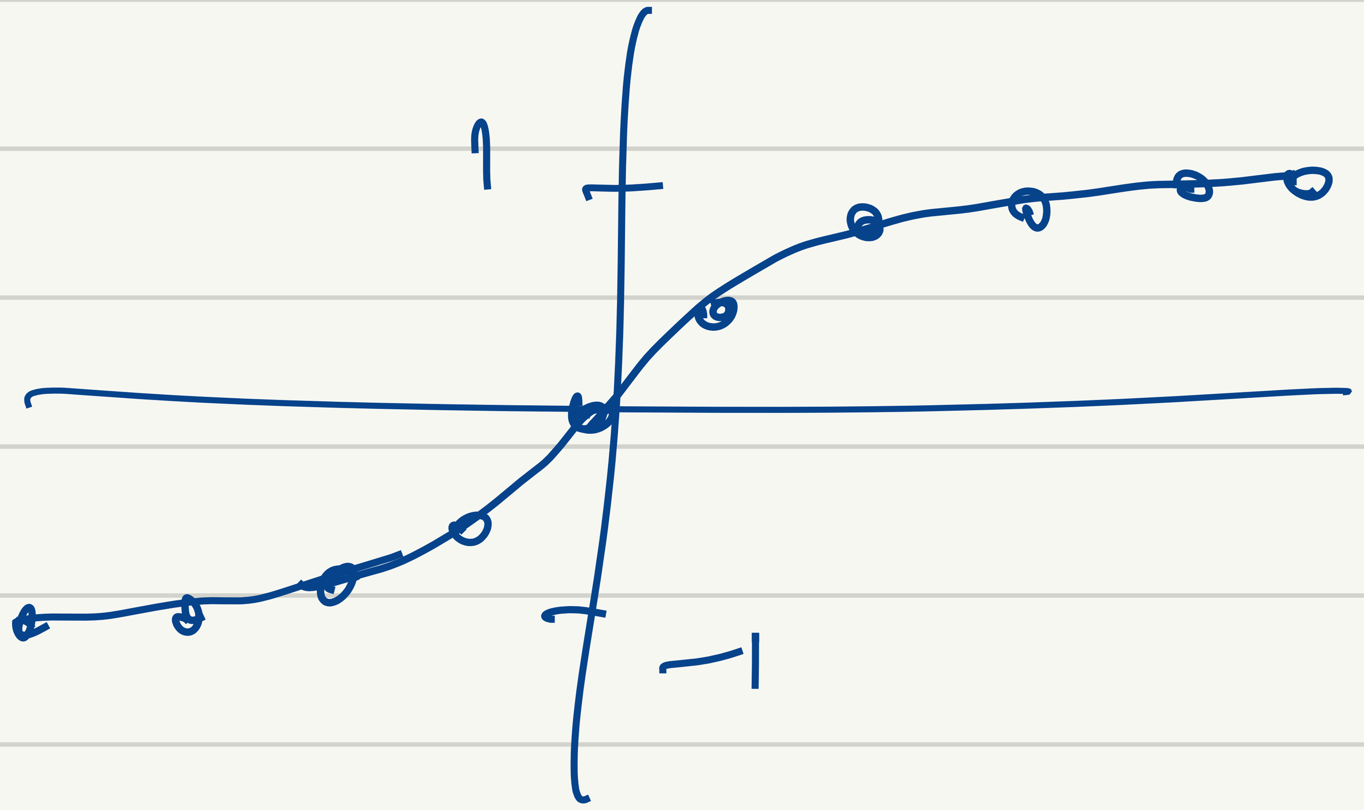
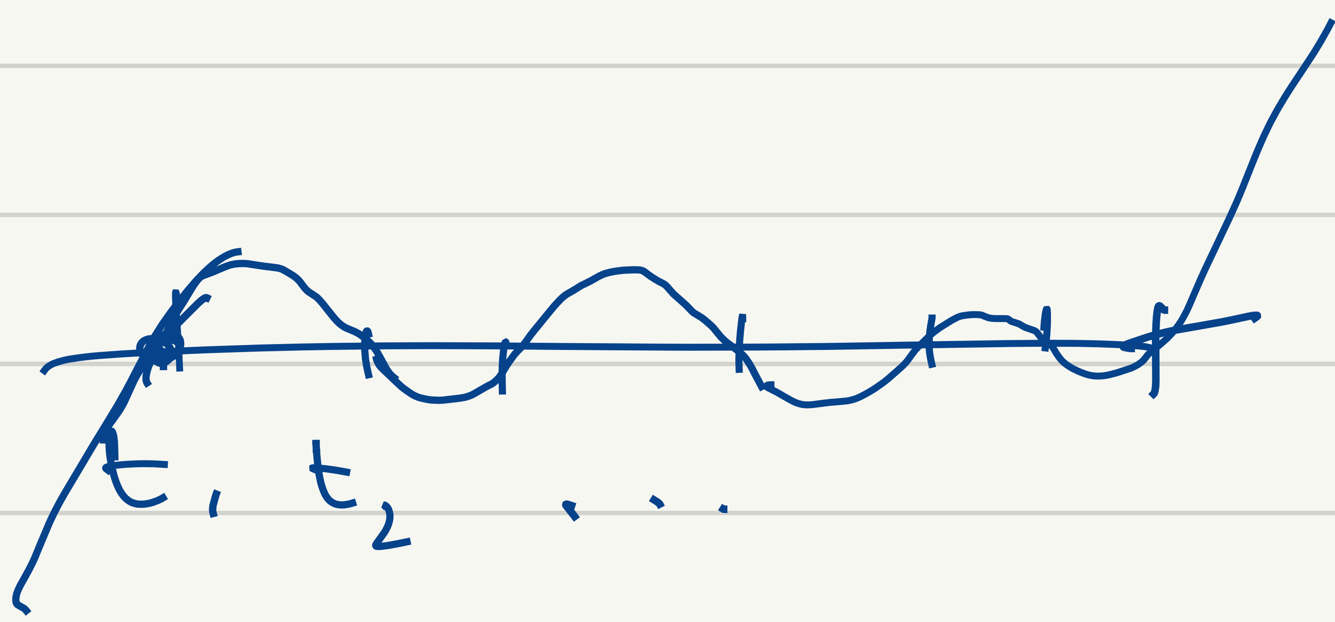
Polynomial Interpolation

$f : \mathbb{R} \to \mathbb{R}$ $\qquad f(x) = \dfrac{e^x - 1}{e^x + 1}$



We're approximating f with degree-$(n-1)$ polynomial p.



$$w(t) = \prod_{i=1}^{n} (t - t_i)$$

Degree $(n-1)$ poly $\iff$ nth derivative $\equiv 0$

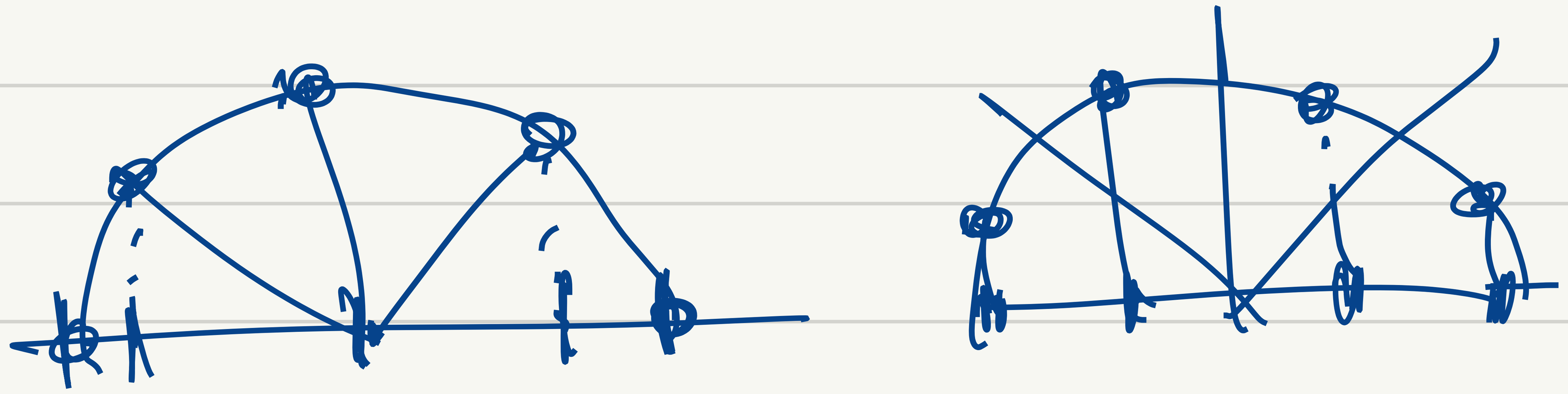
$$M_n = \max_{t \in [a,b]} |f^{(n)}(t)| \quad \longrightarrow \quad |r(t)| \le \frac{M_n}{n!} |w(t)|$$

for uniformly spaced t_i, $\le \dfrac{M_n h^n}{4n}$

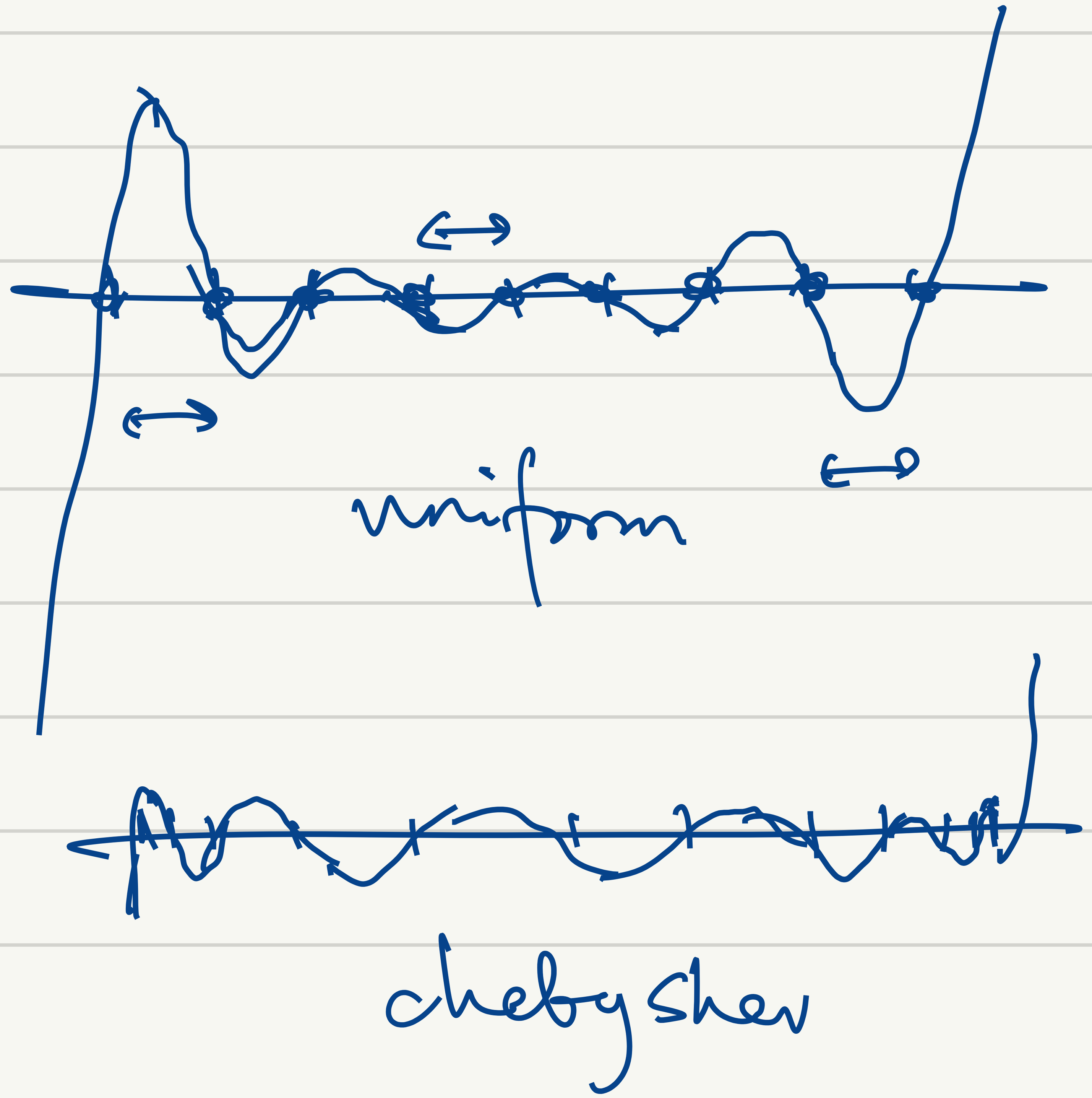
Runge's phenomenon

Runge's function $f(x) = \dfrac{1}{1 + 25x^2}$ on $[-1, 1]$

<u>Moral</u>: <u>High-order</u> <u>polynomial</u> interpolation on <u>uniformly spaced points</u> is a bad idea.



$$w(t) = \prod (t - t_i)$$



<u>Chebyshev points</u>: Take θ_i uniformly in $[0, \pi]$,

$t_i = \cos\theta_i \longrightarrow t_i \in [-1, 1]$

Zeros or extrema of Chebyshev polynomials $\Rightarrow$ makes $w(t)$ as small as possible

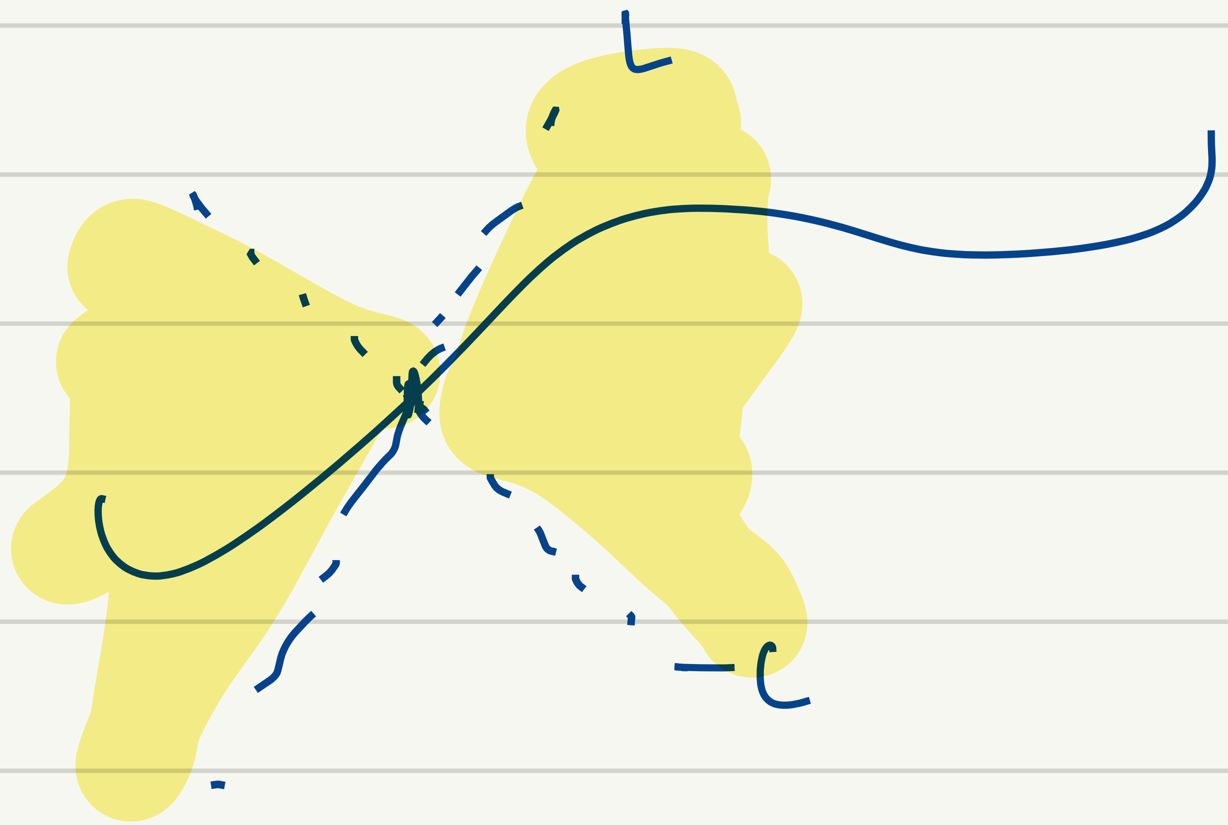
Chebyshev points eliminate Runge's phenomenon!

$|r(t)| \to 0$ as $n \to \infty$ for any Lipschitz continuous f

$\exists\, L \geq 0$ s.t.

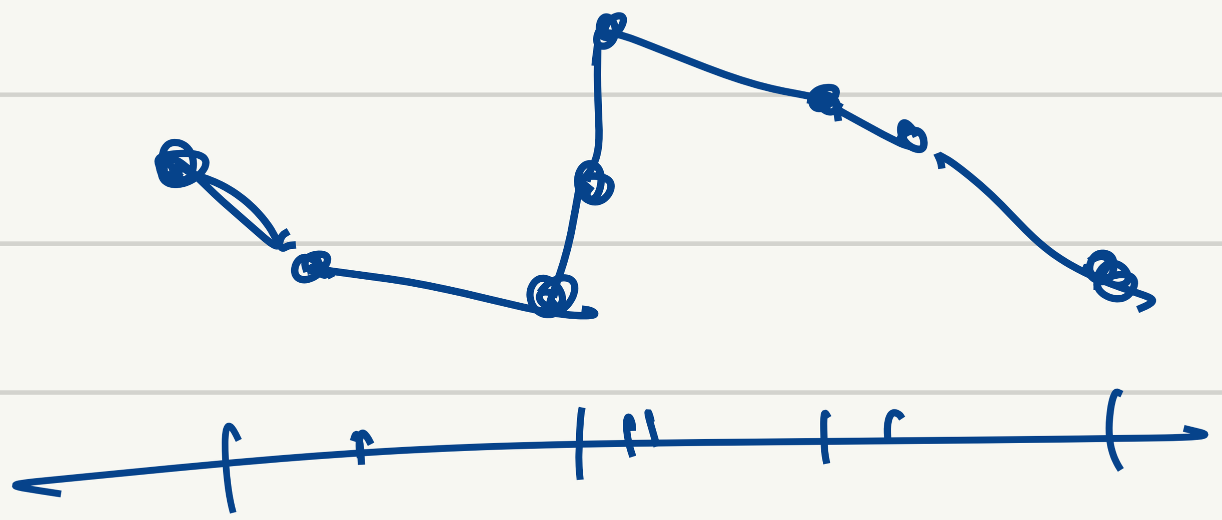
$$|f(t_1) - f(t_2)| \leq L\,|t_1 - t_2|$$

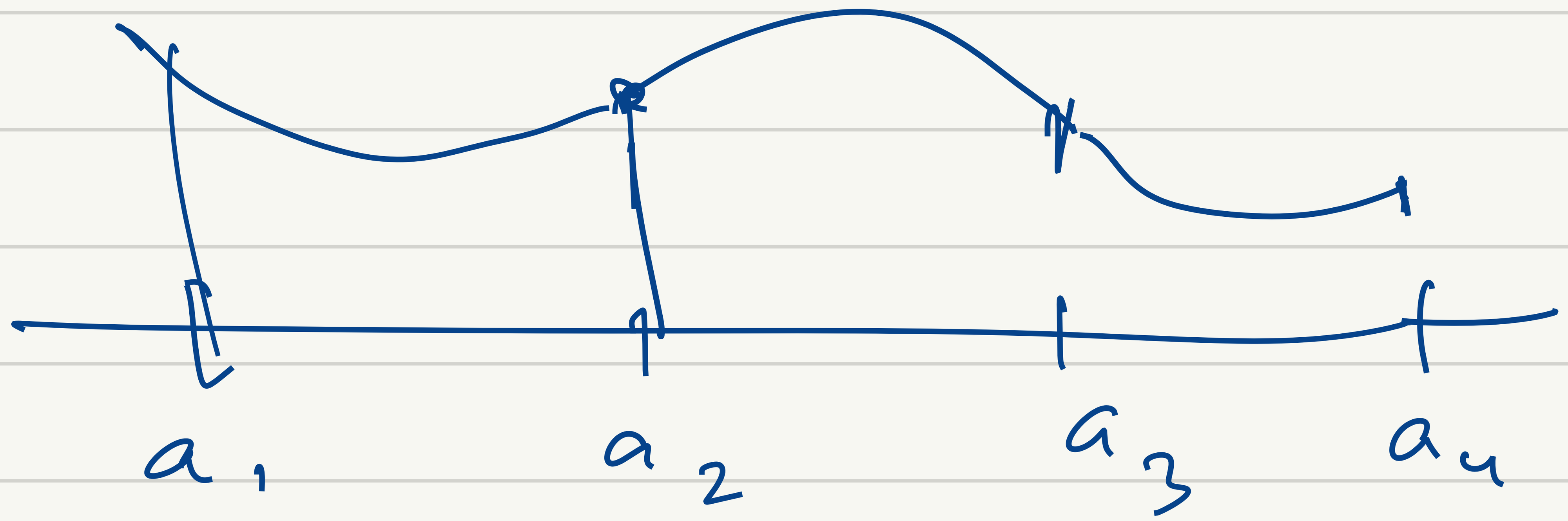
$$t \to \boxed{f} \to \left\{\begin{array}{c}(t_1, y_1)\\ (t_2, y_2)\\ \vdots \\ (t_n, y_n)\end{array}\right\}$$



low-degree polynomials glued together

$\Rightarrow$ $\boxed{\text{piecewise polynomial interpolation}}$



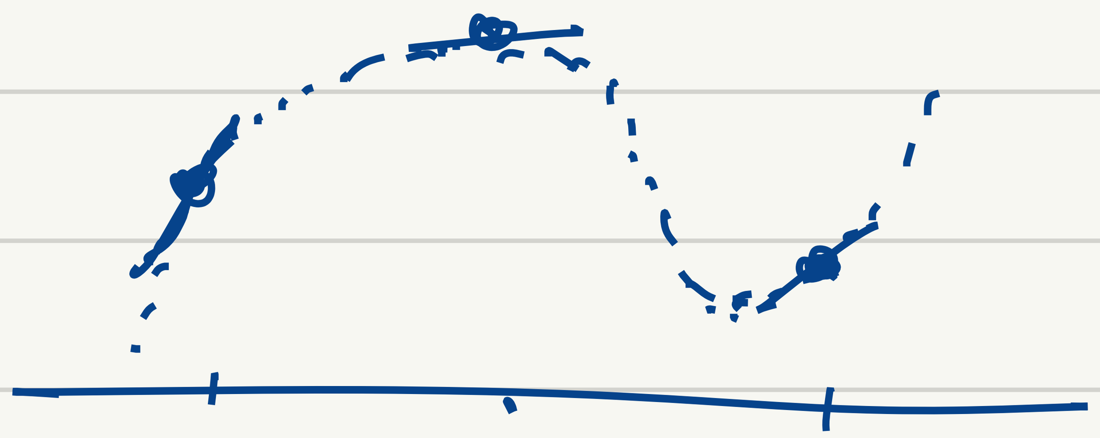


Separate polynomial $p_i(t)$
on each interval $[a_i, a_{i+1}]$

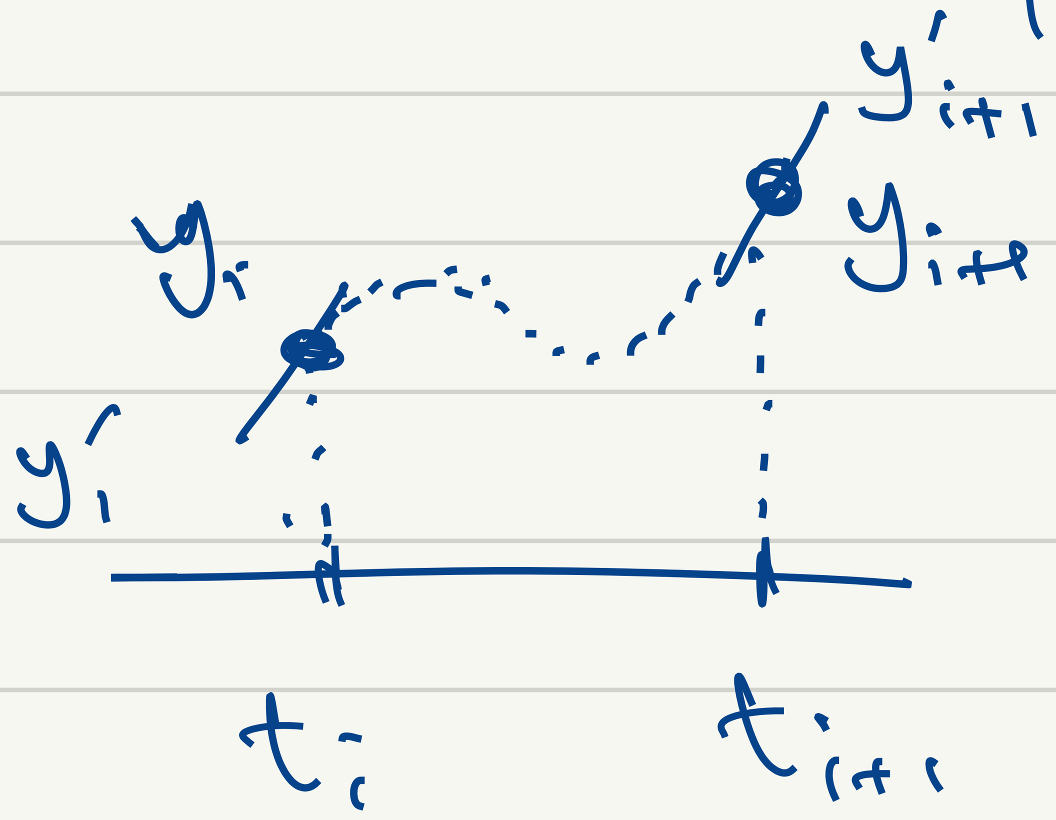
a_i are called $\boxed{\text{knots}}$ or $\boxed{\text{breakpoints}}$

we'll just choose $a_i = t_i$

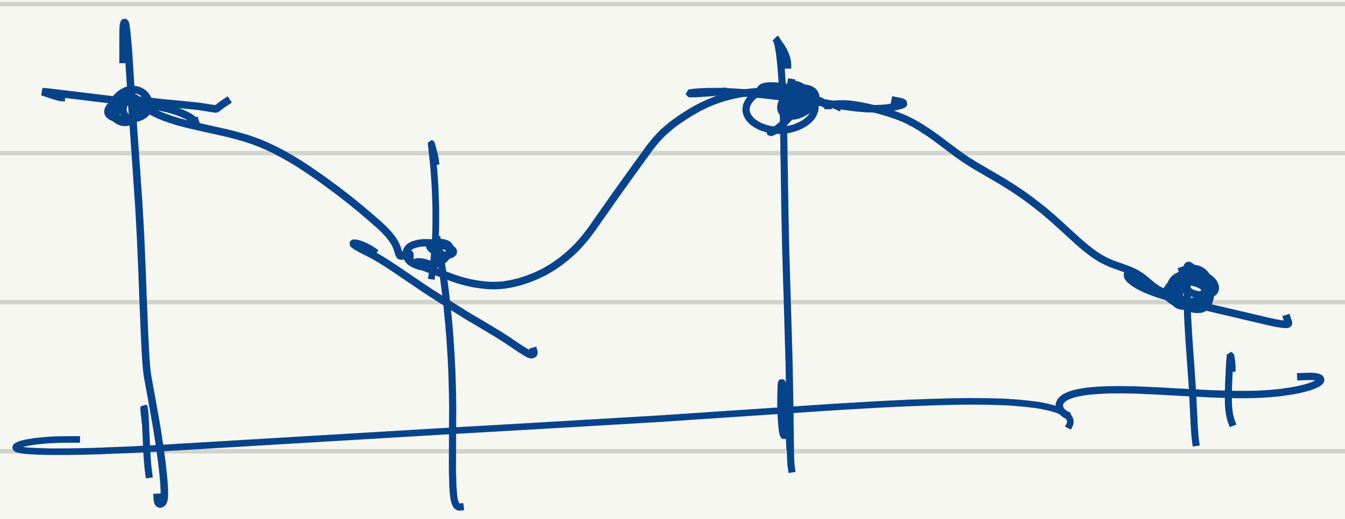
$\boxed{\text{Hermite interpolation}}$: suppose $(t_i, y_i, y_i', \dots, y_i^{(k)})$



$(k+1)n$ equations $\Rightarrow$ degree $(k+1)n-1$ polynomial



fit a cubic polynomial on each interval



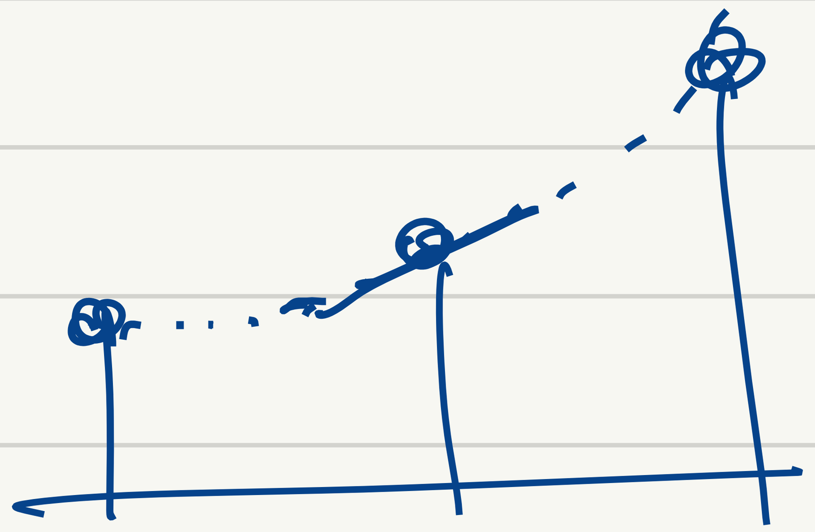
$\tilde{f}$ is continuous and once differentiable

Piecewise cubic Hermite interpolation

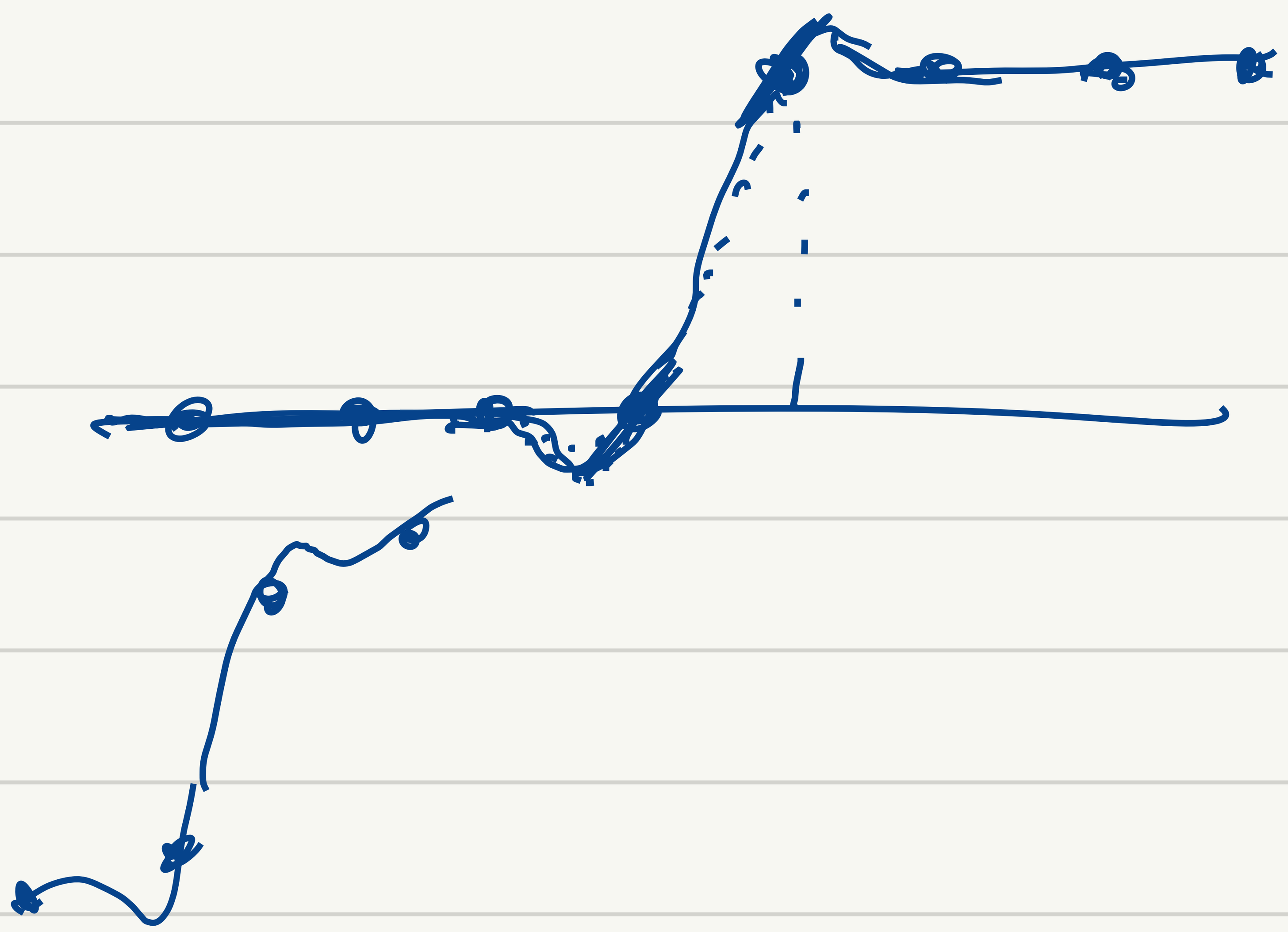
what if you don't have y_i'?

n free parameters. How to choose?

- Estimate from data e.g. fit parabola to y_{i-1}, y_i, y_{i+1}, take slope at t_i



- Akima interpolation: more robust to outliers, irregular point distrib. [Akima 1970]

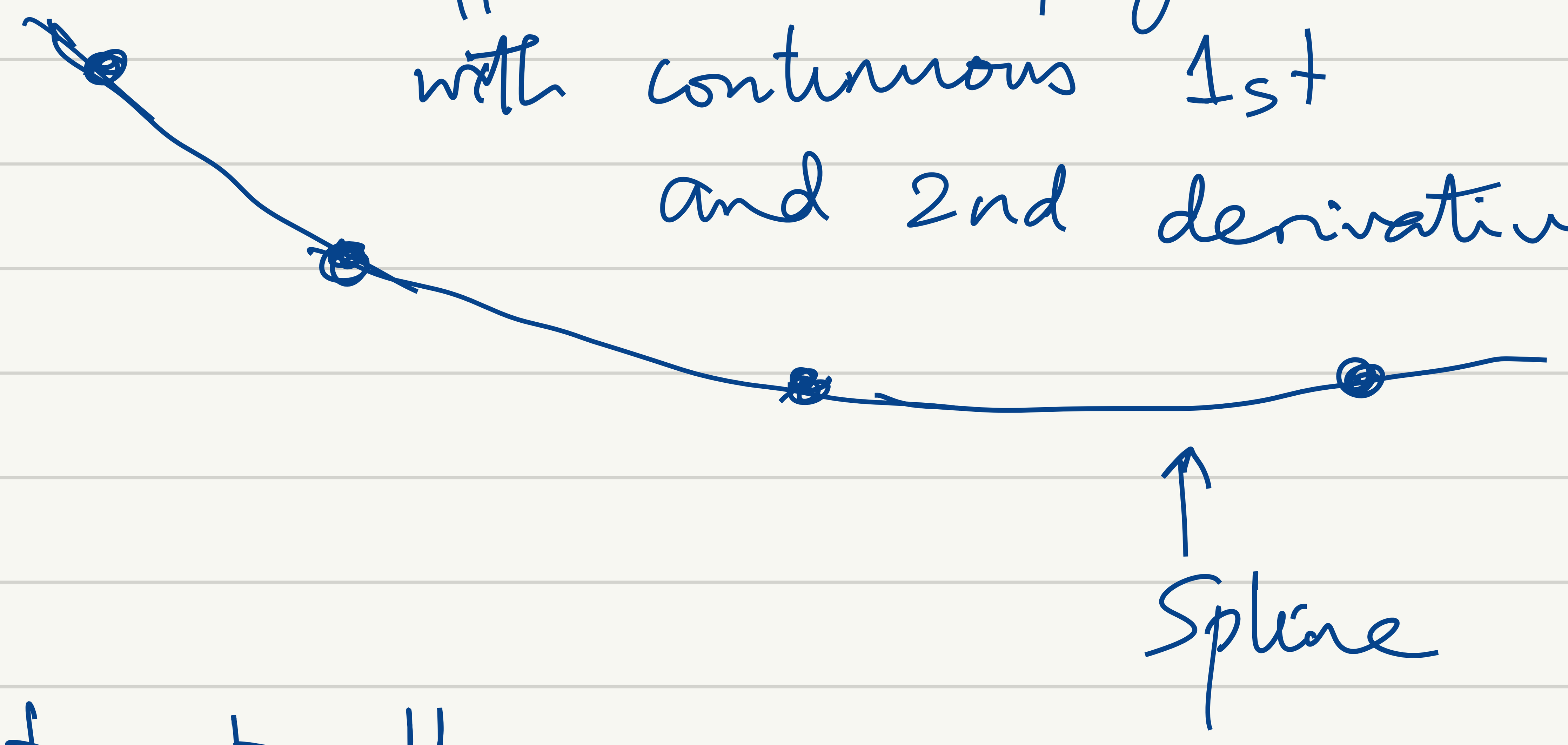


- Monotonic cubic interp.: [Fritsch & Carlson 1980]
- <u>Cubic spline</u>: maximizes smoothness

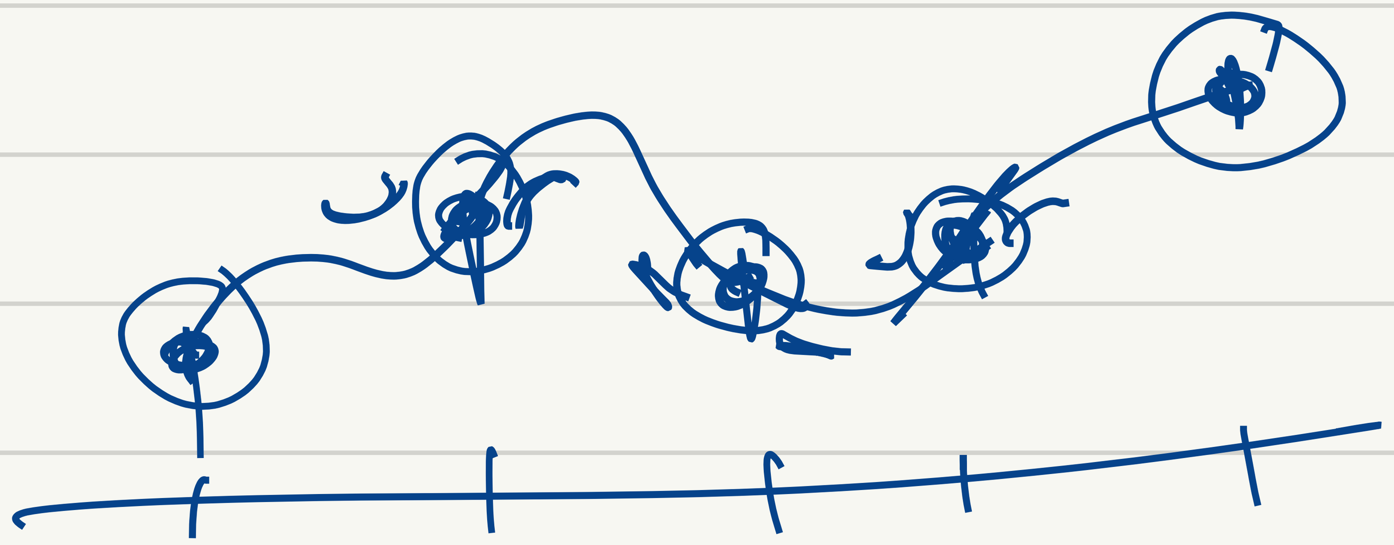
Cubic splines

$\boxed{\text{Spline}}$: degree k piecewise polynomial
with $(k-1)$ continuous derivatives

approx. cubic polynomial
with continuous 1st
and 2nd derivatives



Piecewise cubic Hermite : 1 continuous deriv automatically
make 2nd deriv continuous



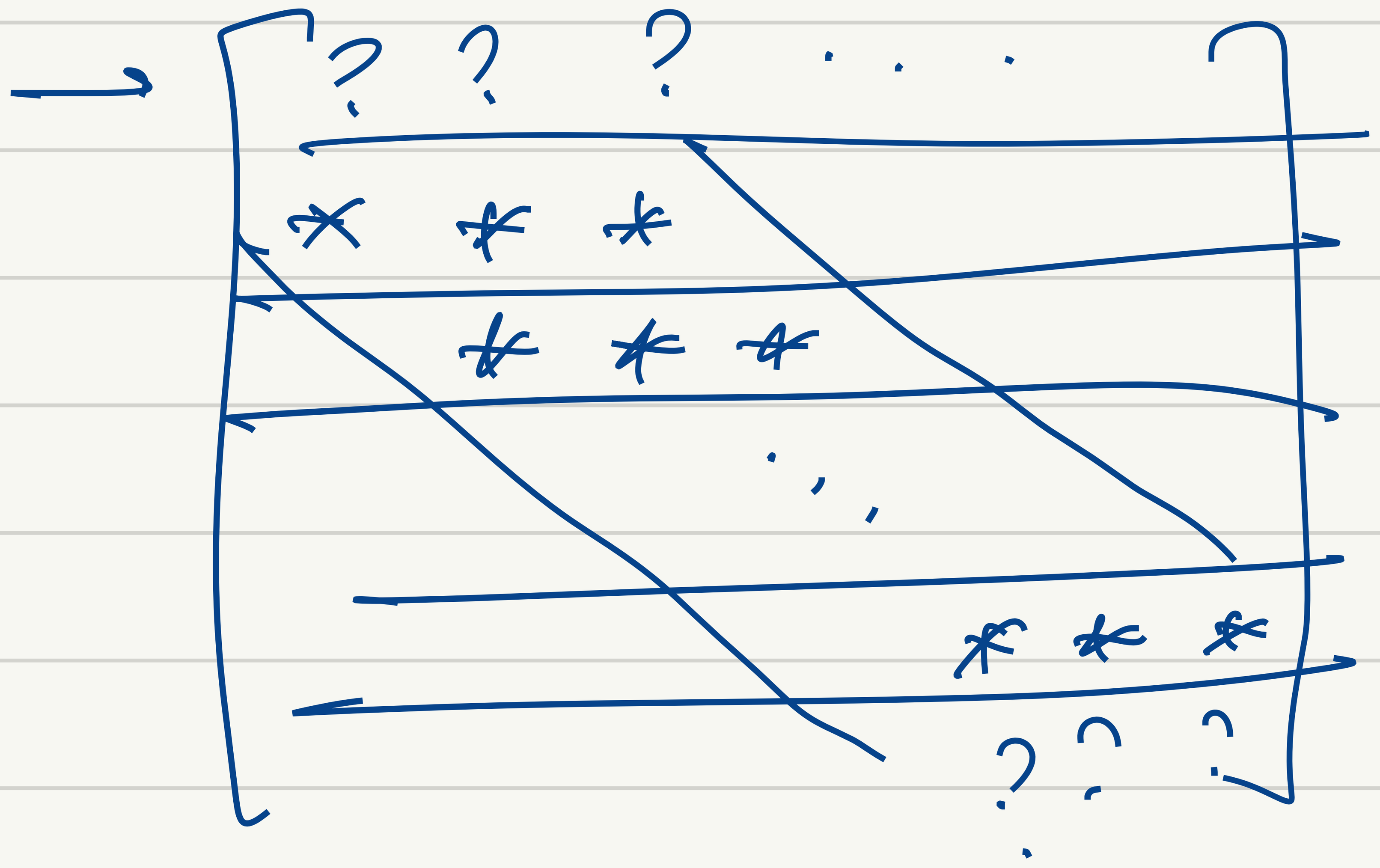
$(n-2)$ equations $\Rightarrow$ 2 free params
Add boundary conditions to fix unique sol.

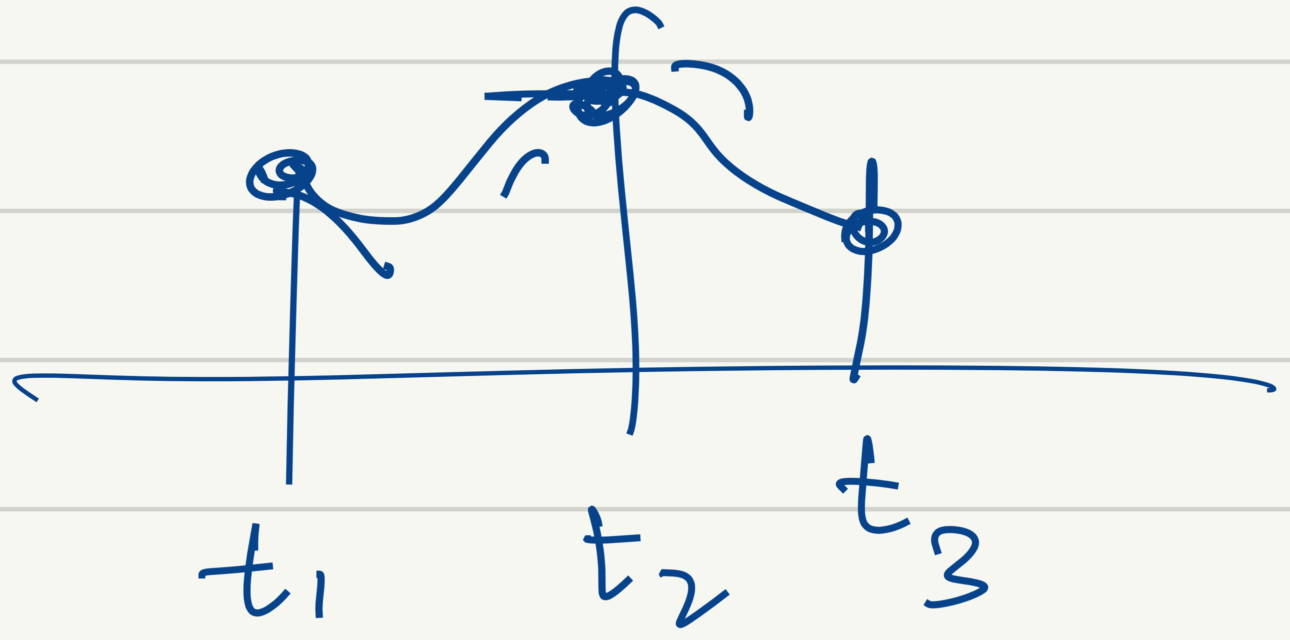
- Natural spline : 2nd derivative $= 0$ at t_1, t_n

n unknowns $(y'_1, y'_2, \ldots, y'_n)$

n equations $(BC_1,\ f''_{2-} = f''_{2+},\ \ldots,\ f''_{(n-1)-} = f''_{(n-1)+},\ BC_n)$

depends on y'_1, y'_2 $\qquad y'_2, y'_3$





$\rightarrow$ get $y'_1, \ldots, y'_n$

$\rightarrow$ piecewise cubic Hermite

Summary: Have f or $\{(t_i, y_i)\}$ $\longrightarrow$ piecewise poly

↳ chebyshev

Numerical Integration and Differentiation

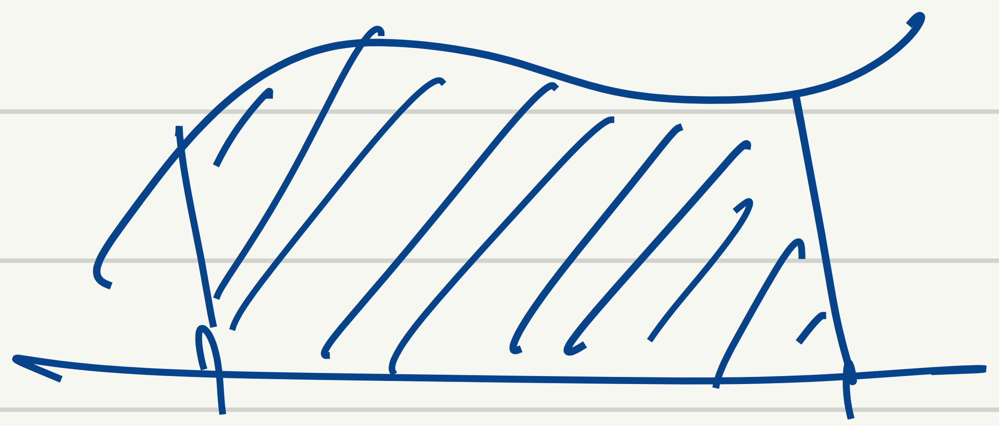
$f : \mathbb{R} \to \mathbb{R}$. find $\int_a^b f(x)\,dx$ or $\frac{df}{dx}(x)$

Computer Algebra System (CAS)

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx \stackrel{?}{=} \sqrt{\pi}$$

$\int_a^b e^{-x^2}\,dx$: no closed form

$t \to \boxed{f} \to f(t) \to$



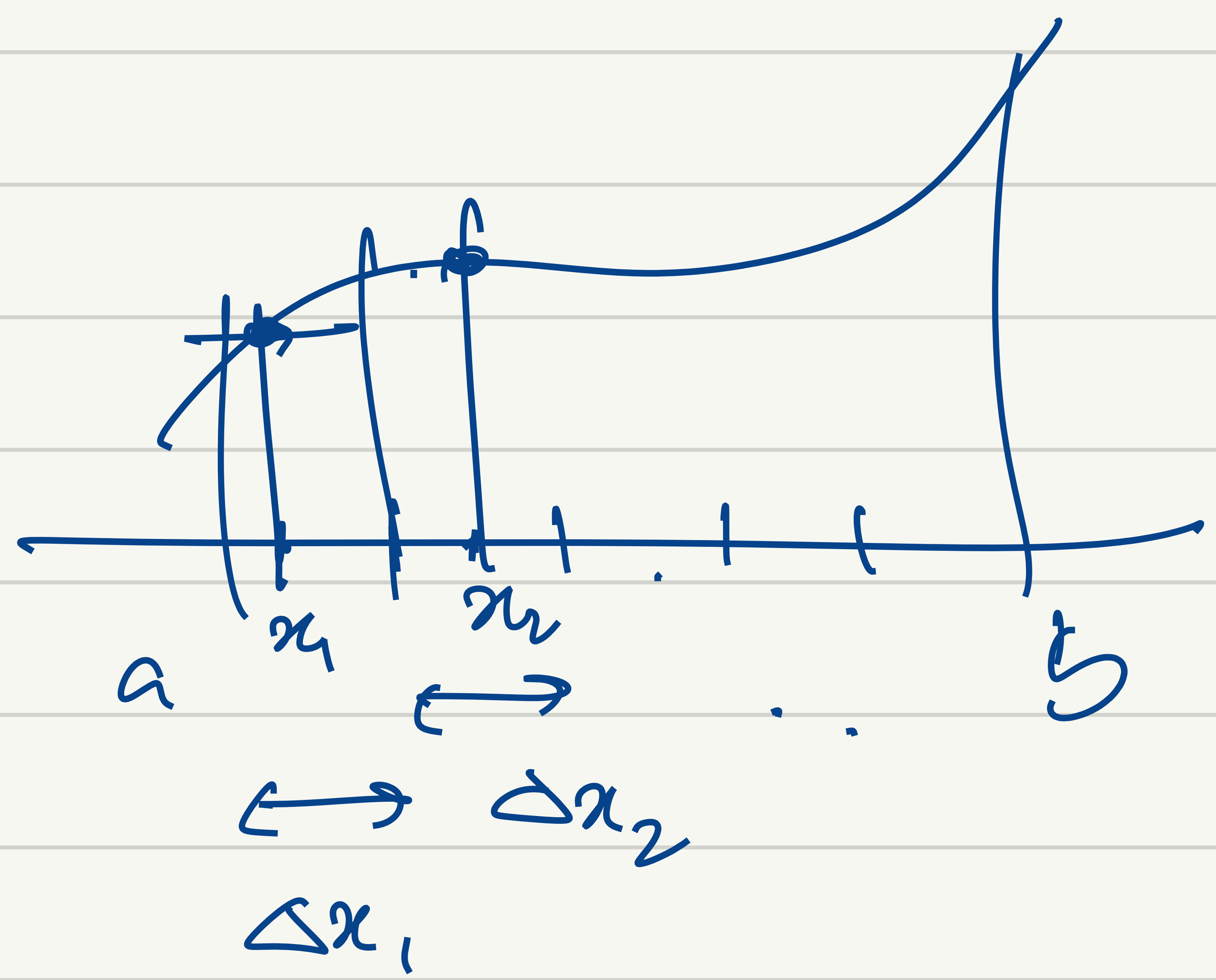
$[a, b]$

Compute $I = \int_a^b f(x)\,dx$

$\boxed{\text{Numerical integration}}$ a.k.a. $\boxed{\text{quadrature}}$

Existence, uniqueness, conditioning

$$I(f) = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x_i$$



Existence : iff f is bounded, continuous a.e.

Uniqueness : By def.

Conditioning

$$I(f, a, b)$$

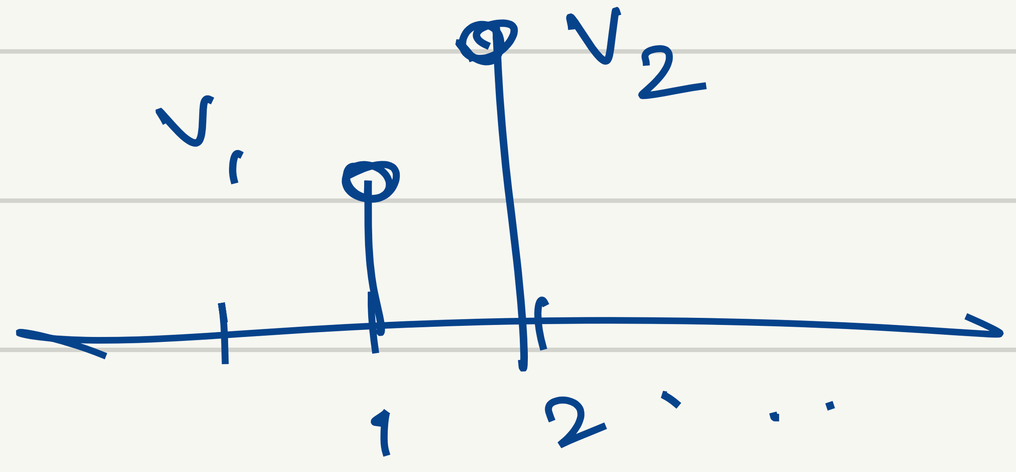
$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

Need norm on f.

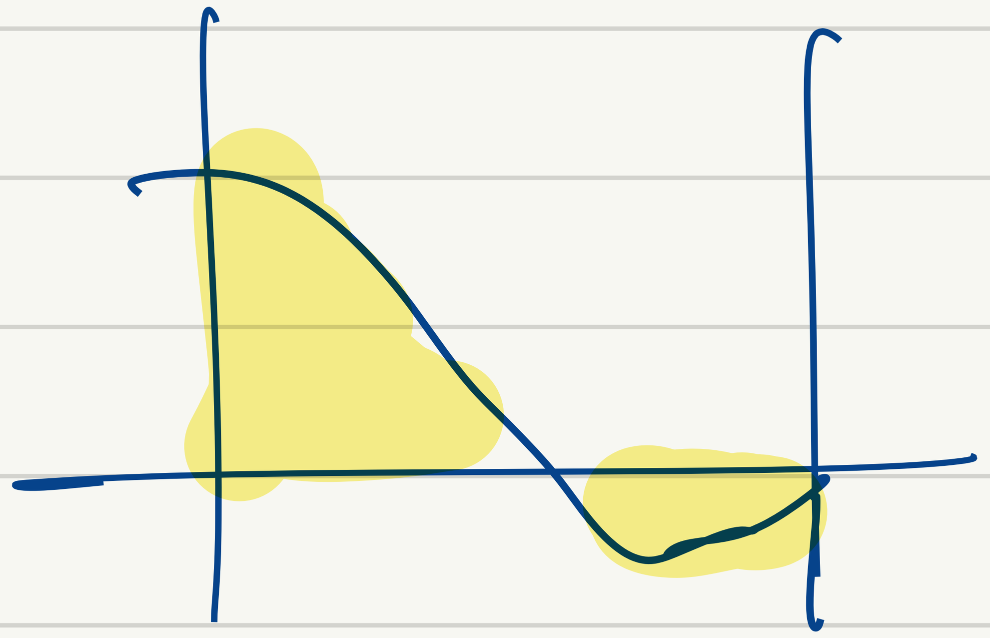
Simplest choice: $\|f\|_\infty = \max_{x\in[a,b]} |f(x)|$

$v : \{1, \ldots, n\} \to \mathbb{R}$

$\|f\|_2 = \sqrt{\int_a^b |f(x)|^2\,dx}$, etc.



$$I = \int_a^b f$$



$$\int_a^b (f + \delta f) = \tilde{I} = I + \delta I$$

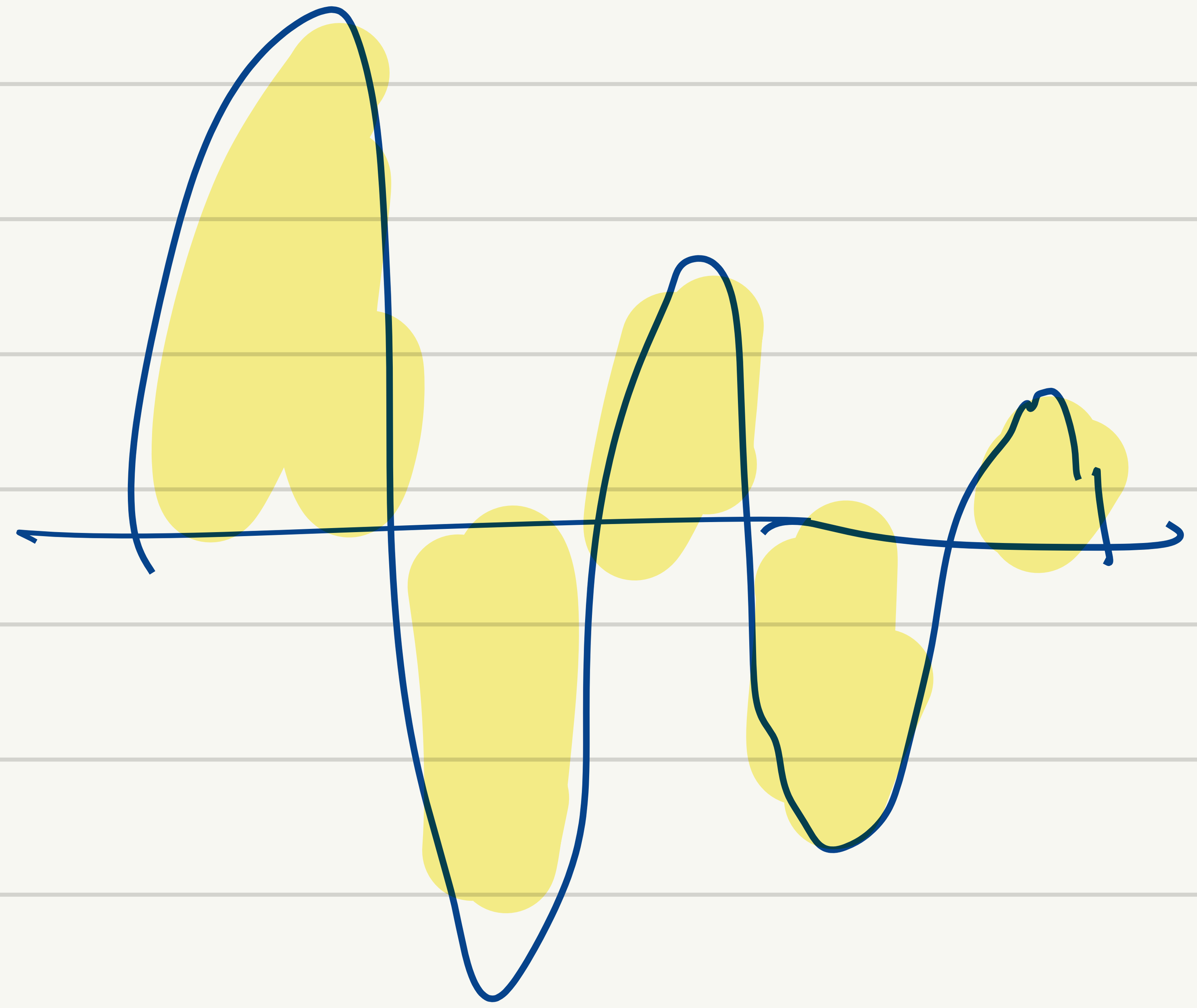
$$|\delta I| = \left| \int f + \delta f - \int f \right| = \left| \int \delta f \right| \le \int \underbrace{|\delta f|}_{} \le \int_a^b \|\delta f\|_\infty$$

$$= (b-a) \|\delta f\|_\infty$$

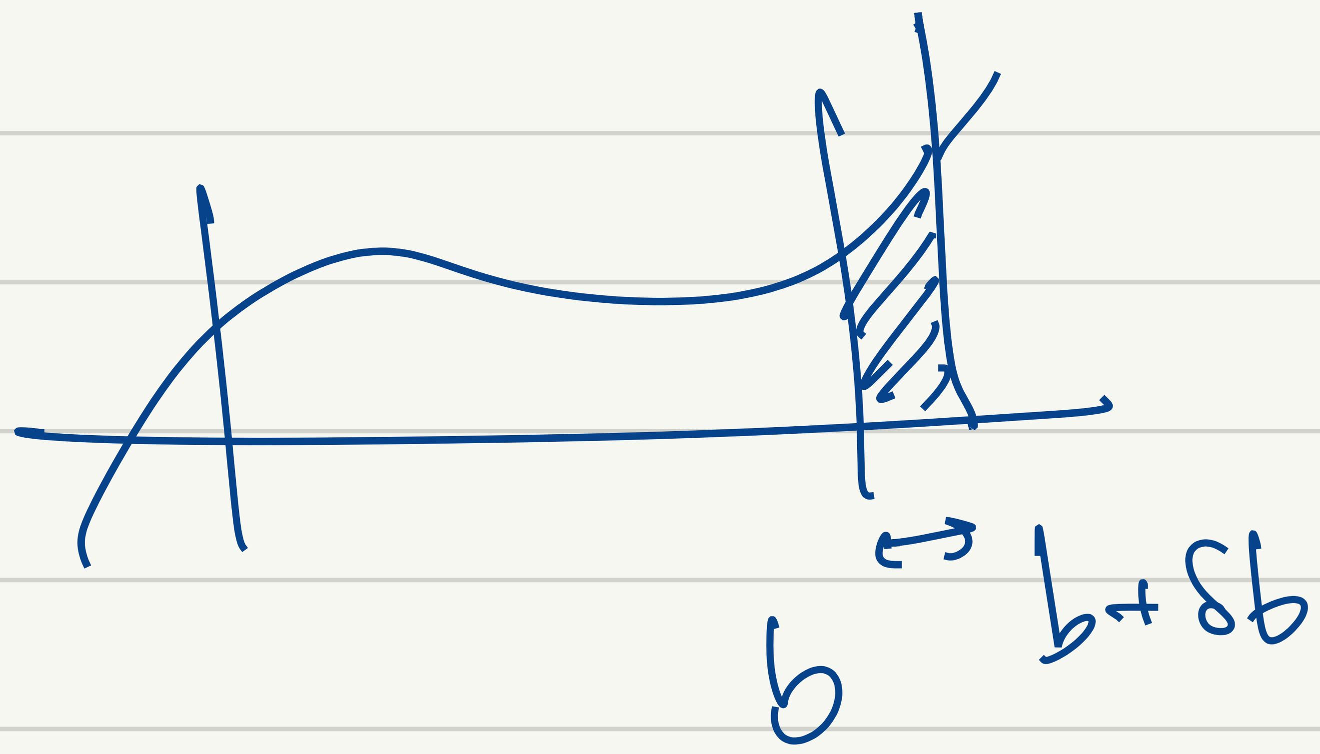
$$\left[\ \|\delta f\|_\infty = \max |\delta f| \ge |\delta f(x)| \ \right]$$

$\Rightarrow$ abs. cond. no. of I $= (b-a)$ w.r.t. ∞-norm

rel. cond. no. $= \boxed{(b-a) \cdot \dfrac{\|f\|_\infty}{|I|}}$



Conditioning w.r.t. a, b :



$$\delta I = f(b)\,\delta b$$