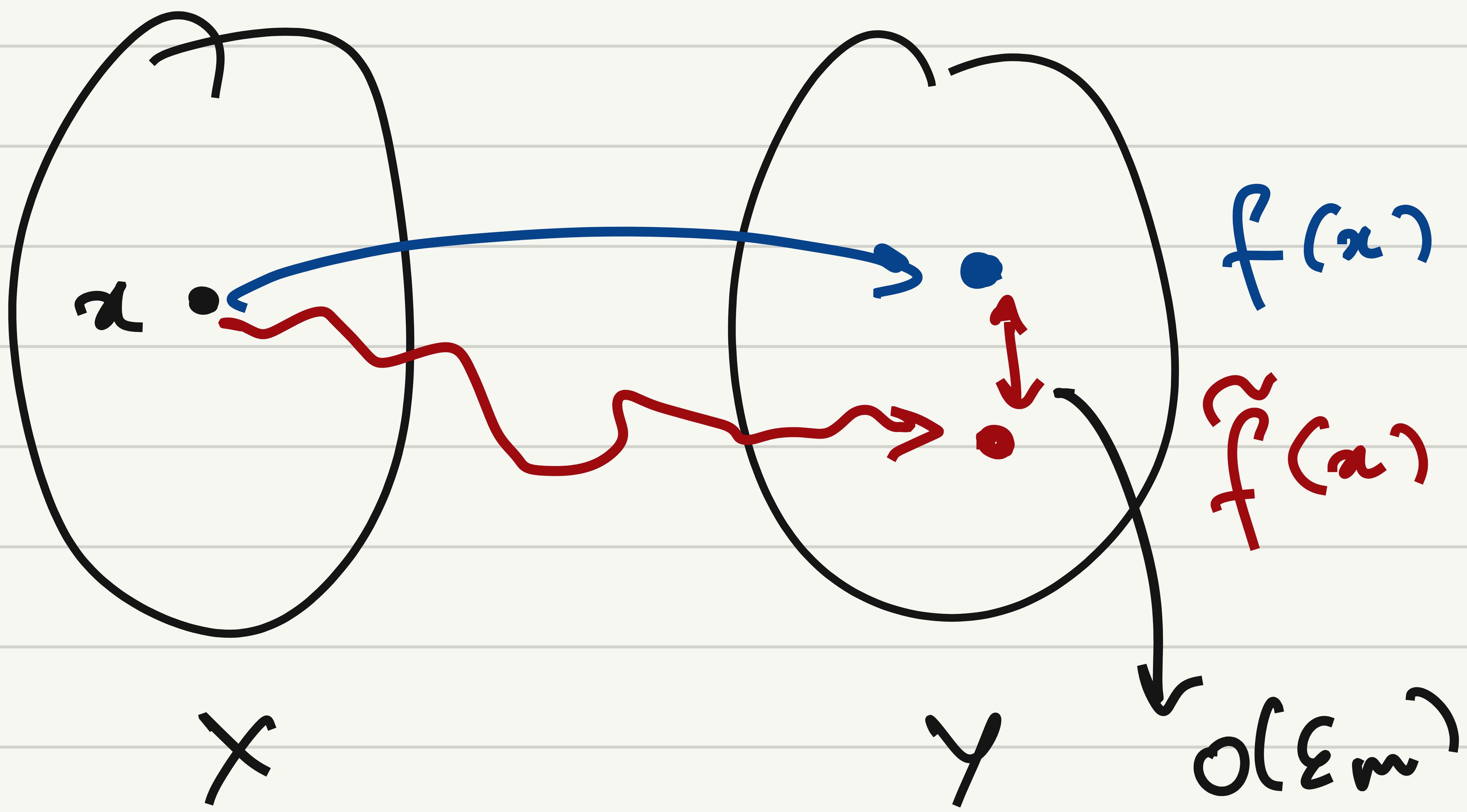
COL726: Stability, Linear Algebra

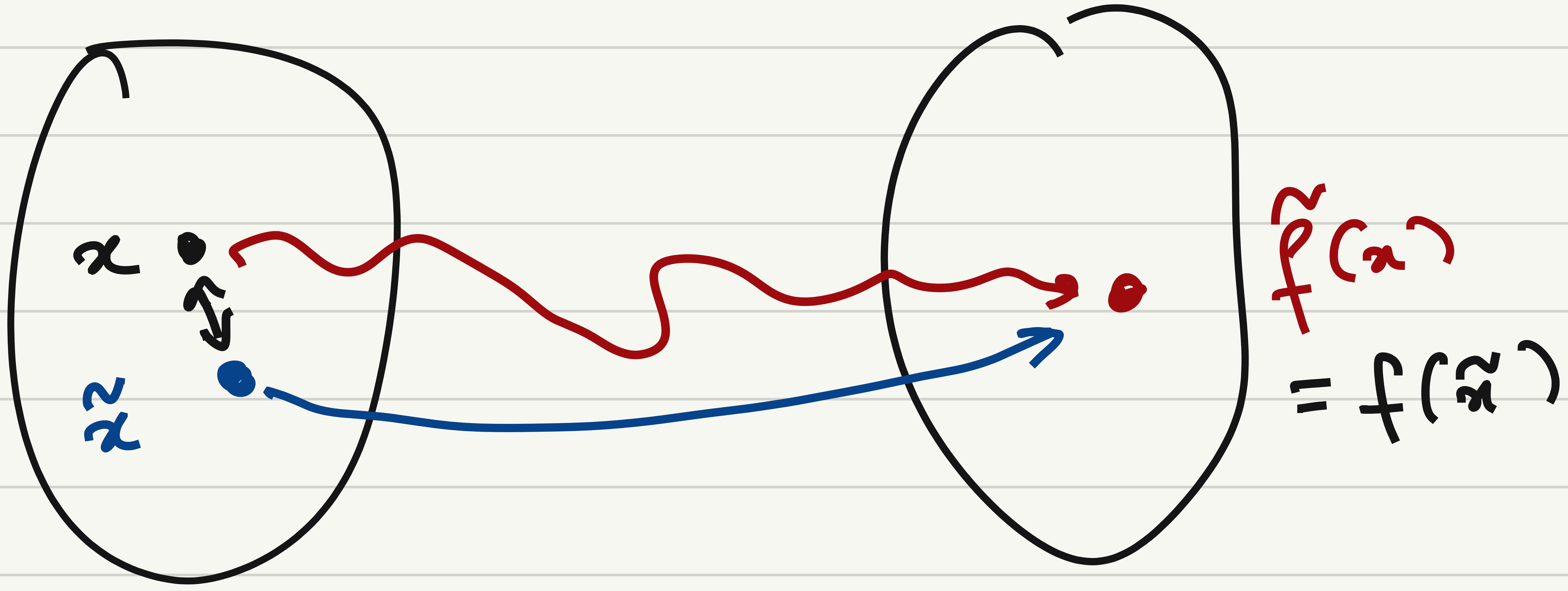
floating-point numbers $F = \{ \pm M \times \beta^E : M = d_0 . d_1 \ldots d_{p-1},$
$E \in \mathbb{Z} \}$

$\forall x \in \mathbb{R}, \quad fl(x) = x(1+\varepsilon), \quad |\varepsilon| \leq \varepsilon_m$

$\forall x_1, x_2 \in F, \quad x_1 \circledast x_2 = (x_1 * x_2)(1+\varepsilon), \quad |\varepsilon| \leq \varepsilon_m$



accuracy



Backward stable

$x \ominus y$ in floating point : accurate?

If $x, y \in F$, then $x \ominus y = (x-y)(1+\varepsilon)$, $|\varepsilon| \leq \varepsilon_m$

If $x, y \in \mathbb{R}$, $\underbrace{fl(x)}_{O(\varepsilon_m)} \ominus \underbrace{fl(y)}_{O(\varepsilon_m)}$

$$\begin{array}{r} 1.2345\overbrace{\#\#\#\cdots} \\ -\ 1.2321\#\#\# \\ \hline 2.4\underbrace{000} \end{array}$$

$x \circledast y$ is backward stable :

$\exists\ \underbrace{\tilde{x}, \tilde{y}}$ near x, y s.t. $\tilde{x} - \tilde{y} = fl(x) \ominus fl(y)$

$fl(x) = x + \delta x_1$ $\quad |\delta x_1| \leq |x|\,\varepsilon_m$

$fl(y) = y + \delta y_1$ $\quad |\delta y_1| \leq |y|\,\varepsilon_m$

$fl(x) \ominus fl(y) = (x + \delta x_1) - (y + \delta y_1) + \delta z$ $\quad |\delta z| \leq (x + \delta x_1 - y - \delta y_1)\,\varepsilon_m$

$$\underline{fl(x) \ominus fl(y)} = \underbrace{\overbrace{x+\delta x_1} - \overbrace{y-\delta y_1}}_{z} + \widetilde{\delta_2} \quad , \quad |\delta_2| \le |x+\delta x_1 - y - \delta y_1|\, \varepsilon_m$$

$$\le \underbrace{|x+\delta x_1|\,\varepsilon_m}_{\delta x_2} + \underbrace{|y+\delta y_1|\,\varepsilon_m}_{\delta y_2}$$

$$= \underbrace{(x+\delta x_1) + \delta x_2}_{\tilde{x}} - \underbrace{(y-\delta y_1) + \delta y_2}_{\tilde{y}}$$

$$|a+b| \le |a| + |b|$$

$$|\delta x_1| \le |x|\,\varepsilon_m \qquad \lim \to 0$$

$$|\delta x_2| \le |x+\delta x_1|\,\varepsilon_m$$

$$|x| \underbrace{(2+\varepsilon_m)\,\varepsilon_m}_{O(\varepsilon_m)}$$

$f : X \to Y$

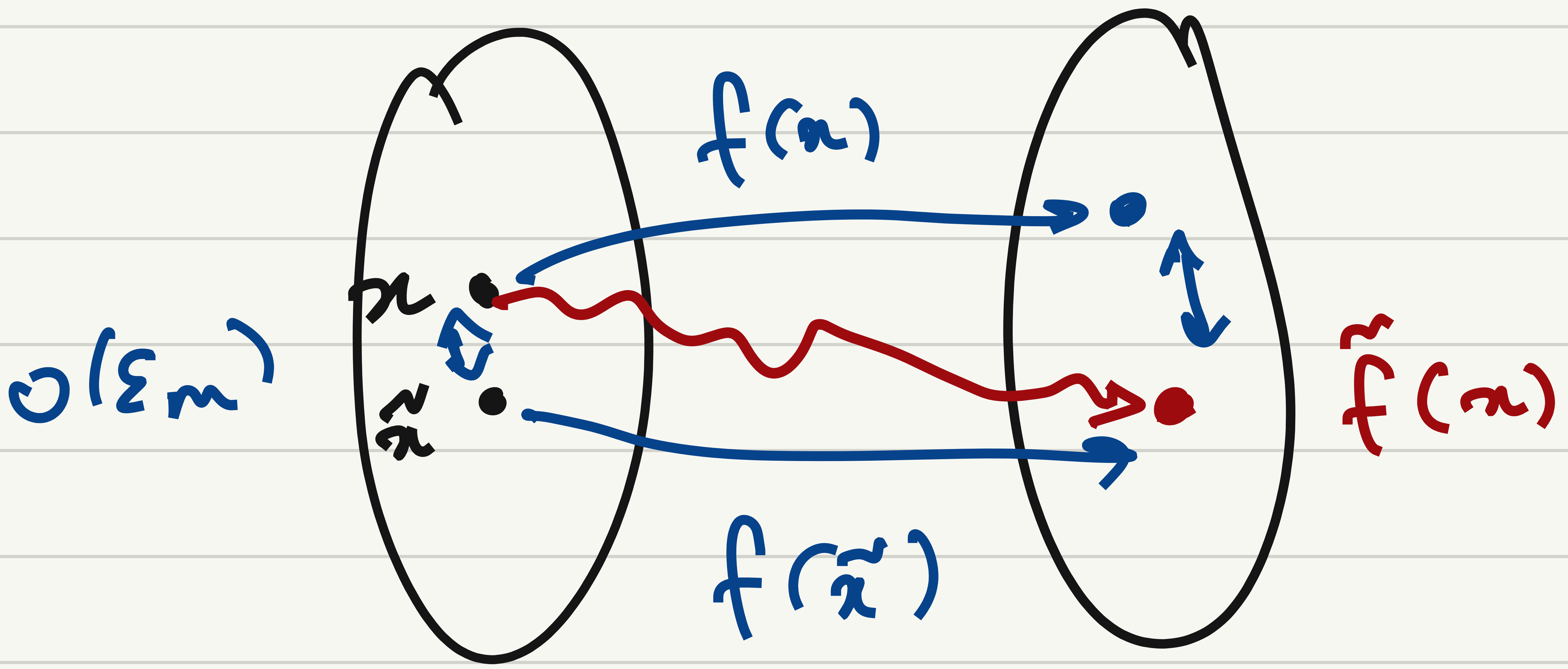
Backward stable alg. $\tilde{f}$ $\iff$ $\forall x \in X$, $\exists \tilde{x} \in X$ s.t. $\tilde{f}(x) = f(\tilde{x})$,

$$\frac{\|\tilde{x} - x\|}{\|x\|} = O(\varepsilon_m)$$

r.b.e.

What can I say about $\dfrac{\|\tilde{f}(x) - f(x)\|}{\|f(x)\|}$?

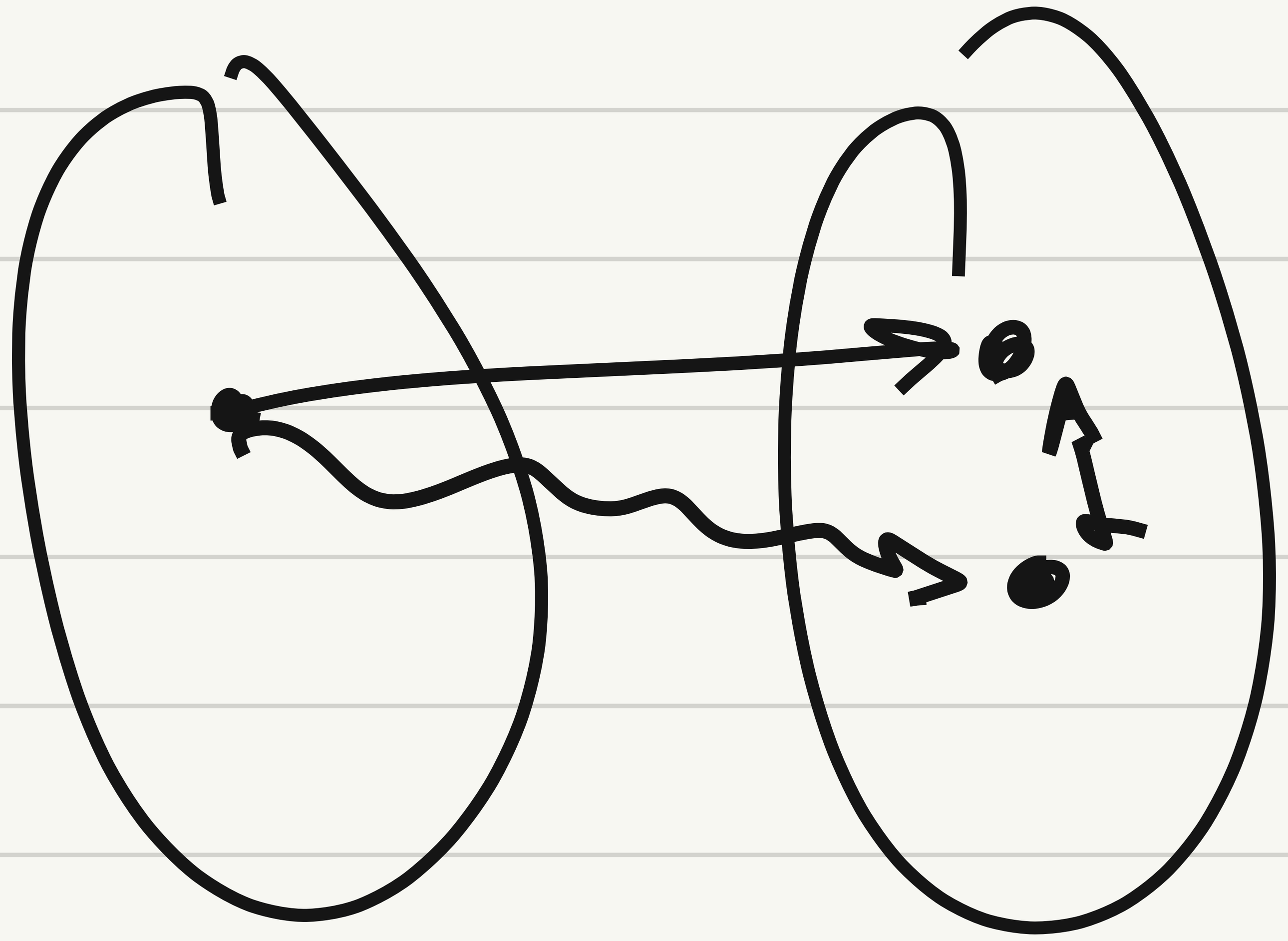
r.f.e

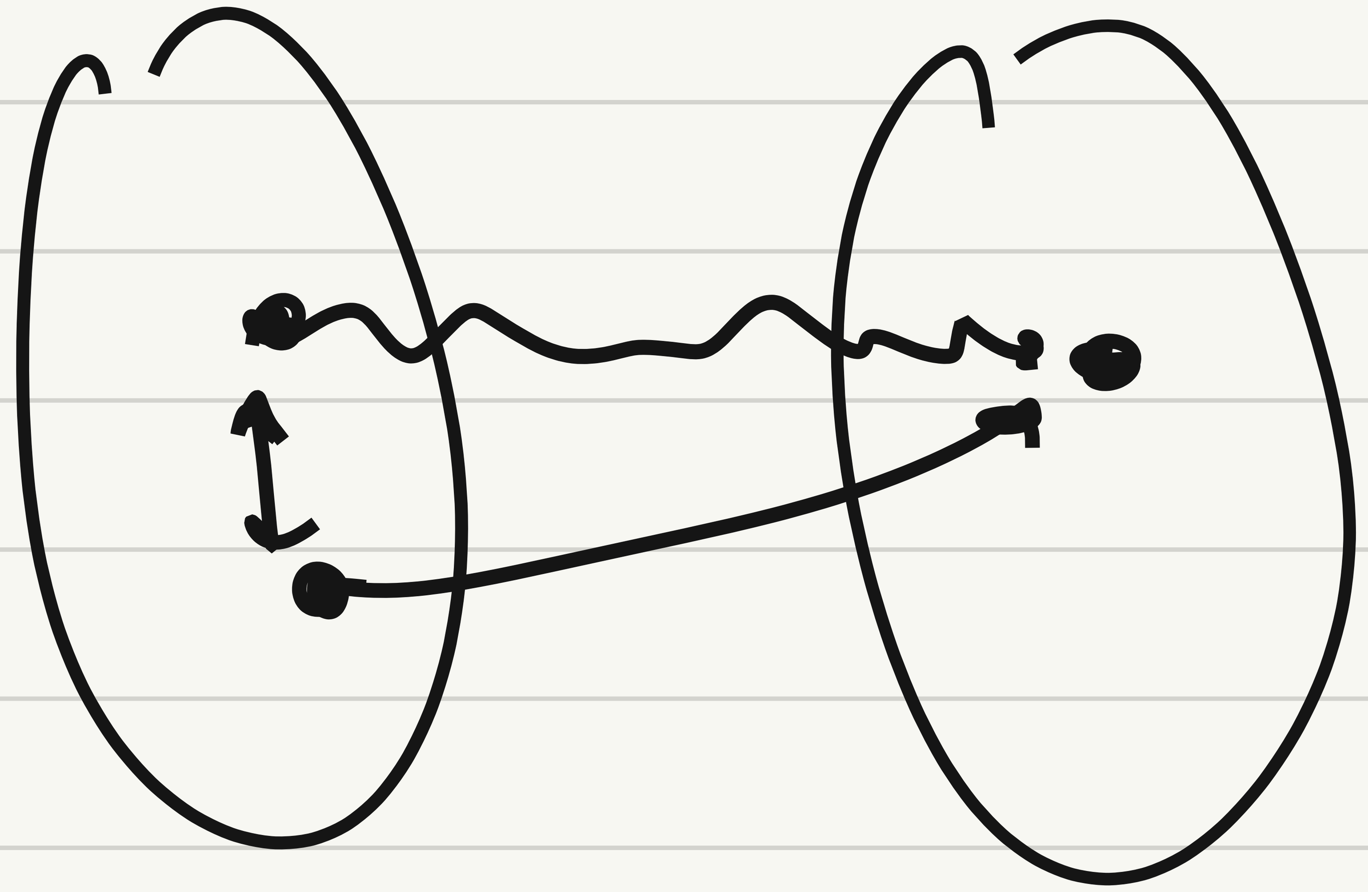


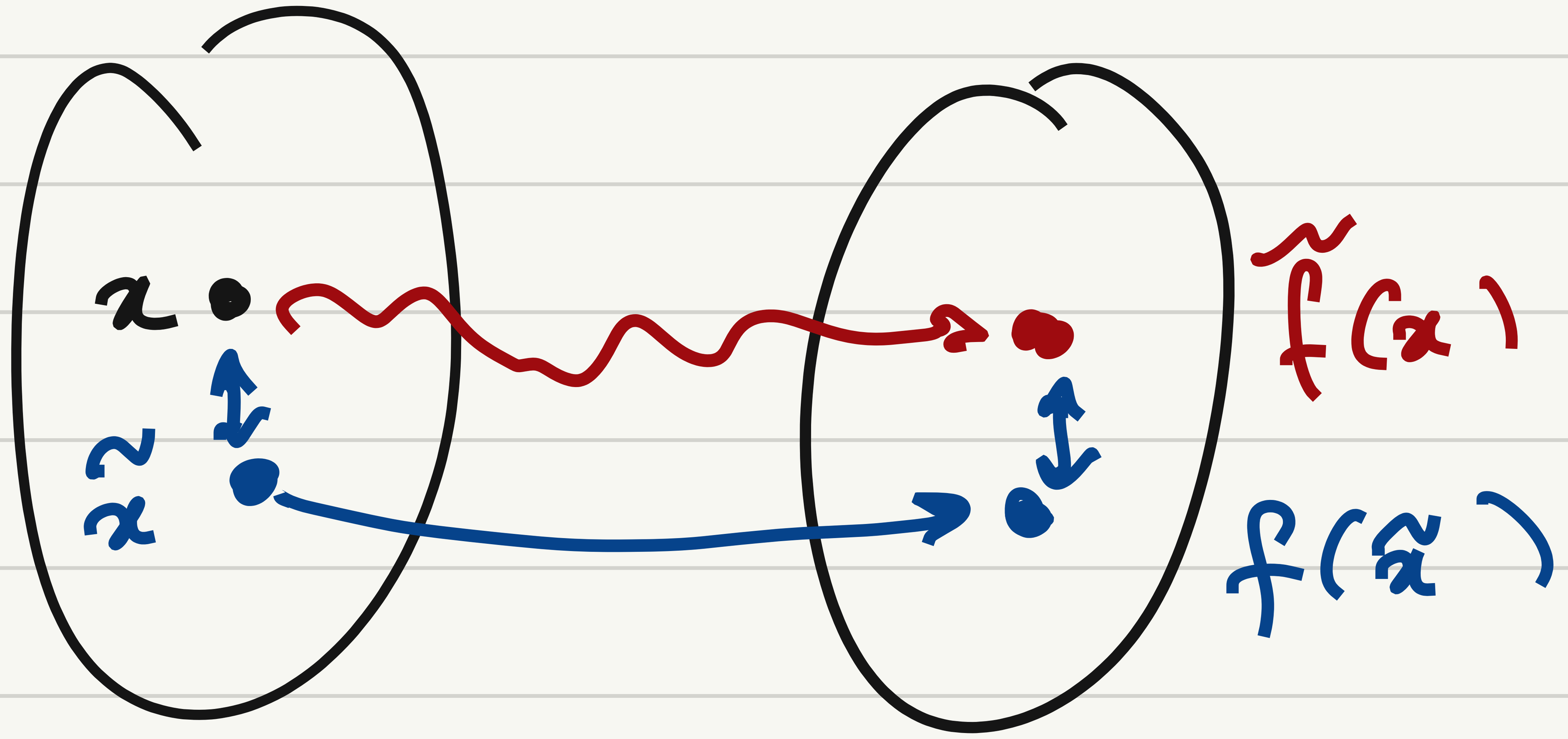
as $\varepsilon_m \to 0$, r.f.e. $\lesssim K(x) \cdot$ r.b.e.

$= O(K(x)\, \varepsilon_m)$

r.f.e. $= O\big((K(x) + o(1))\, \varepsilon_m \big)$







$\forall x \in X, \; \exists \, \tilde{x} \in X$ s.t. mixed $\boxed{\text{Stability}}$

$$\frac{\|\tilde{x} - x\|}{\|x\|} = O(\varepsilon_m), \quad \frac{\|\tilde{f}(x) - f(\tilde{x})\|}{\|f(x)\|} = O(\varepsilon_m)$$

$$\left.\begin{aligned} f(\theta) &= (\cos(\theta), \sin(\theta)) \\ f(x) &= 1 + x \end{aligned}\right\} \text{ stable but not backward stable}$$

Linear algebra

Vector = list ~~of n number~~ → too specific

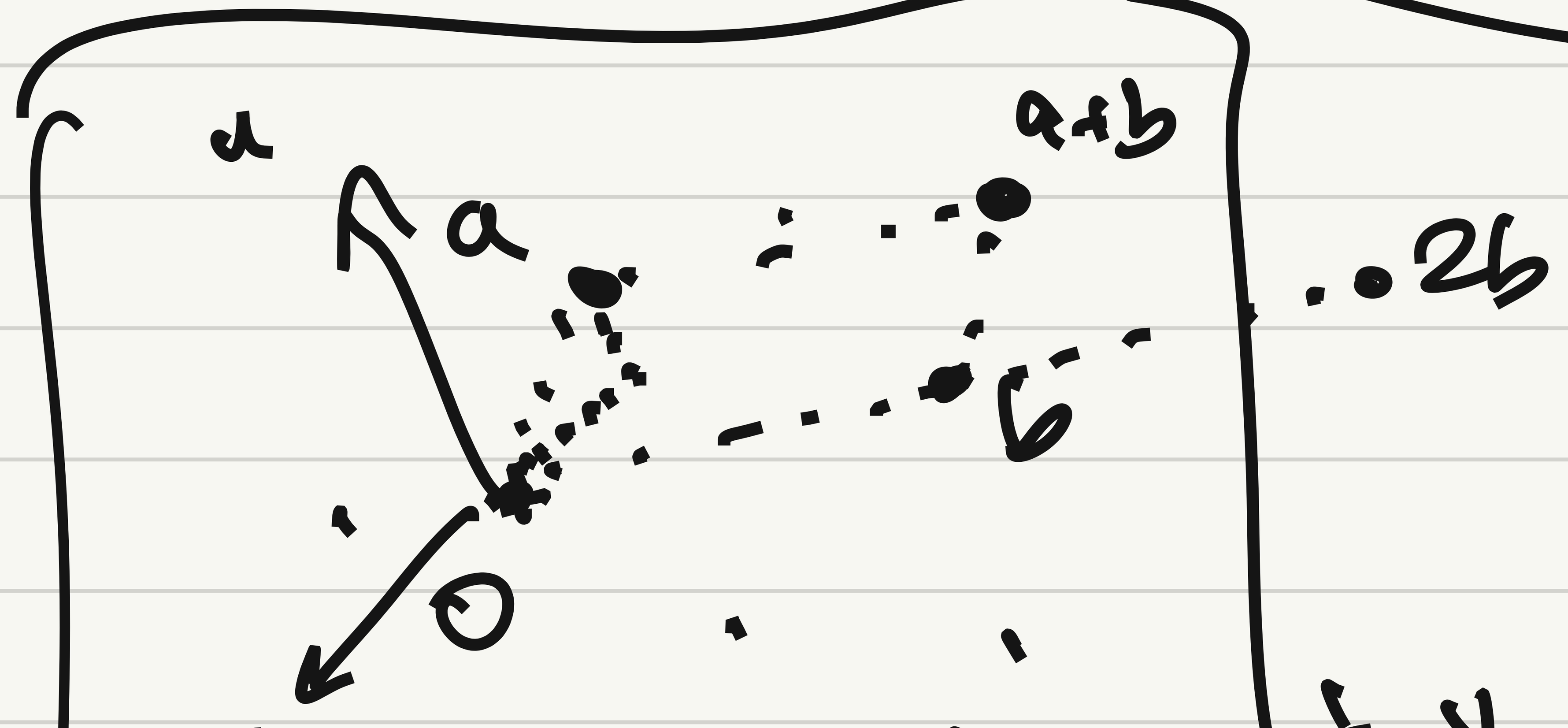
= ~~qty with mag. & dir.~~

(informally)

Vector space: set of things you can add ($x, y \in V$ then $x+y \in V$)

over a field $\mathbb{K}$ and mult. by numbers ($x \in V$, $s \in \mathbb{K}$ then $sx \in V$)

$\mathbb{R}$ or $\mathbb{C}$



$a = \frac{1}{2} u - \frac{1}{2} v$

$a : \begin{bmatrix} 1/2 \\ -1/2 \end{bmatrix}$

P^n : set of polynomials of degree $\leq n$

What is a basis?

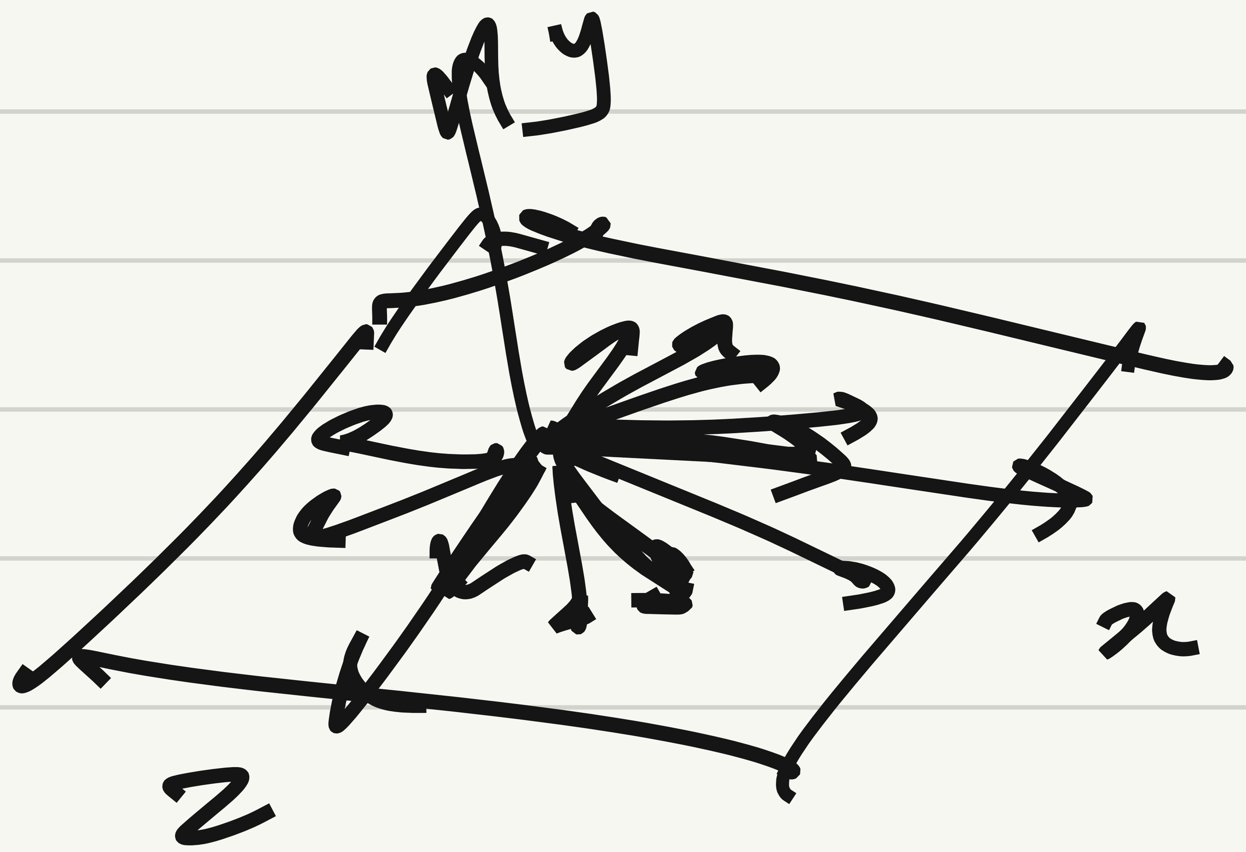
linear combination. $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n \in V$

$$\left(s_2 \left(s_1 \vec{v}_1 + \vec{v}_3 \right) + s_3 \vec{v}_6 \right). \longrightarrow \quad s_1 \vec{v}_1 + s_2 \vec{v}_2 + \cdots + s_n \vec{v}_n : \boxed{\text{linear combination}}$$

$$s_1, s_2, \ldots, s_n \in \mathbb{K}$$

$\boxed{\text{Span}}$ of $\{\vec{v}_1, \ldots, \vec{v}_n\}$ is the set of all lin. comb.

$$= \{ s_1 v_1 + \cdots + s_n v_n : s_1, \ldots, s_n \in \mathbb{K} \}$$

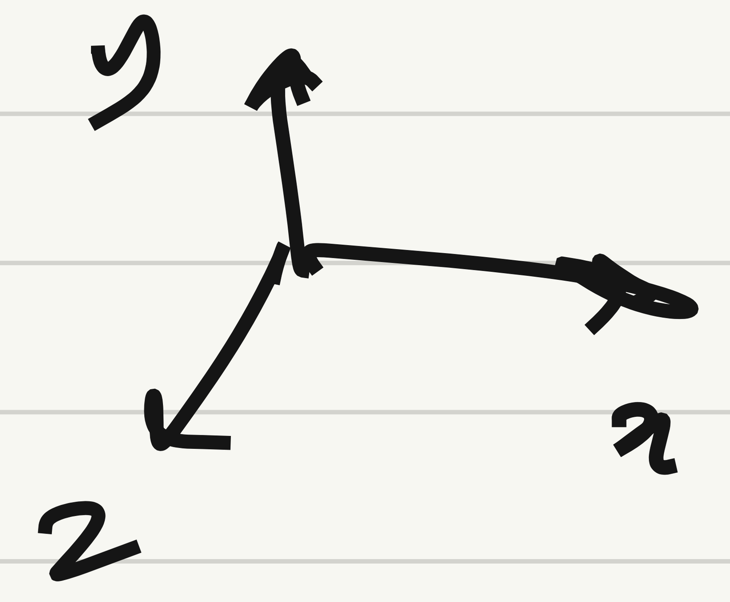


set of vectors is $\boxed{\text{linearly independent}}$

if no nontrivial lin. comb. is $\vec{0}$.

$$s_1 \vec{v}_1 + \cdots + s_n \vec{v}_n = \vec{0} \implies s_1 = \cdots = s_n = 0$$

lin. indep. $\iff$ $\nexists \ \vec{v}_i = \sum_{j \neq i} s_j \vec{v}_j$



$\boxed{\text{Basis}}$ of V = lin. indep. set that spans V.

If I have basis, $(\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n)$

any $\vec{v} \in V$ can be expressed uniquely as $x_1 \vec{v}_1 + \cdots + x_n \vec{v}_n = \vec{v}$.

So $\vec{v}$ can be represented as $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{K}^n$.

$\boxed{\text{Coordinate vector}}$ of $\vec{v}$

Eg. quadratic polynomials $p(x) = ax^2 + bx + c$

$v_1 = 1$

$v_2 = x$

$v_3 = x^2$

$p(x) = x^2 - 2 \longleftrightarrow$

coordinate vector : $\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$

$p(x) = ax^2 + bx + c$

$$\vec{v} = \begin{bmatrix} p(0) \\ p(1) \\ p(2) \end{bmatrix}$$

Does this still corresp. to some choice of basis v_1, v_2, v_3 ?

Now we can with only $\mathbb{R}^n$, $\mathbb{C}^n$.

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} \in \mathbb{R}^m$$

standard basis of $\mathbb{R}^m / \mathbb{C}^m$: $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, $\dots$, $\vec{e}_m = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$

$$= x_1 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \dots + x_m \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

U, V $\boxed{\text{linear transformation :}}$: function that respects vector space structure

$T : U \to V$

$$\left.\begin{aligned} T(\vec{u}_1 + \vec{u}_2) &= T(u_1) + T(u_2) \\ T(s\vec{u}) &= sT(u) \end{aligned}\right\}$$

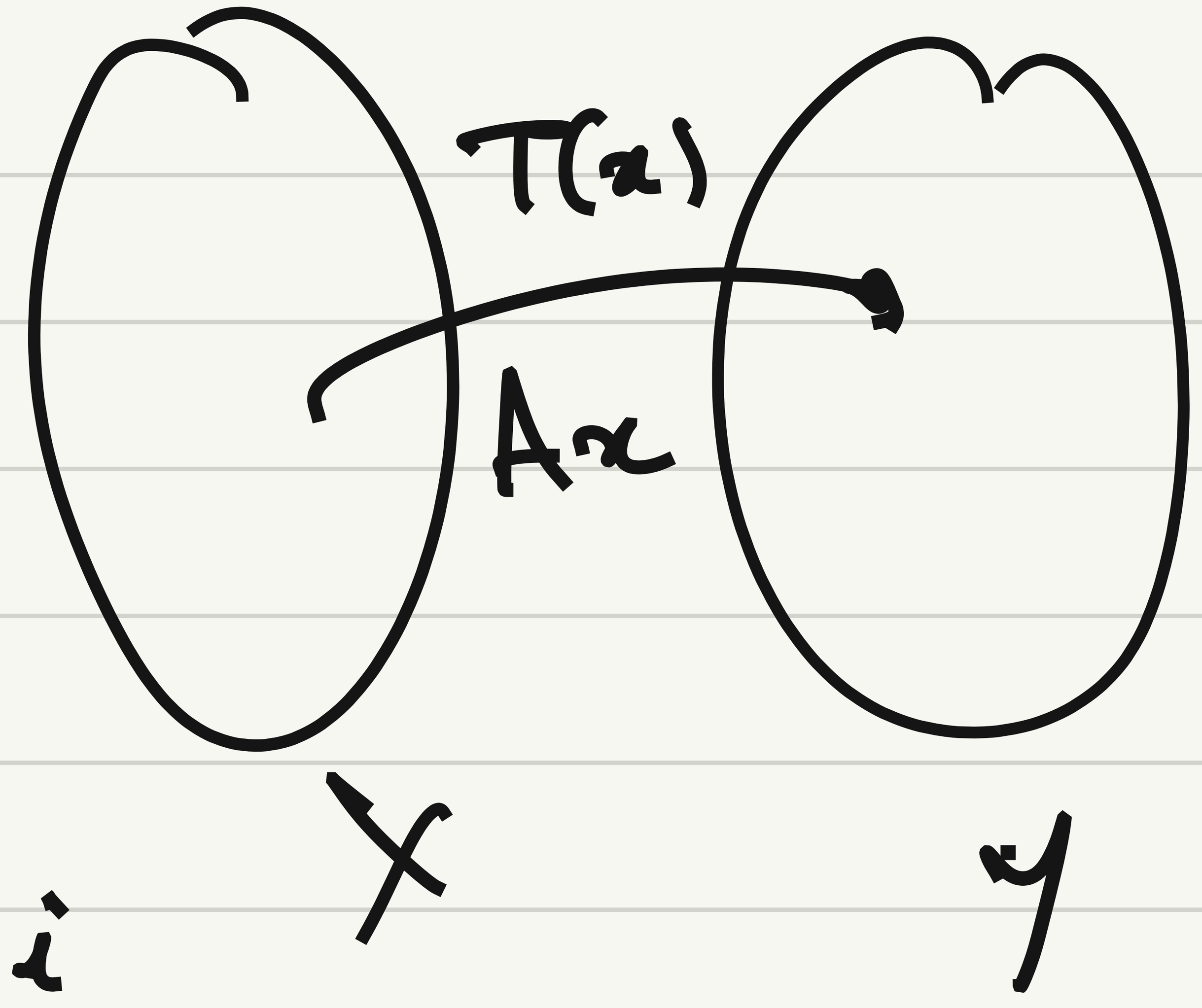
$$T(s_1 u_1 + \cdots + s_n u_n) = s_1 T(u_1) + \cdots + s_n T(u_n)$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad T(\vec{x}) = \vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

$$T(\vec{x}) = T(x_1\vec{e}_1 + \cdots + x_n\vec{e}_n)$$

$$= x_1 \underbrace{T(\vec{e}_1)}_{\vec{a}_1} + \cdots + x_n \underbrace{T(\vec{e}_n)}_{\vec{a}_n} \qquad \vec{a}_j = T(\vec{e}_j)$$

$$T(\vec{x}) = x_1\vec{a}_1 + \cdots x_n\vec{a}_n = \vec{y}$$



$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} \leftarrow i$$

$$y_i = \underbrace{a_{i1}x_1 + \cdots + a_{in}x_n}_{\text{matrix vector mult}}$$

$$A = \underbrace{\begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \end{bmatrix}}$$

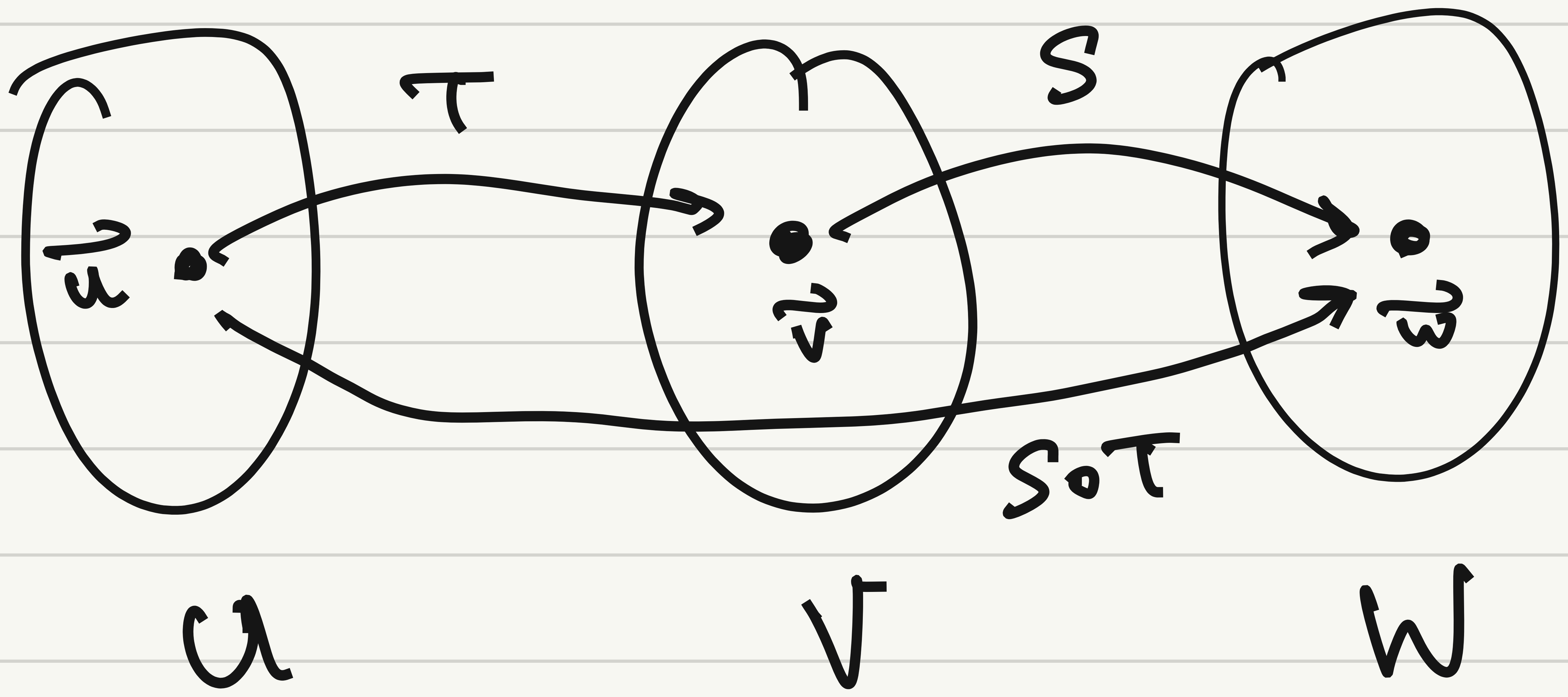
$$\vec{y} = \underbrace{T}(\vec{x}) = \underbrace{A}\vec{x}$$

$$\vec{a}_j = T(\vec{e}_j)$$

Properties of mat-vec mult:

1. $A\vec{x} = x_1\vec{a}_1 + \cdots + x_n\vec{a}_n$: $A\vec{x}$ is <u>lin. comb.</u> of columns of A

2. jth column of A = $A\vec{e}_j$



$T(\vec{u}) = B\vec{u}$

$S(\vec{v}) = A\vec{v}$

$$\begin{aligned}(S\circ T)(\vec{u}) &= S(T(\vec{u}))\\ &= S(B\vec{u})\\ &= A(B\vec{u})\\ &= \underline{(AB)}\,\vec{u}\end{aligned}$$

$$\underbrace{(S\circ T)\vec{u} = C\vec{u}}_{c_{ij} = \sum a_{ik} b_{kj}}$$

$$C = \begin{bmatrix} \vec{c}_1 & \vec{c}_2 & \cdots & \end{bmatrix}$$

$$C\vec{x} = A(B\vec{x})$$

choose $\vec{x} = \vec{e}_j \Rightarrow Ce_j = \vec{c}_j$

$$= A(Be_j) = A\vec{b}_j$$

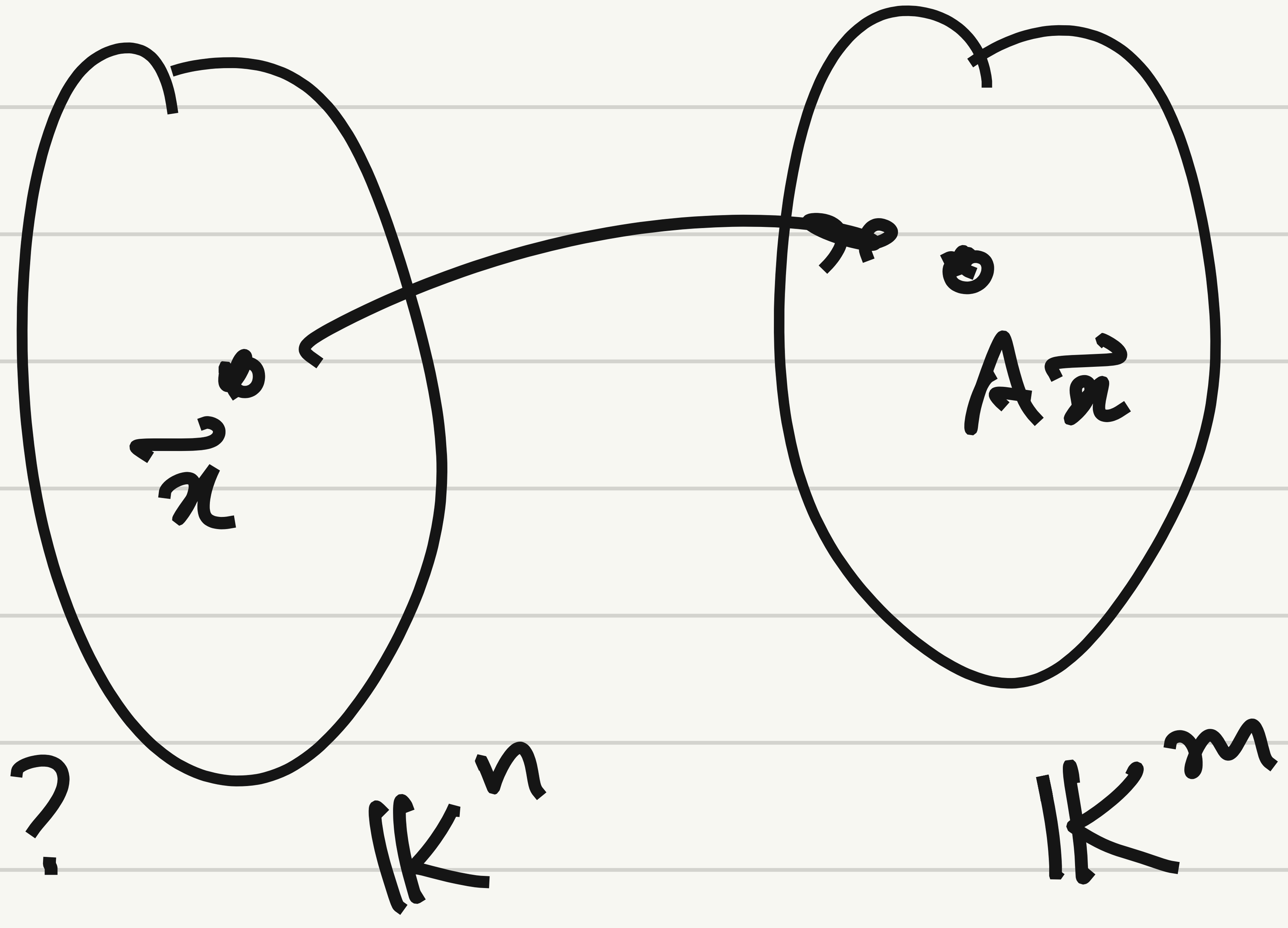
$$= \underbrace{\begin{bmatrix} A\vec{b}_1 & A\vec{b}_2 & \cdots & A\vec{b}_n \end{bmatrix}}$$

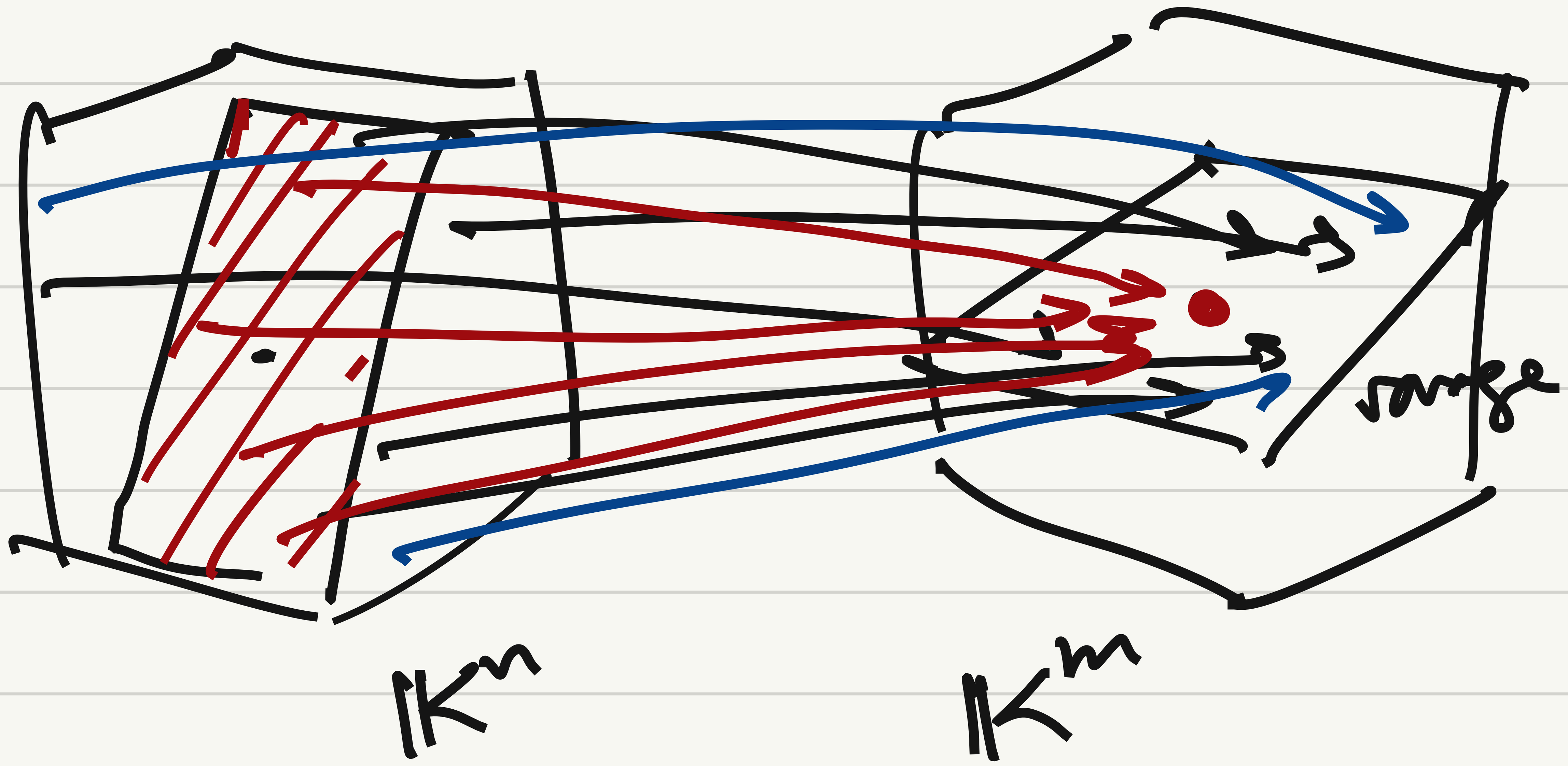
$$\Leftrightarrow \quad c_{ik} = \sum a_{ij} b_{jk}$$

$$\begin{bmatrix} \vec{c}_1 \\ \hline \vec{c}_2 \\ \hline \vdots \\ \hline \end{bmatrix} = \begin{bmatrix} a_1 B \\ \hline a_2 B \\ \hline \vdots \\ \hline a_m B \end{bmatrix}$$

$A \in \mathbb{K}^{m \times n}$

What is the range of the fn $\vec{x} \mapsto A\vec{x}$?





$\boxed{\text{range}}$ of A $= \{ A\vec{x} : \vec{x} \in \mathbb{K}^n \} = \{ \text{lin. comb. of } \vec{a}_1, \dots, \vec{a}_n \}$

$= \text{span} \{ \vec{a}_1, \dots, \vec{a}_n \} = \boxed{\text{column space}}$ of A

range(A)

col(A)

$\boxed{\text{null space}} = \{ \vec{x} : A\vec{x} = \vec{0} \}$

Solve $A\vec{x} = \vec{b}$. ① for which $\vec{b}$ does sol. exist? ② given one sol. $\vec{x}$. what are all other sol?

Subspace = Subset that is also vector space

$S \subseteq V$, $\quad x, y \in S \Rightarrow x+y \in S$

$ax \in S$ for any $a \in \mathbb{K}$