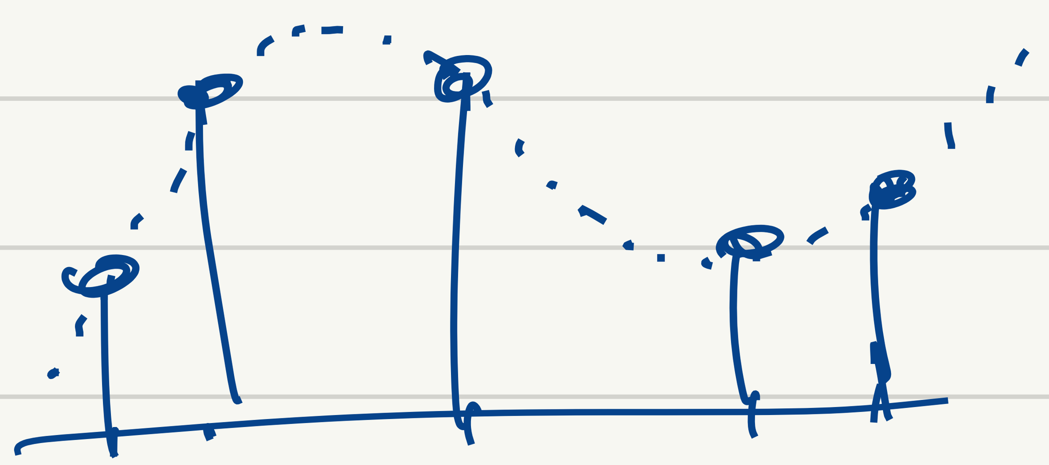
COL726

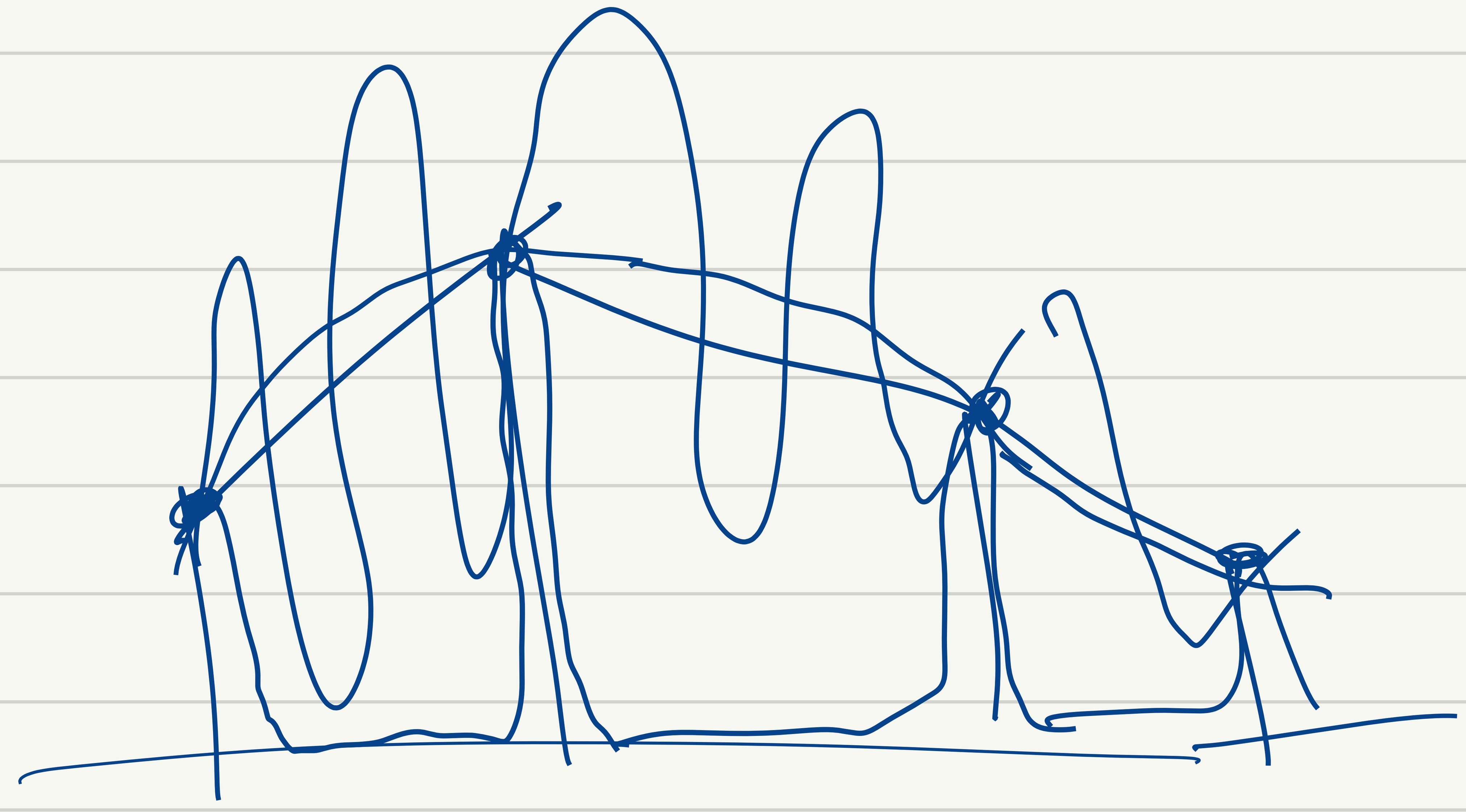
- A4 updated
- Deadline: Mon, 21 March

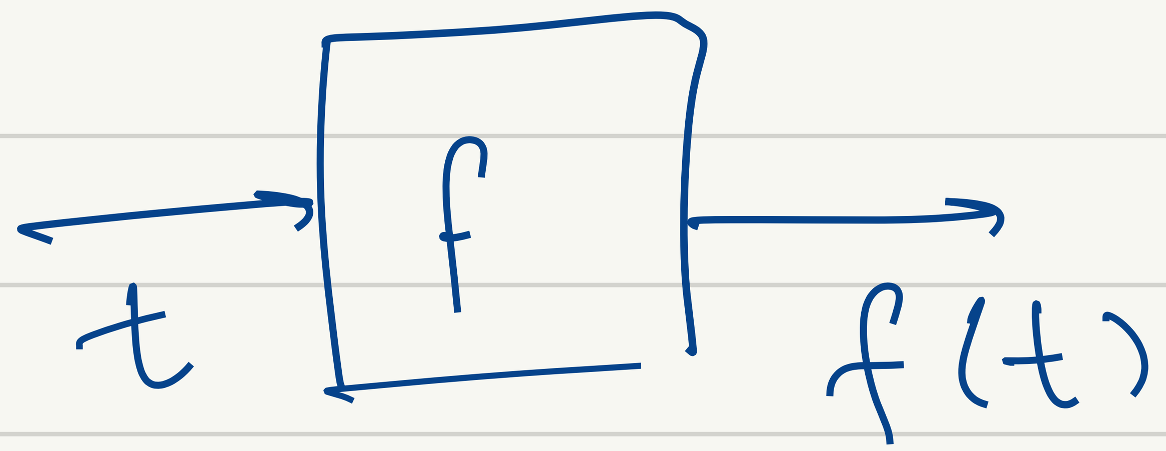
Interpolation



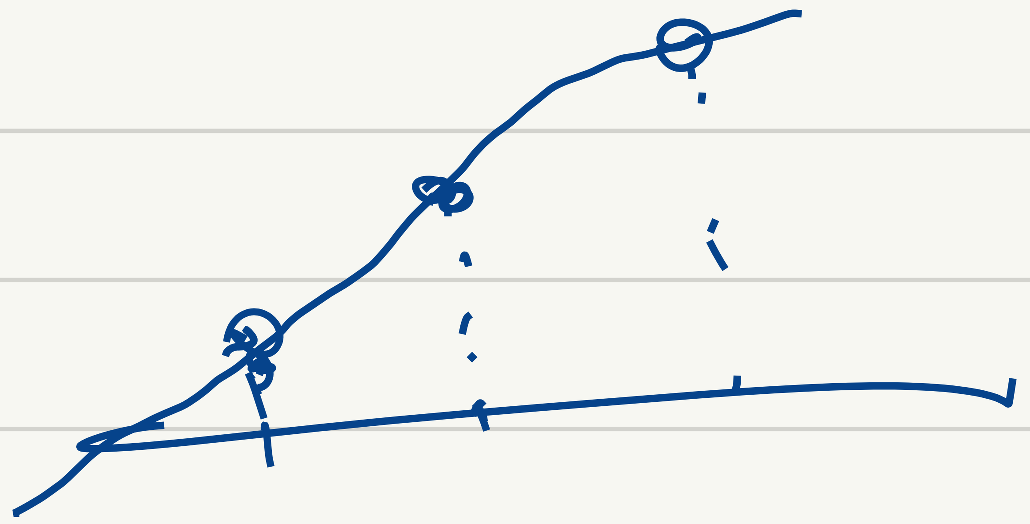
$(t_1, y_1), (t_2, y_2), \ldots$

find $\tilde{f}$ s.t. $\tilde{f}(t_i) = y_i$ for all i





$\tilde{f} \approx f$



$\boxed{\text{interpolating function}}$

or $\boxed{\text{interpolant}}$

① what is a good choice of t_i?

② what is a good choice of function?

$\tilde{f}$ should be easy to compute and easy to evaluate

$$\boxed{\begin{matrix}(t_1, y_1)\\ (t_2, y_2)\\ \vdots\end{matrix}} \xrightarrow[\text{compute}]{} \boxed{\tilde{f}} \xrightarrow[\text{evaluate}]{} \boxed{\tilde{f}(t)}$$

$\in \mathbb{R} \times \mathbb{R}$

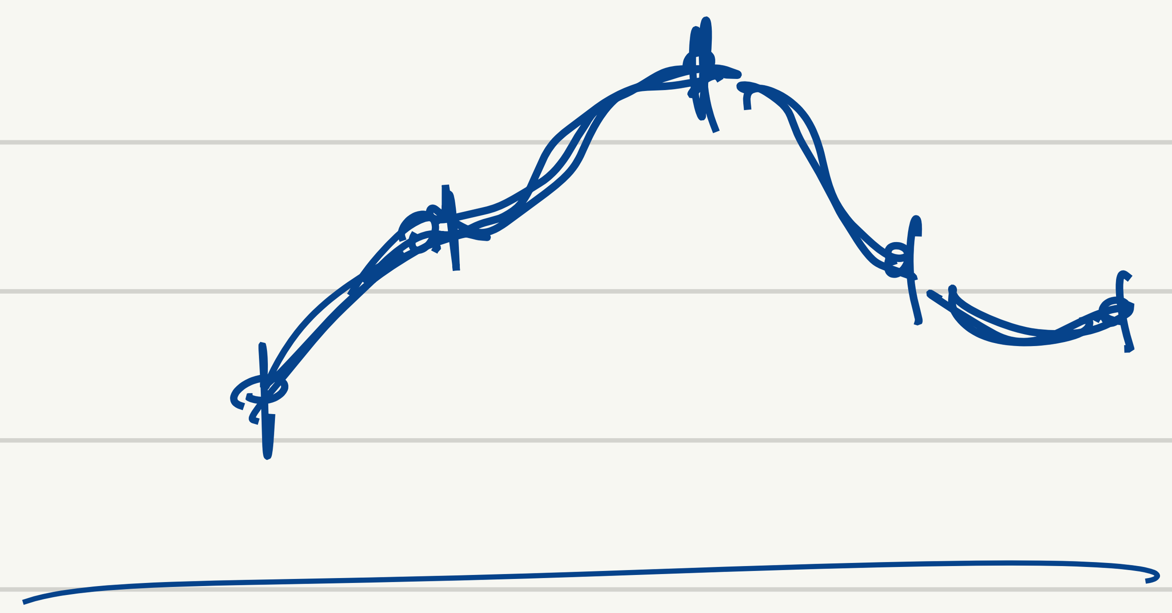
Many choices of $\tilde{f}$

$$\begin{cases} \text{① polynomial} \\ \text{② piecewise polynomial} \end{cases}$$

③ trigonometric

④ rational fns

$$\frac{p(x)}{q(x)}$$



$f : \mathbb{R} \to \mathbb{R}$

(1D interp.)

$f : \mathbb{R} \to \mathbb{R}^n$

$\equiv n \times$ 1D interp

$f : \mathbb{R}^n \to \mathbb{R}$

multivariate interp.

out of scope

Basis function interpolation

choose a set of basis functions $\phi_1, \phi_2, \dots, \phi_n : \mathbb{R} \to \mathbb{R}$

Choose $\tilde{f}$ as linear combination

$$\tilde{f} = x_1 \phi_1 + x_2 \phi_2 + \cdots + x_n \phi_n \iff \tilde{f}(t) = x_1 \phi_1(t) + \cdots$$

m data points $(t_1, y_1), \dots, (t_m, y_m)$

$$y_i = \tilde{f}(t_i) = \sum_{j=1}^{n} x_j \phi_j(t_i) \;:\quad m \text{ linear eqs. in } n \text{ unknowns}$$

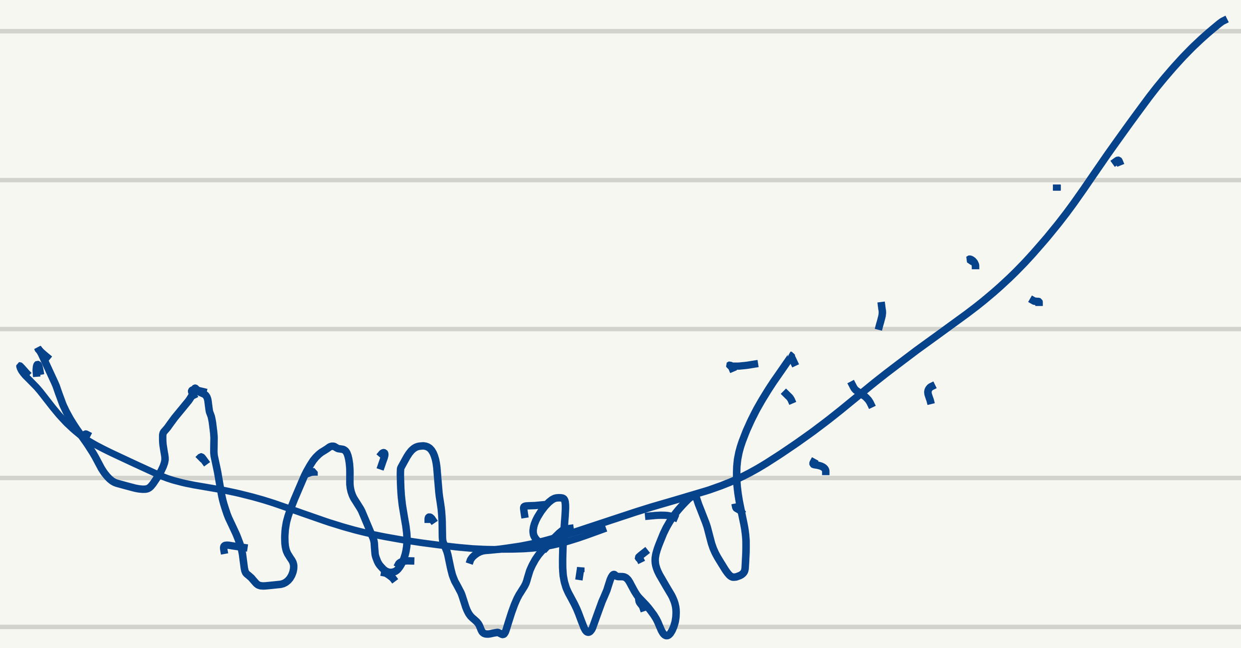
$$y = Ax \quad \text{where} \quad a_{ij} = \phi_j(t_i)$$

If $n = m$ and A is full rank — usually true for most basis fns

$\Rightarrow$ sol. exists & is unique

If $n < m$: least-squares

If $n > m$: underdetermined



<u>Conditioning</u> depend on $\kappa(A)$ which depends on $\phi_j(t_i)$

depends on both ϕ_j and t_i

$\sim = P$

<u>Polynomial interpolation</u> : n data points, $\tilde{f}$ is polynomial of degree $n-1$

Monomial basis : $\phi_1(t) = 1, \ \phi_2(t) = t, \ \phi_3(t) = t^2, \ \dots, \ \phi_j(t) = t^{j-1}$

$$p(t) = x_1 + \overbrace{x_2 t + x_3 t^2 + \cdots + x_n t^{n-1}}$$

$$= x_1 + t(x_2 + t(x_3 + \cdots (t \cdot x_n) \cdots)) \quad : \quad \boxed{\text{Horner's rule}}$$

$\Rightarrow$ evaluate in $O(n)$ time

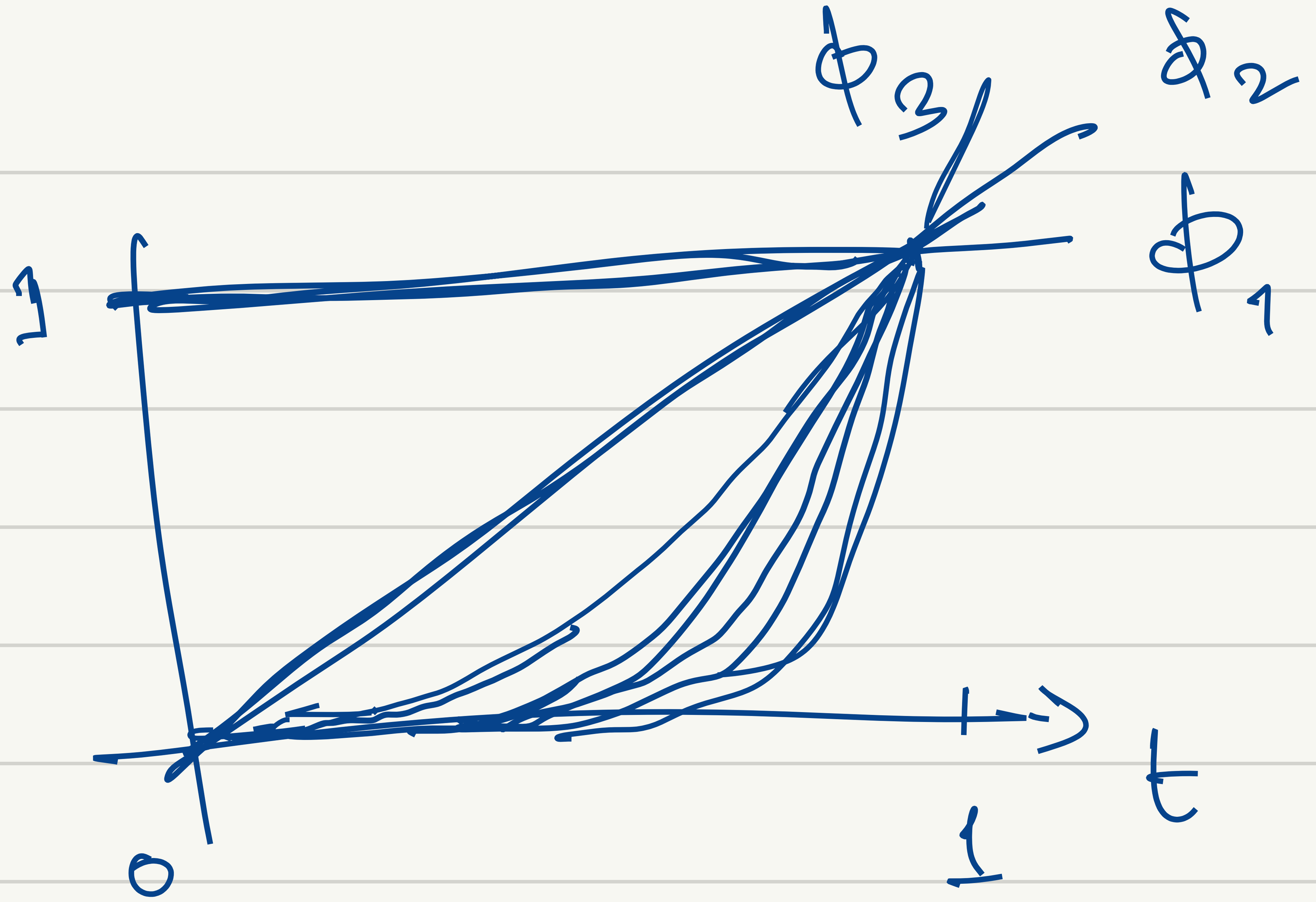
$$a_{ij} = t_i^{j-1}$$

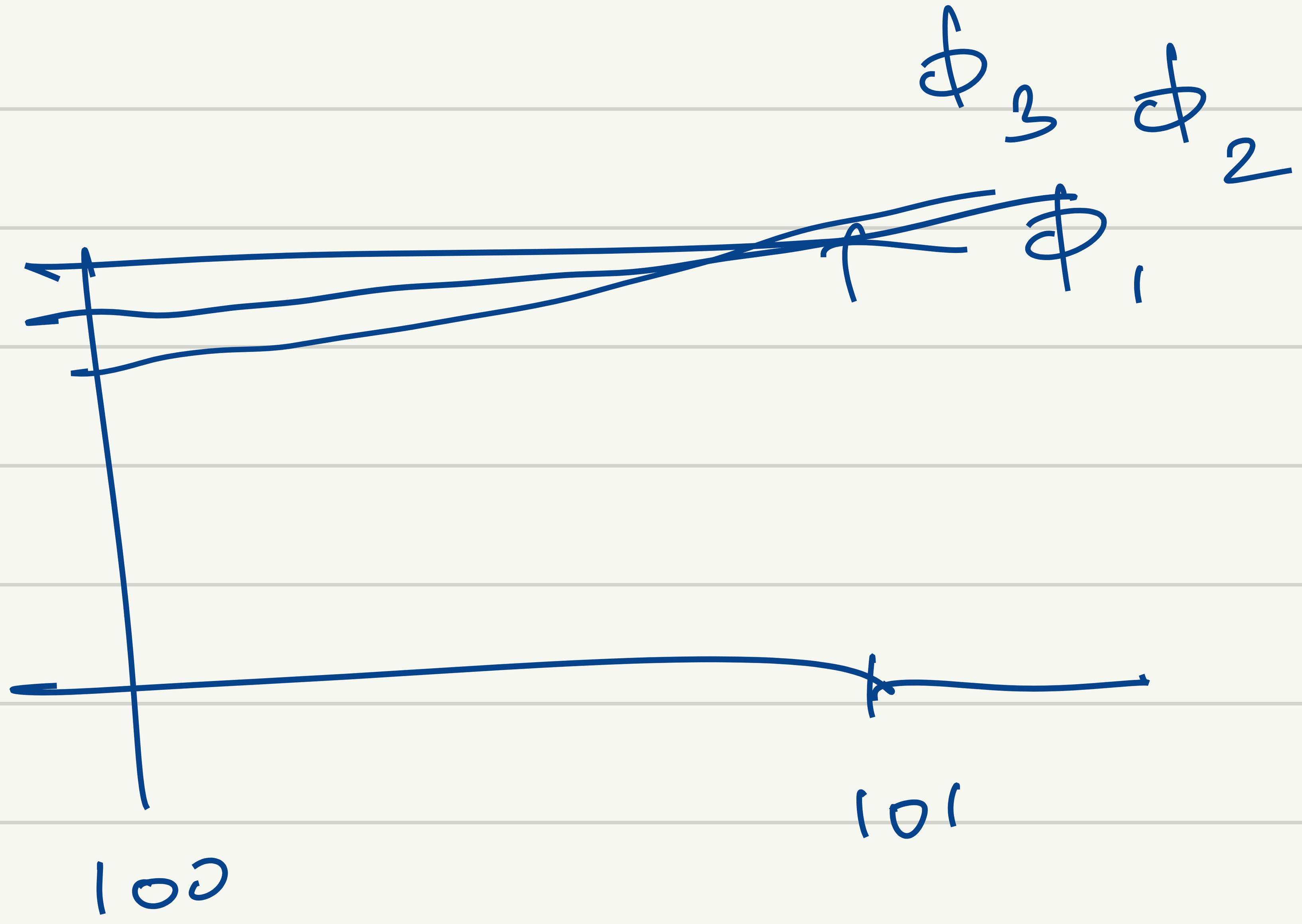
$$A = \begin{bmatrix} 1 & t_1 & t_1^2 & \cdots & t_1^{n-1} \\ 1 & t_2 & t_2^2 & \cdots & t_2^{n-1} \\ 1 & t_3 & t_3^2 & \cdots & \vdots \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & t_n & t_n^2 & \cdots & t_n^{n-1} \end{bmatrix} \qquad \boxed{\text{Vandermonde matrix}}$$

nonsingular as long as $t_i \neq t_j$ for $i \neq j$

To compute p, solve $Ax = y$: $O(n^3)$ time

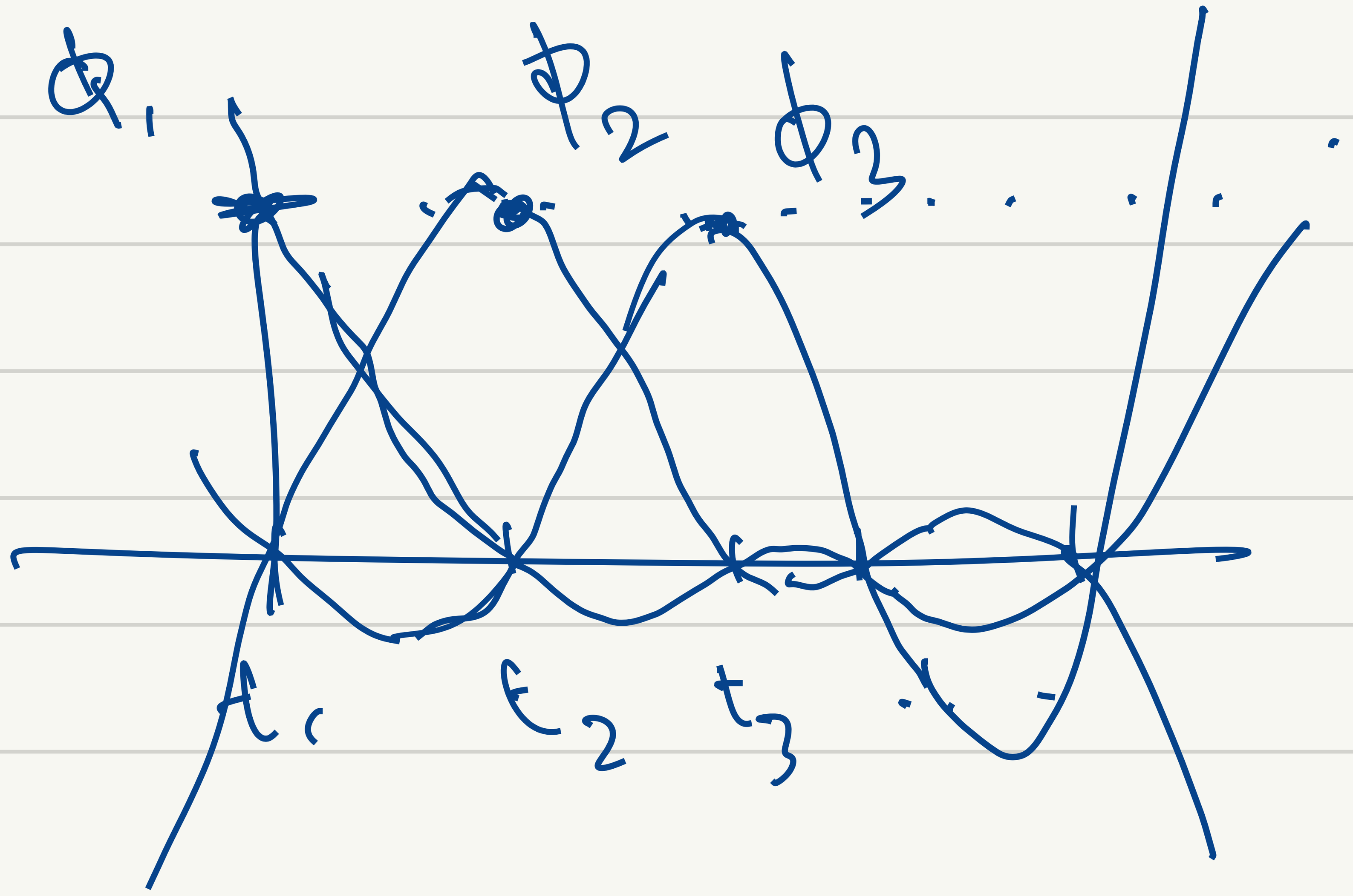
A is very badly conditioned for large n!





$\kappa(A)$ is at least exponential in n

Lagrange basis



$$\phi_i(t_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

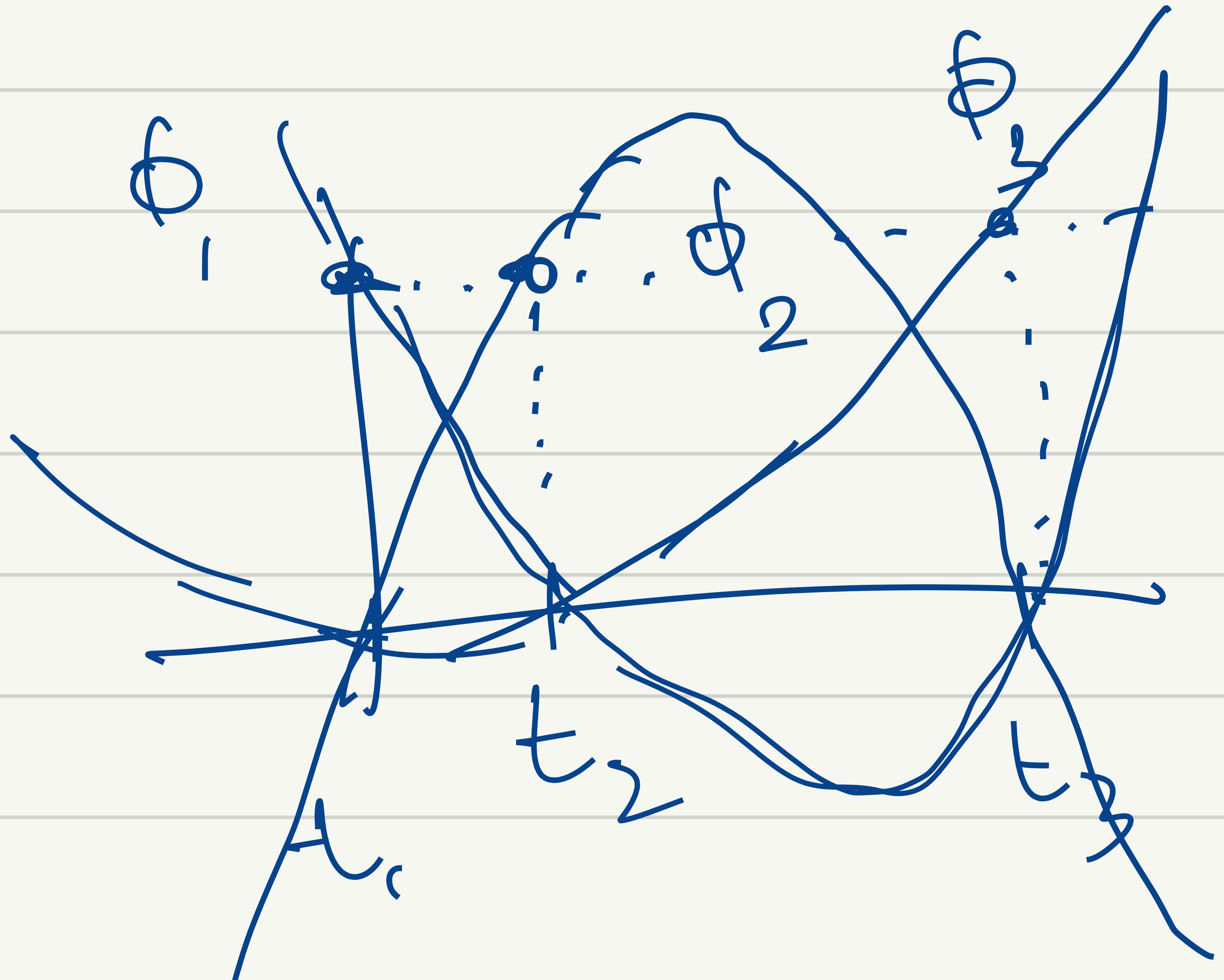
$$p(t) = \sum y_i \phi_i(t)$$

$$a_{ij} = \phi_i(t_j) \Rightarrow A = I$$

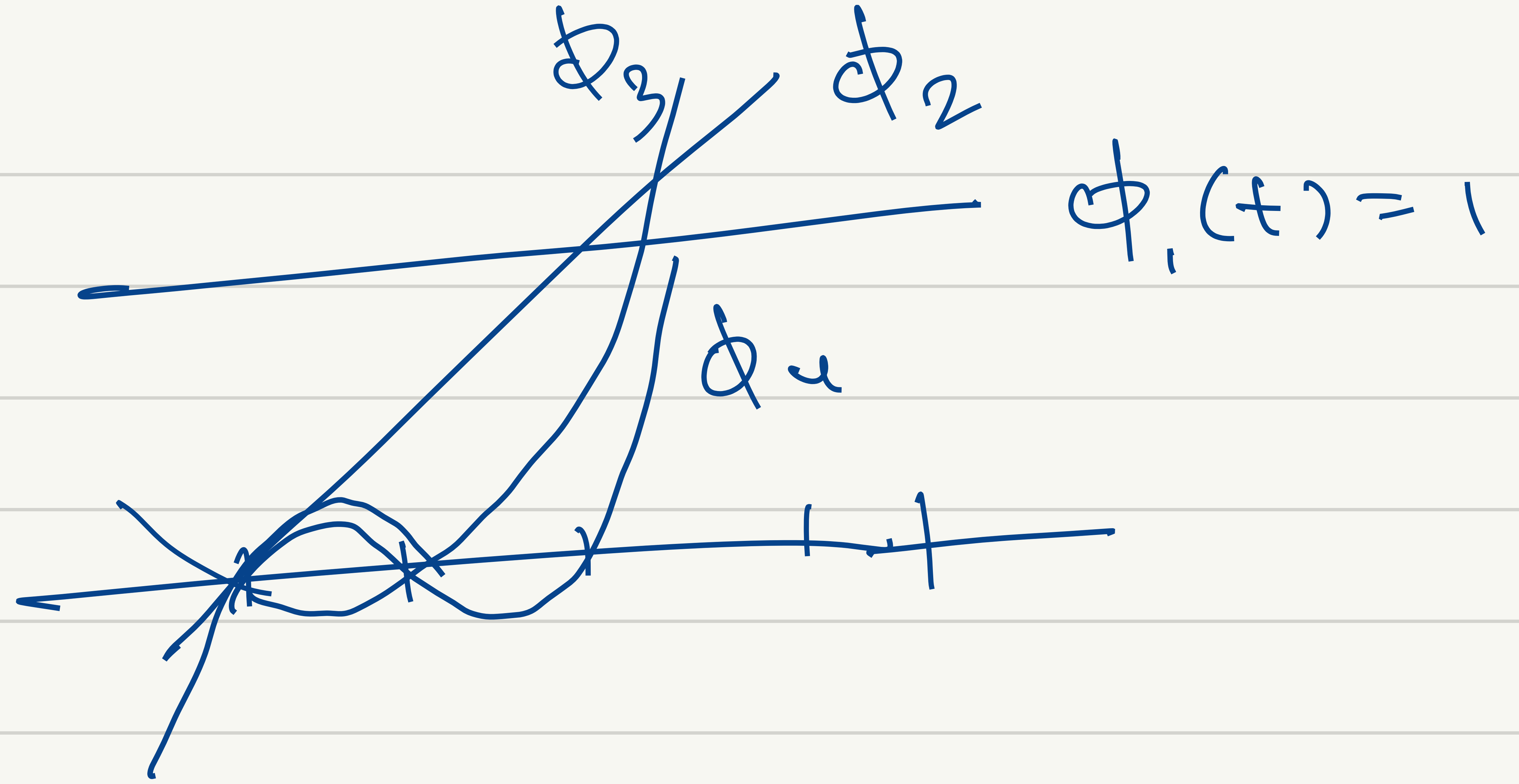
$$\phi_j(t_i) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq 0 \end{cases}$$

$$\phi_j(t) = \frac{\prod_{i \neq j} (t - t_i)}{\prod_{i \neq j} (t_j - t_i)} \quad \Rightarrow \text{ lagrange basis}$$

$$p(t) = \underbrace{y_1 \phi_1(t)} + \cdots + y_n \phi_n(t) \quad : \ O(n^2) \text{ time to evaluate}$$



Newton basis



$\phi_j(t_i)$ if $i < j$

$\phi_1(t) = 1$

$\phi_2(t) = t - t_1$

$\phi_3(t) = (t - t_1)(t - t_2)$

$\vdots$

$\phi_j(t) = \prod_{i<j} (t - t_i)$

$$A = \begin{bmatrix} 1 & 0 & 0 & & \\ 1 & t_2 - t_1 & 0 & & 0 \\ 1 & t_3 - t_1 & * & & \\ \vdots & \vdots & \vdots & \ddots & \\ 1 & t_n - t_1 & * & \cdots & * \end{bmatrix}$$

$\phi_1(t_i)$ $\phi_2(t_i)$ $\phi_3(t_i)$

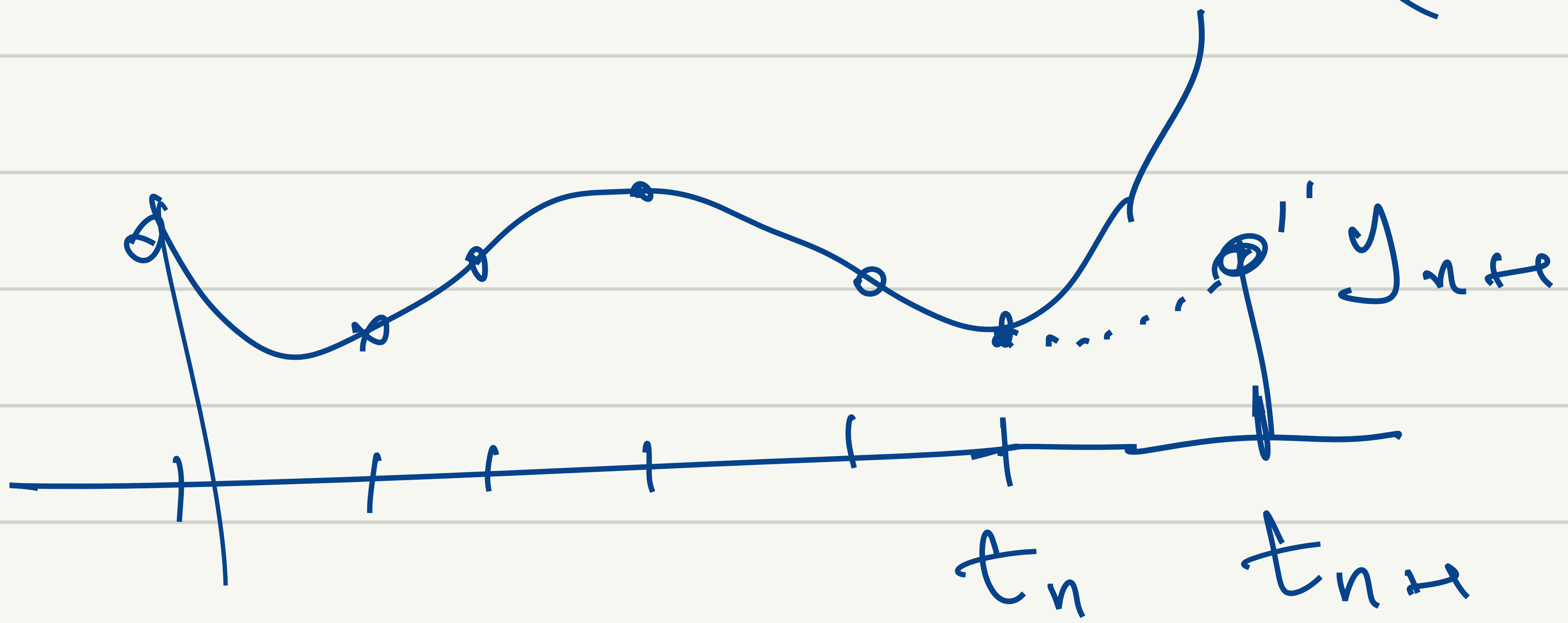
Computing p via $Ax = y$ takes $O(n^2)$ time

$$p(t) = x_1\phi_1(t) + x_2\phi_2(t) + x_3\phi_3(t) + \cdots + x_n\phi_n(t)$$

$$= x_1 + x_2(t-t_1) + x_3(t-t_1)(t-t_2) + \cdots + x_n(t-t_1)(t-t_2)\cdots(t-t_{n-1})$$

$$= x_1 + (t-t_1)\Big(x_2 + (t-t_2)\big(x_3 + \cdots \big(x_n(t-t_{n-1})\big)\cdots\big)\Big)$$

evaluate in $O(n)$ time



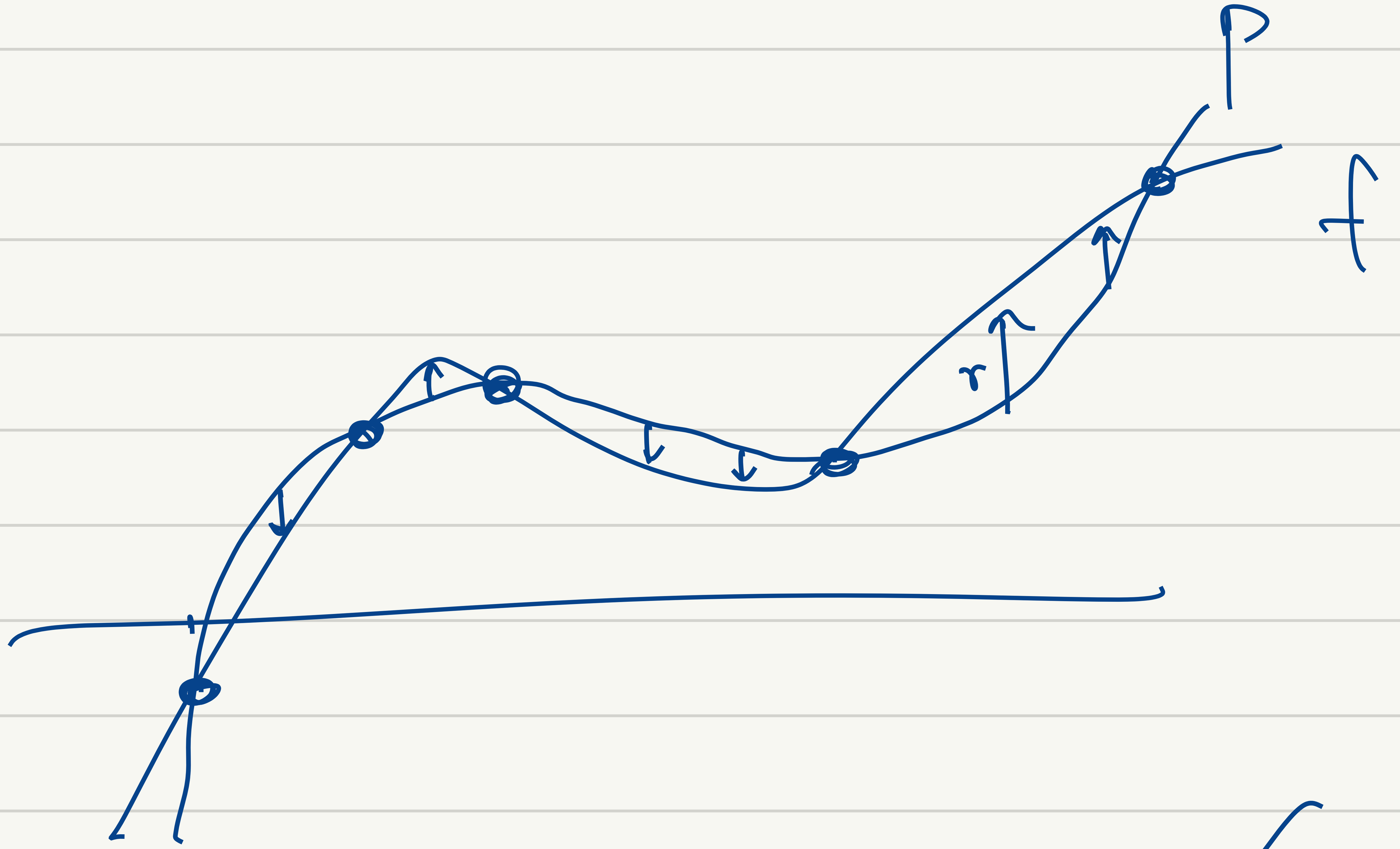
$$\phi_{n+1}(t) = (t-t_1)\cdots(t-t_n)$$

Just compute x_{n+1}

Bounds on polynomial interpolation

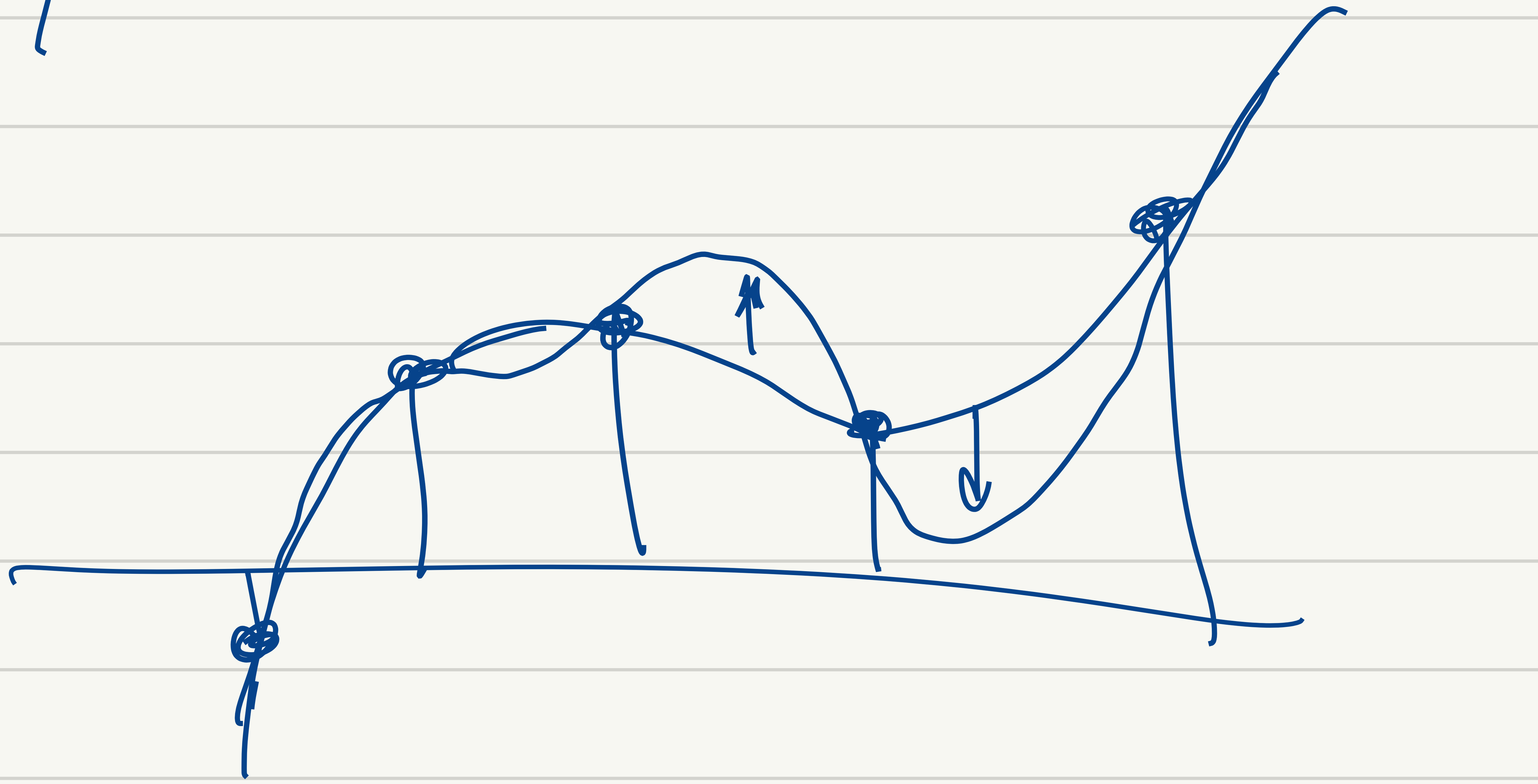
p exactly matches f at t_i

How close is it elsewhere?



Define residual fn $r = f - p$

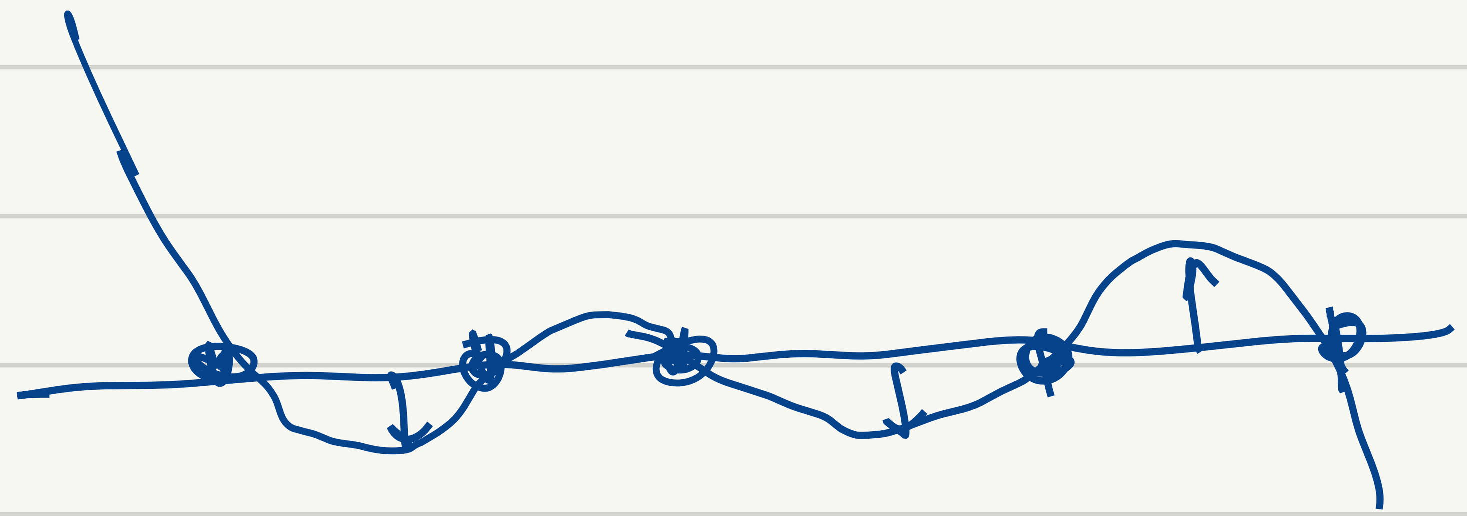
$r = 0$ at $t_1, t_2, \dots, t_n$



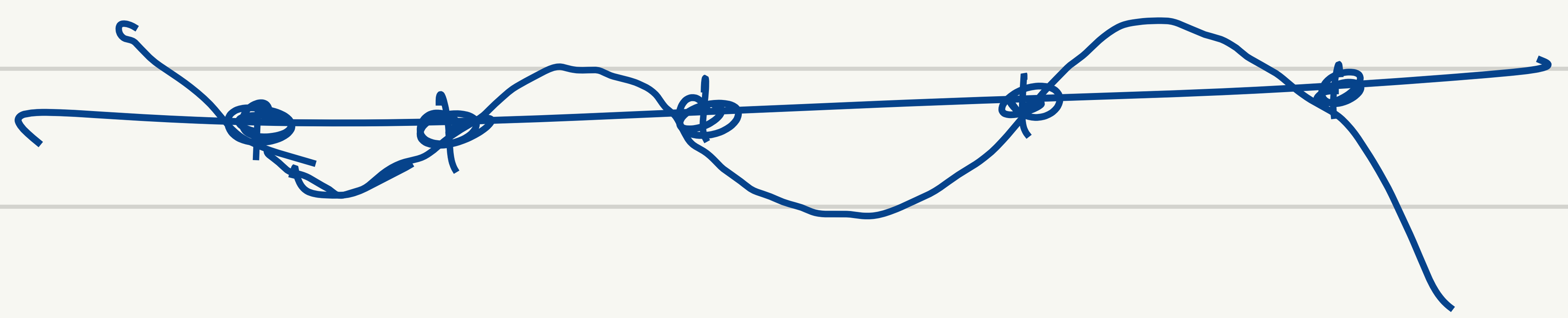
Suppose $|f^{(n)}(t)| \le M_n$ for all $t \in [t_1, t_n]$

$$|r^{(n)}(t)| = |f^{(n)}(t) - \cancelto{0}{p^{(n)}(t)}| = |f^{(n)}(t)| \le M_n$$

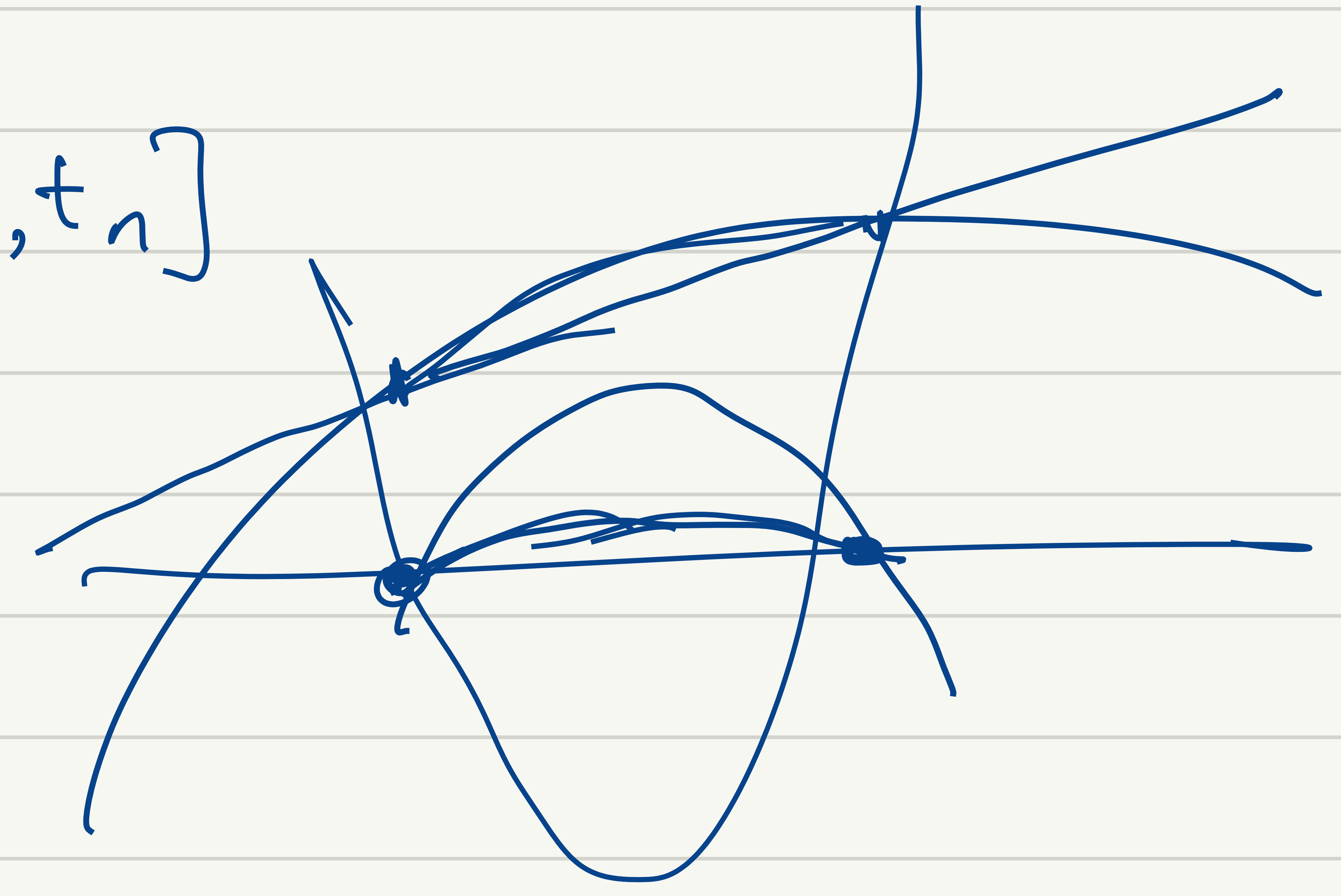
degree $n-1$



$$\text{Assume } |f^{(n)}| \le M_n \implies \left.\begin{aligned} &|r^{(n)}(t)| \le M_n \\ &r^{(n)}(t_i) = 0 \end{aligned}\right\}$$



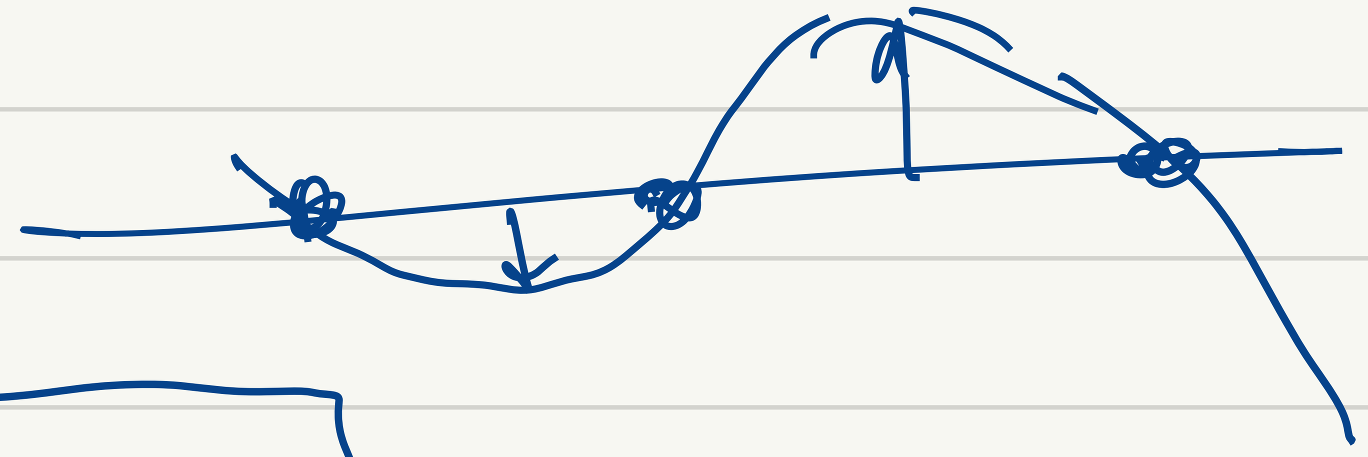
How big can $|r(t)|$ be?



Thm: for any $t \in [t_1, t_n]$, there exists $\theta \in [t_1, t_n]$

s.t. $$r(t) = \frac{r^{(n)}(\theta)}{n!} \underbrace{(t-t_1)\cdots(t-t_n)}_{w(t)}$$

$$\Rightarrow \boxed{|r(t)| \le \frac{M_n}{n!} |w(t)|}$$



If equally spaced points $|t_1 - t_2| = |t_2 - t_3| = \cdots = h$

then $|w(t)| \le \dfrac{h^n (n-1)!}{4} \implies \boxed{|r(t)| \le \dfrac{h^n M_n}{4n}}$