COL726 : Nonlinear Equations

Conditioning of eigenvalues

$A \in \mathbb{C}^{m\times m}$ (may not be Hermitian)

λ eigenvalue

If μ is eigenvalue of $A+\delta A$

$\exists\ \lambda_j$ eigenvalue of A

s.t. $|\mu-\lambda_j| \le \underbrace{\kappa(V)}\, \|\delta A\| + O(\|\delta A\|^2)$

?

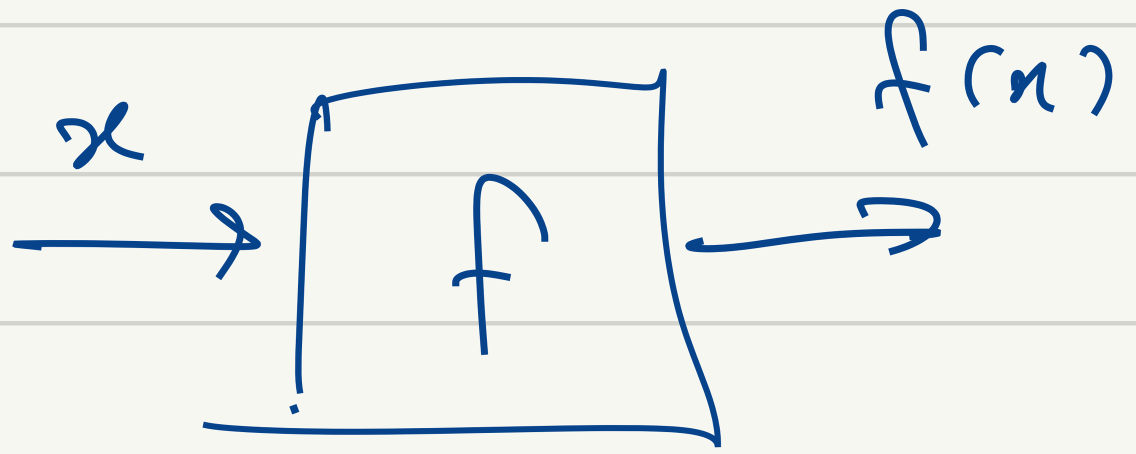
$A = V \Lambda V^{-1}$

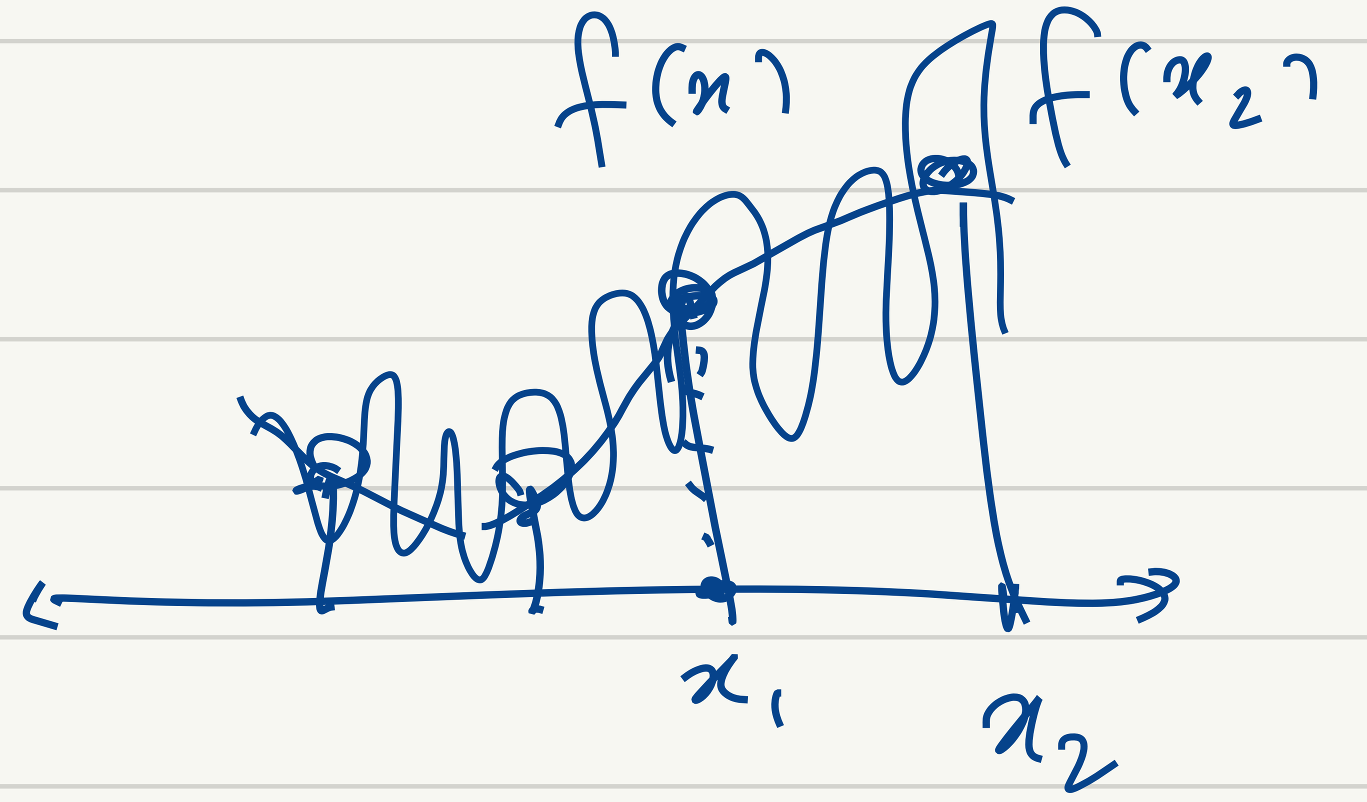
$f : \mathbb{R} \to \mathbb{R}$ $\qquad$ $f : \mathbb{R}^n \to \mathbb{R}^n$

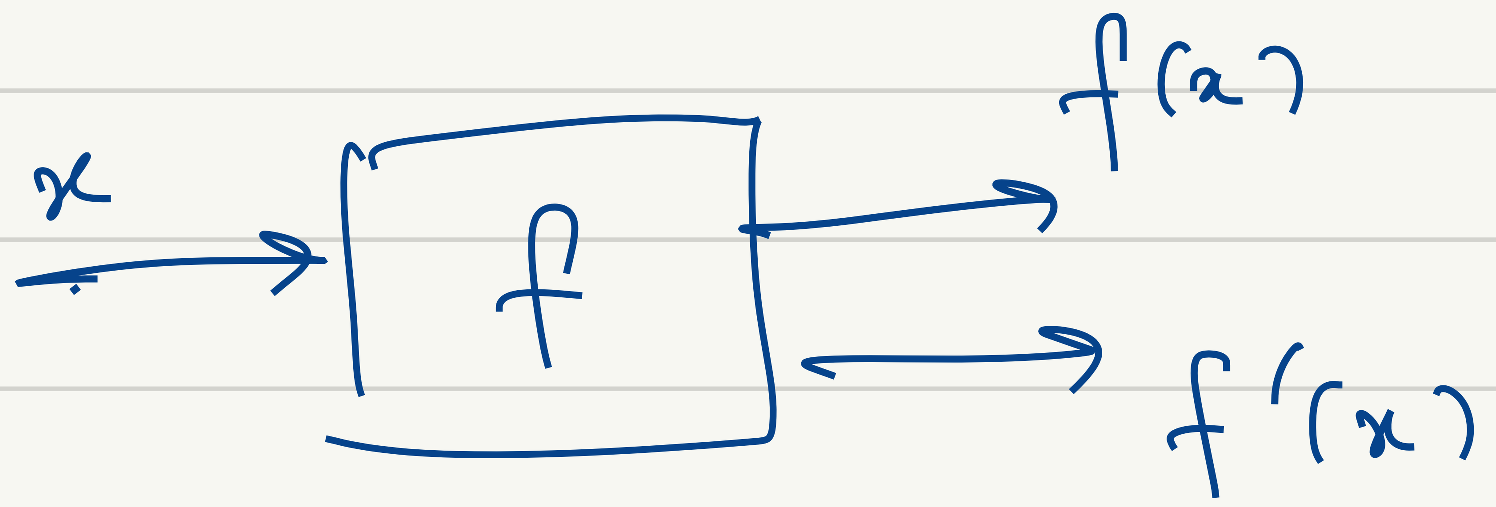
find x s.t. $f(x) = 0$

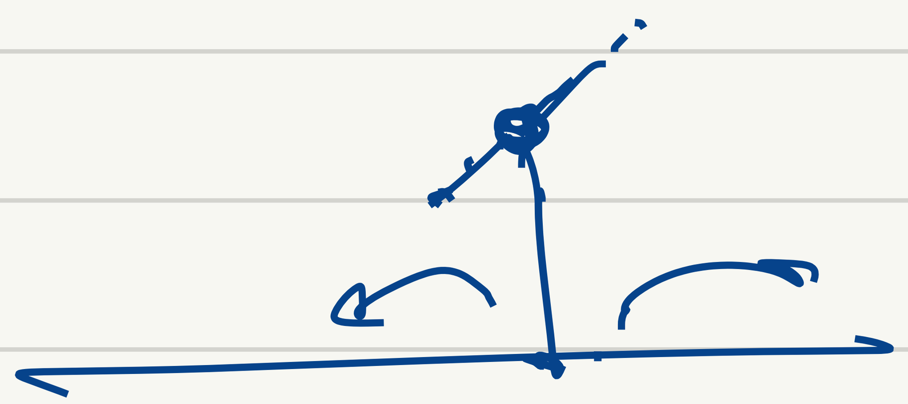
x is a $\boxed{\text{root}}$ of f if $f(x)=0$

$f(x) = \sin(x) + e^x \cos \ldots$









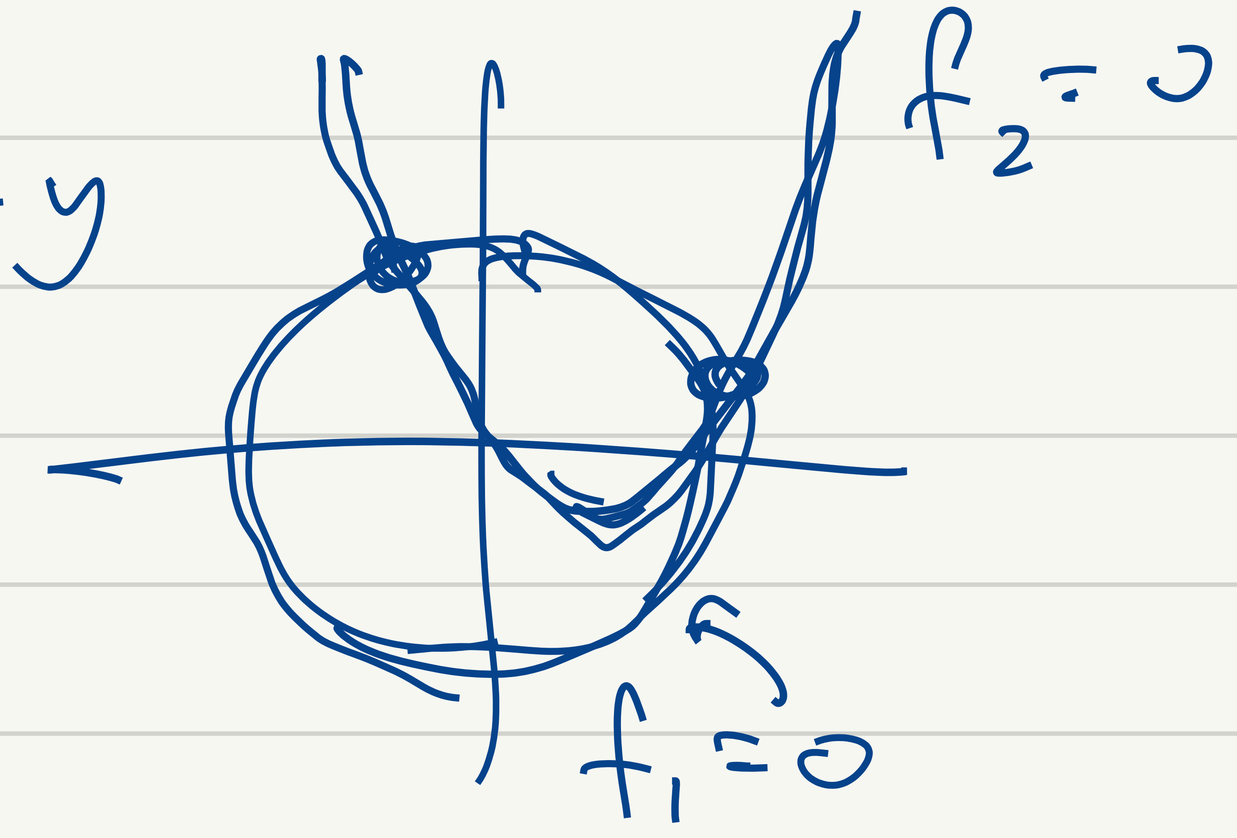
Existence & uniqueness

$f: \mathbb{R}^n \to \mathbb{R}^n$ $\cancel{\mathbb{R}^m \to \mathbb{R}^n}$

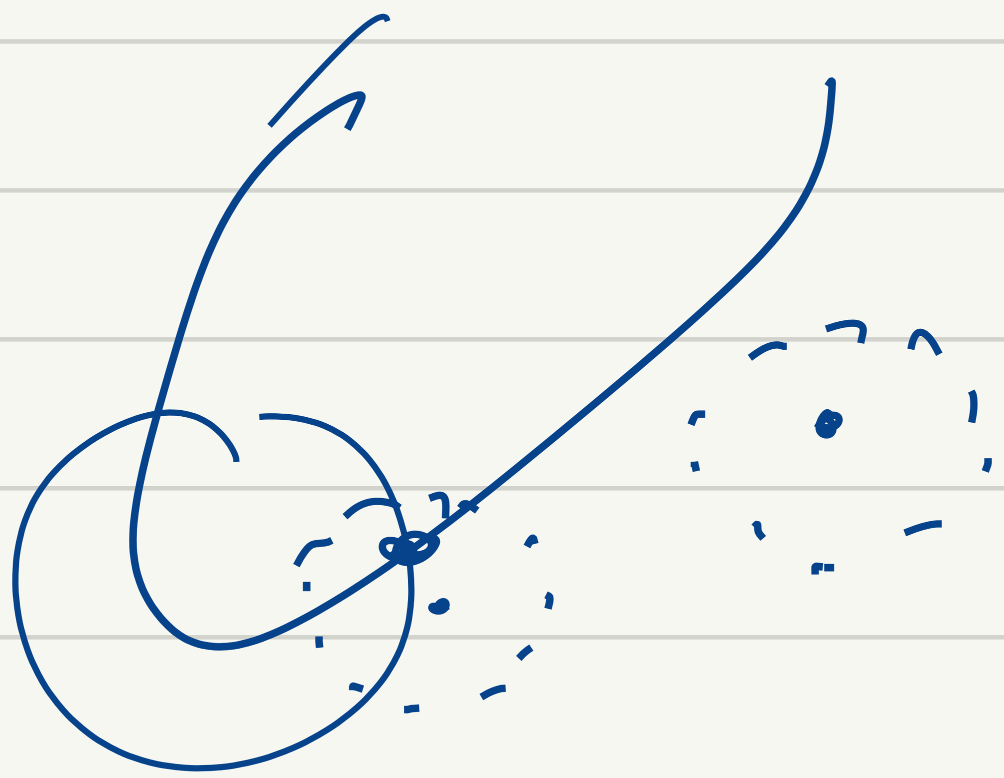
$f: \mathbb{R}^2 \to \mathbb{R}^2$ $f_1(x_1, x_2) = 0$

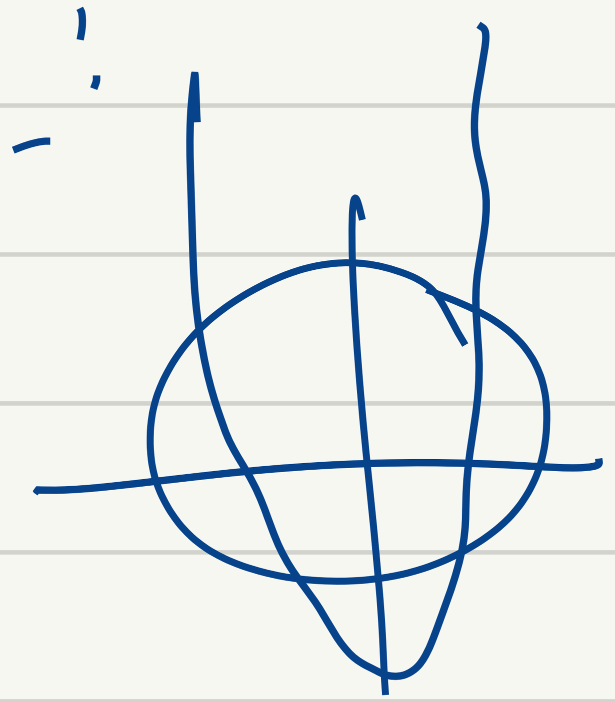
$f_1(x_1, x_2) = x_1^2 + x_2^2 - 1$

$f_2(x_1, x_2) = ax^2 + bx + c - y$



<u>Lo</u>cal guarantees only:

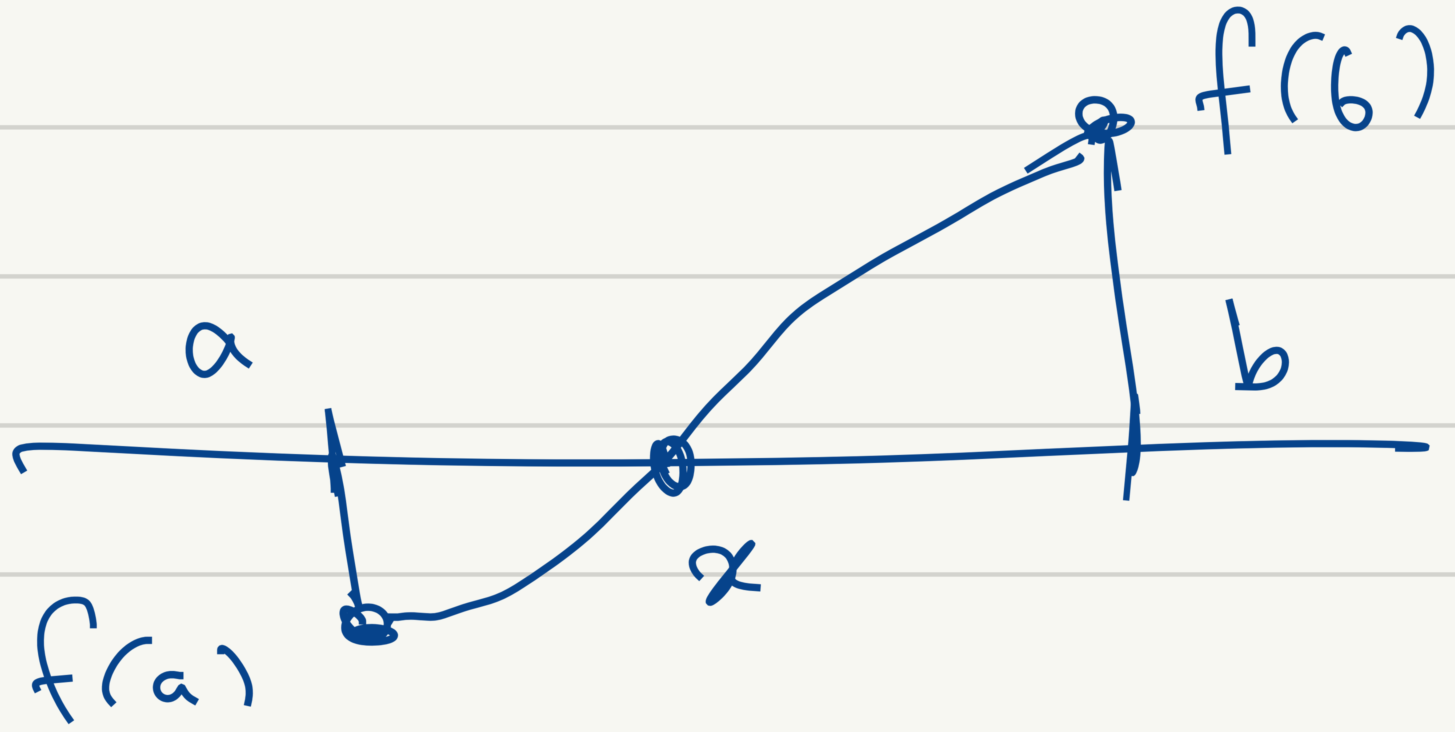




1. Intermediate value theorem:

$f: \mathbb{R} \to \mathbb{R}$ continuous, $f(a) < 0$, $f(b) > 0$

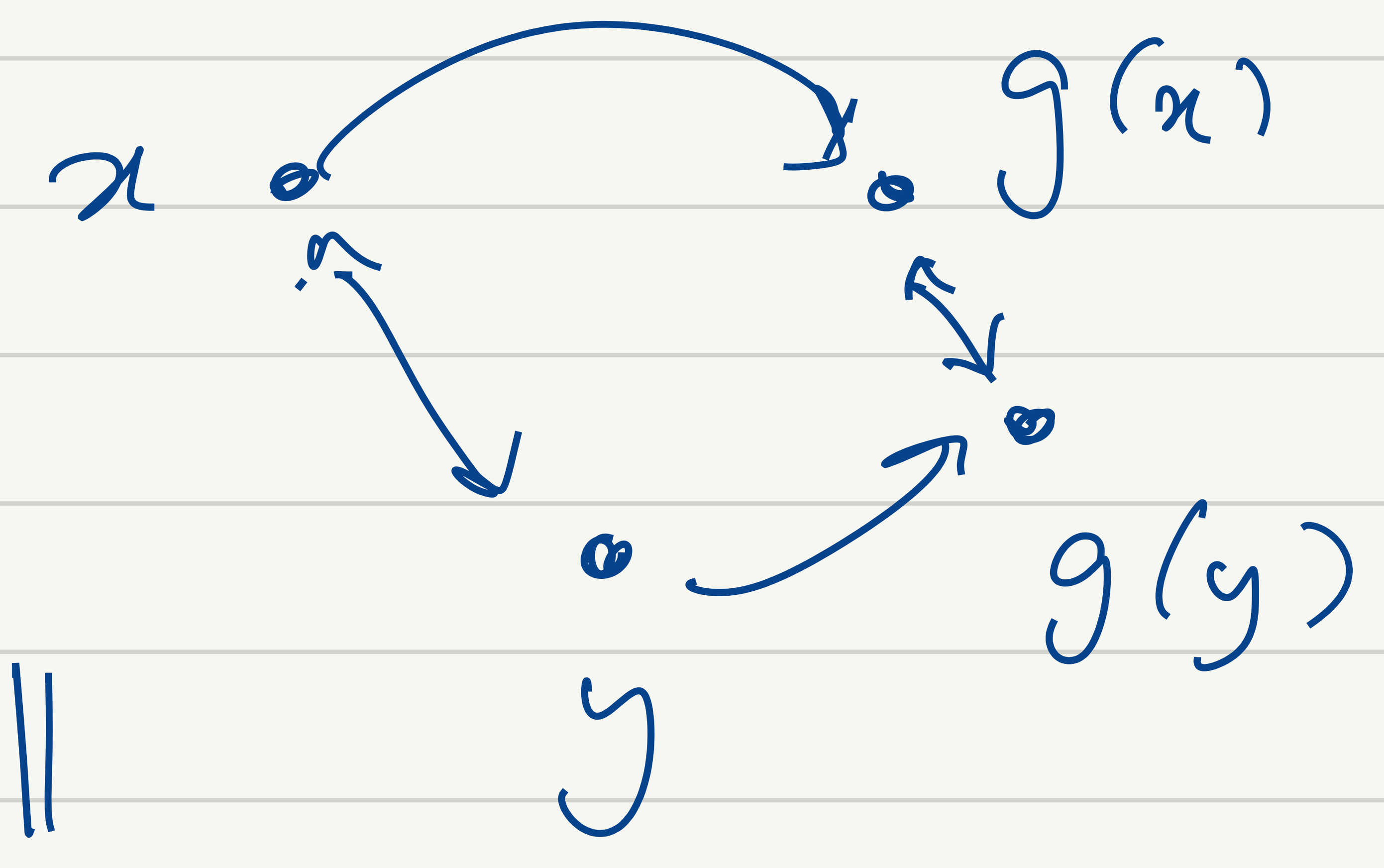
then $\exists\, x \in (a, b)$ s.t. $f(x) = 0$



Interval $[a, b]$ with $f(a) < 0 < f(b)$ or $f(a) > 0 > f(b)$

is called a $\boxed{\text{Bracket}}$

2. <u>Contraction mapping theorem.</u>



$g : \mathbb{R}^n \to \mathbb{R}^n$

g is a $\boxed{\text{contraction}}$ if $\|g(x) - g(y)\|$

$< \|x - y\|$ for all x, y.

compact
= closed
& ~~bounded~~

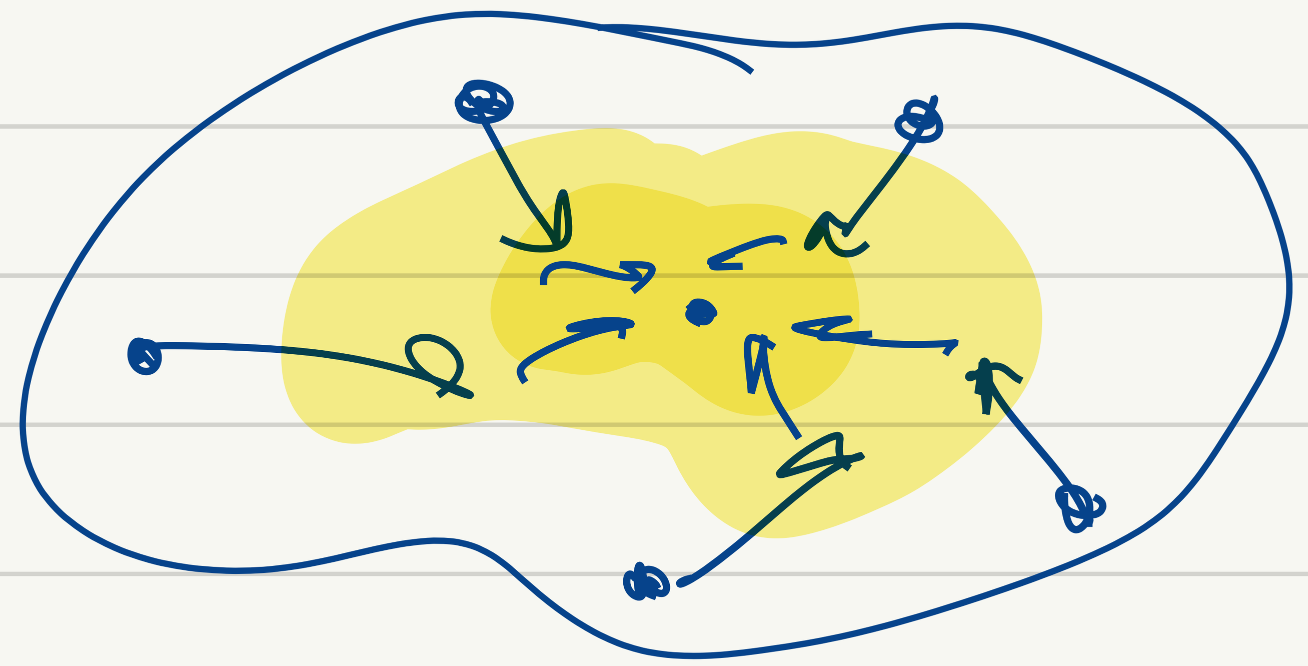
If there is a <u>compact</u> set S s.t.

g is a contraction from S to itself $\longrightarrow$ $\|g(x) - g(y)\| < \|x-y\| \; \forall \, x, y \in S$

then g has a unique $\boxed{\text{fixed point}}$ in S. and $g(x) \in S \; \forall x \in S$

$$g(x_*) = x_*$$

$$\Leftrightarrow f(x) = x - g(x) = 0$$



3. Inverse function theorem

$f : \mathbb{R}^n \to \mathbb{R}^n$

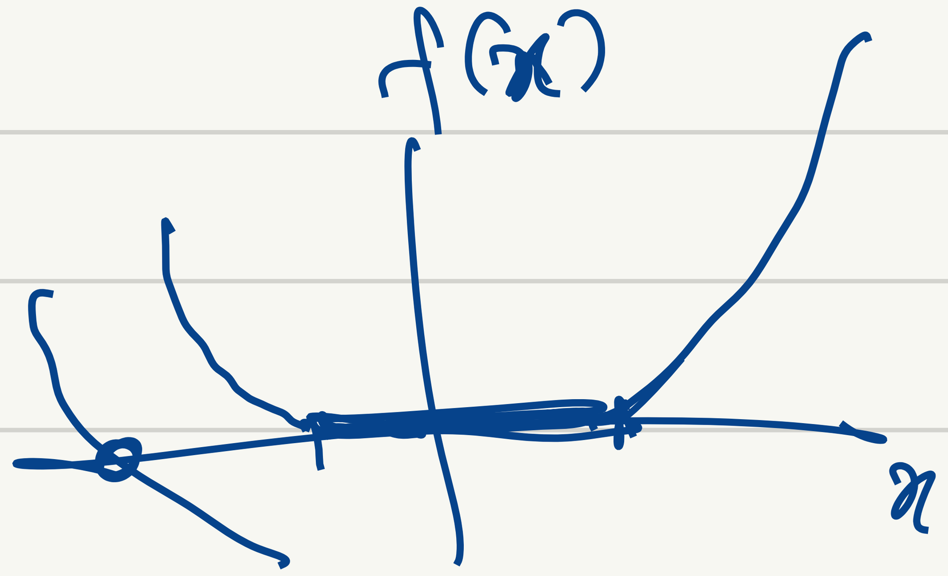
$$f(x_1, \dots x_n) = \begin{bmatrix} f_1(x_1 \dots x_n) \\ f_2(x_1, \dots x_n) \\ \vdots \\ f_n(x_1 \dots x_n) \end{bmatrix}$$

$\boxed{\text{Jacobian}}$ of f = $J(x) =$

$$\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \partial f_2/\partial x_1 & \partial f_2/\partial x_2 & \cdots & \\ \vdots & & \ddots & \\ \partial f_n/\partial x_1 & \cdots & & \partial f_n/\partial x_n \end{bmatrix}$$

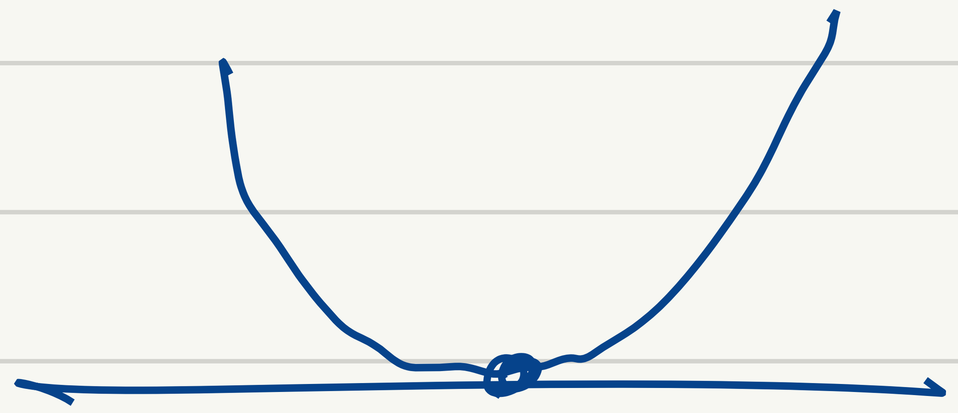
If $J(x)$ is nonsingular
then f is locally invertible
in some neighbourhood of x.

$$f(x + \delta x) \approx f(x) + J(x) \cdot \delta x + O(\|\delta x\|^2)$$



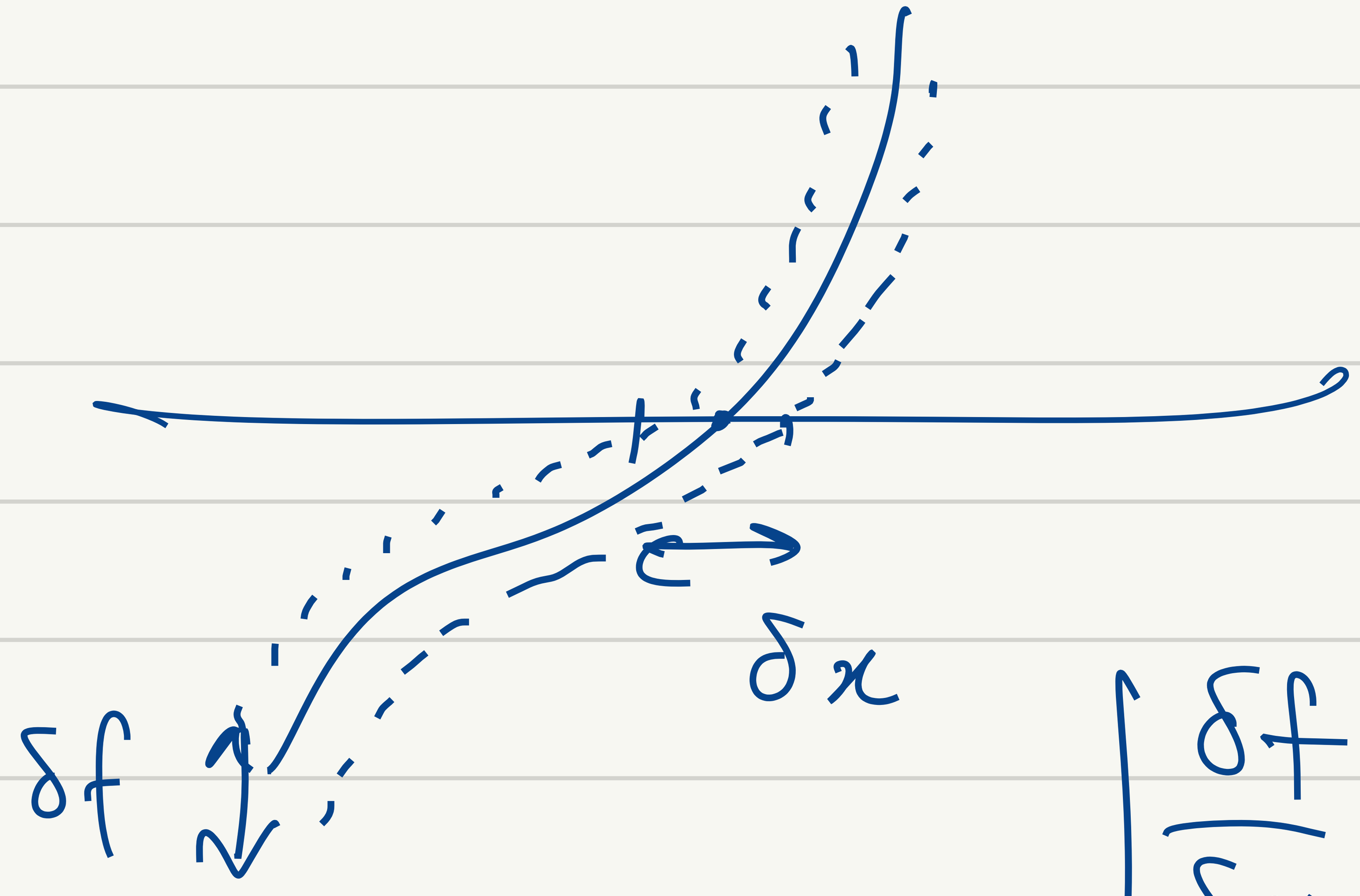
$\Rightarrow$ If x is root $(f(x) = 0)$
Then it is <u>locally unique</u>

If $J(x)$ is singular, system is $\boxed{\text{degenerate}}$ at x



Sensitivity & conditioning

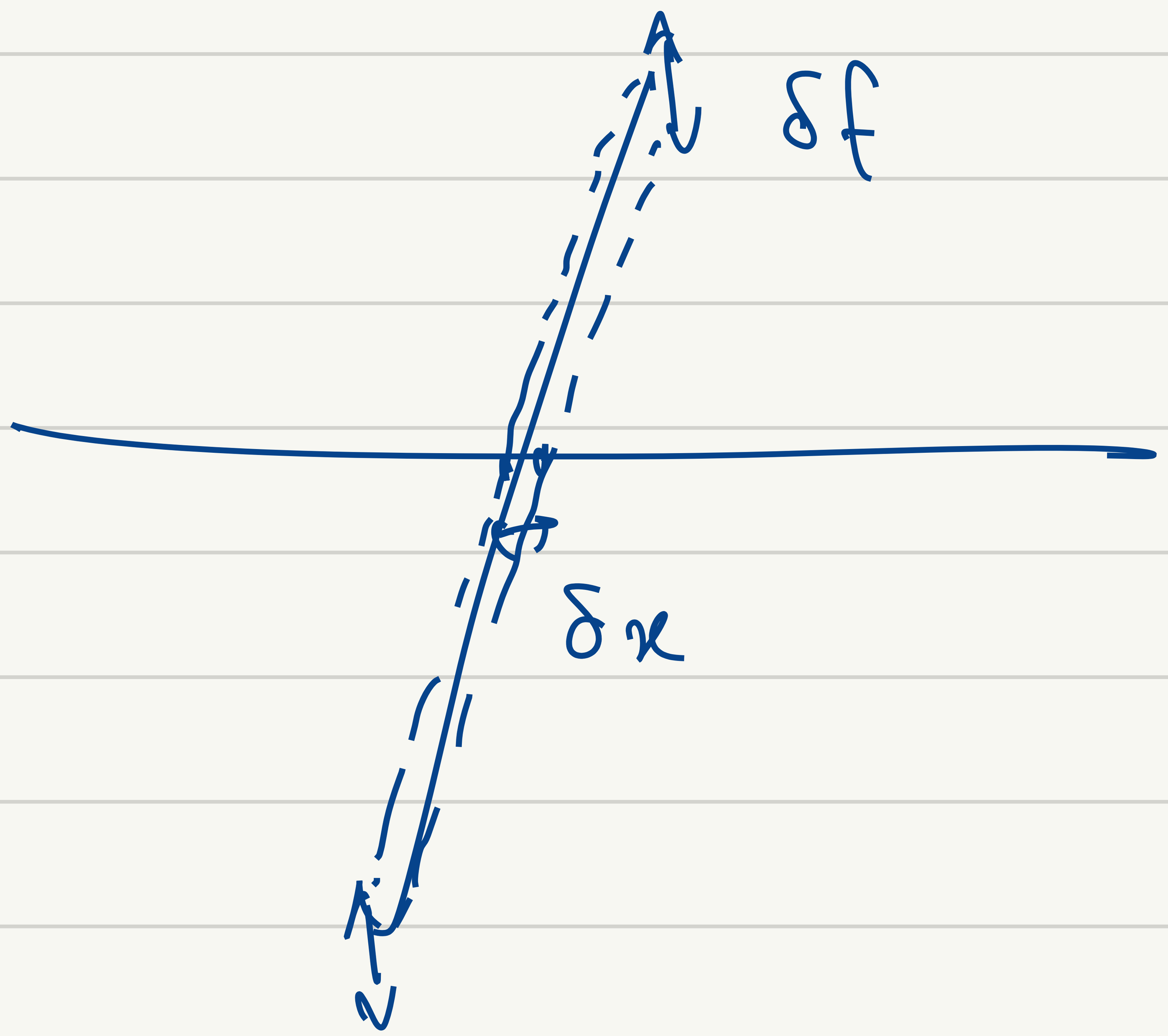
$f(x) + \delta f = 0$ $\qquad$ $f(x) = 0 + \delta y$

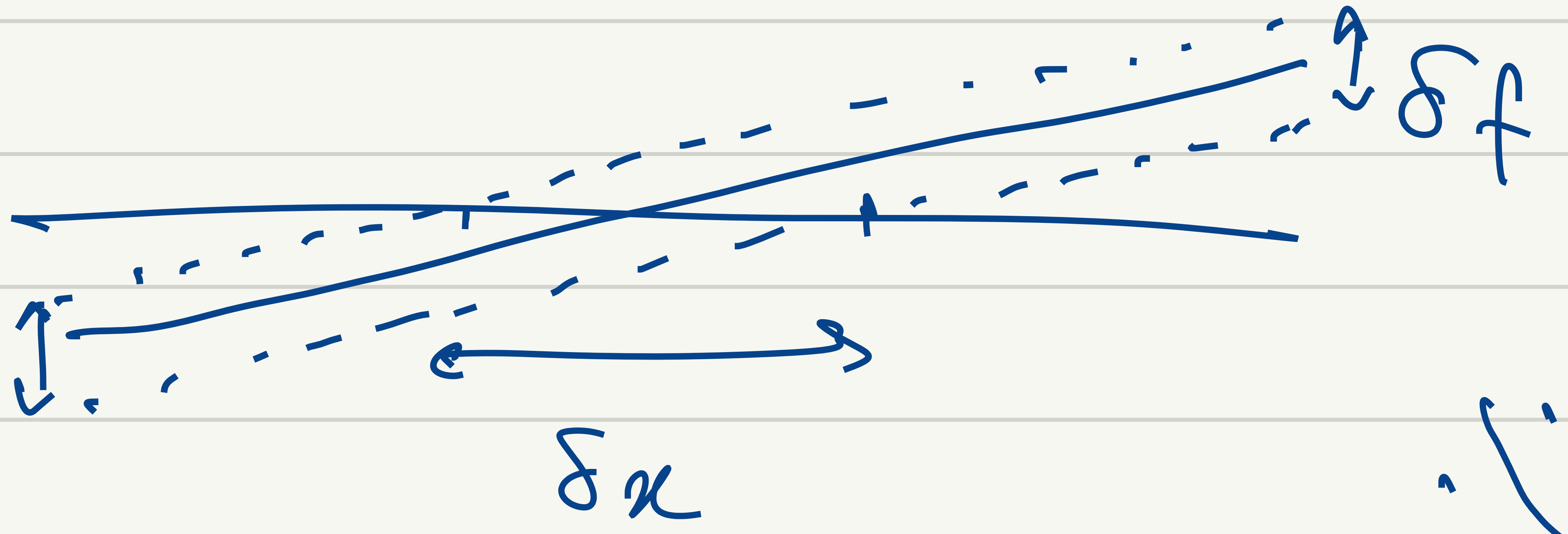


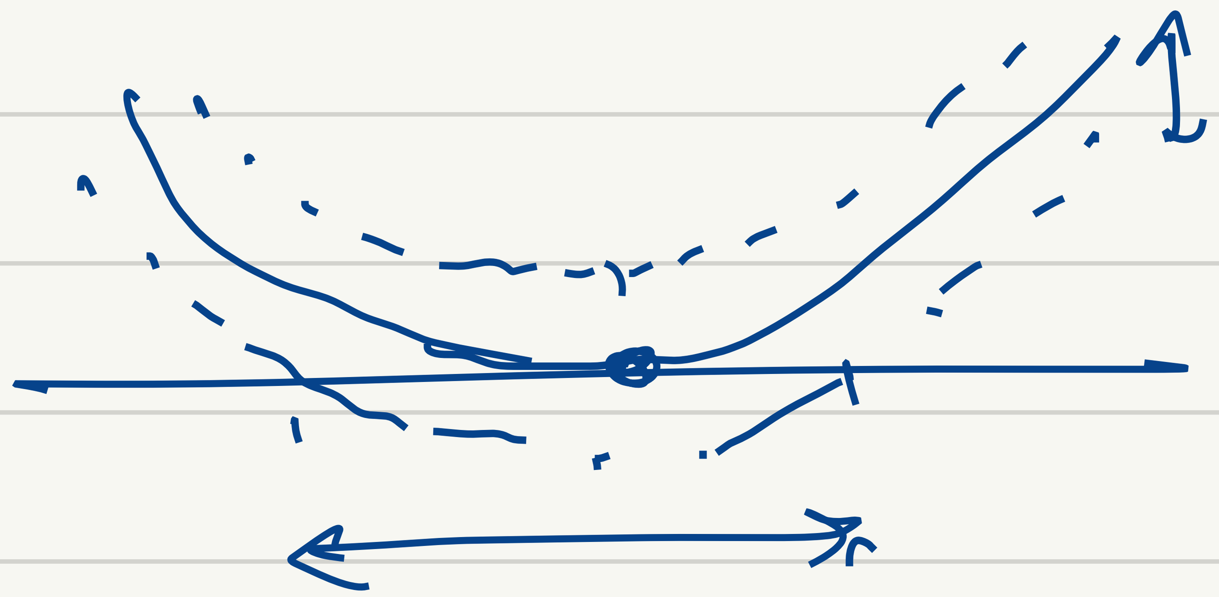
Abs. Cond. num. $= \dfrac{1}{|f'(x)|}$

$\left|\dfrac{\delta f}{\delta x}\right| = |f'(x)|$

$\Rightarrow |\delta x| = \dfrac{|\delta f|}{|f'(x)|}$







$\|\delta f\| = \|J(x)\cdot\delta x\| \Rightarrow$ abs. cond. num. $= \|J^{+}(x)\|$ (not $\kappa(J)$)

$\|f(x)\|$: residual

$\|x - x_*\|$: error

Convergence

When to terminate?

$x_0 \to x_1 \to x_2 \to \ldots \longrightarrow x_*$

$\to \|f(x_k)\| \leq \varepsilon$

$\|x_k - x_{k-1}\| / \|x_k\| \leq \varepsilon_2$

$e_k = x_k - x_*$. we want $e_k \to 0$ quickly.

If $\lim_{k\to\infty} \dfrac{\|e_{k+1}\|}{\|e_k\|} = c \in (0,1)$ then convergence is $\boxed{\text{linear}}$ with rate const. c

If $\lim_{k\to\infty} \|e_{k+1}\| / \|e_k\|^r = c$, $r > 1$: $\boxed{\text{superlinear}}$, $r=2$: quadratic, $r=3$: cubic

1D algorithms

Bisection method :

repeat:

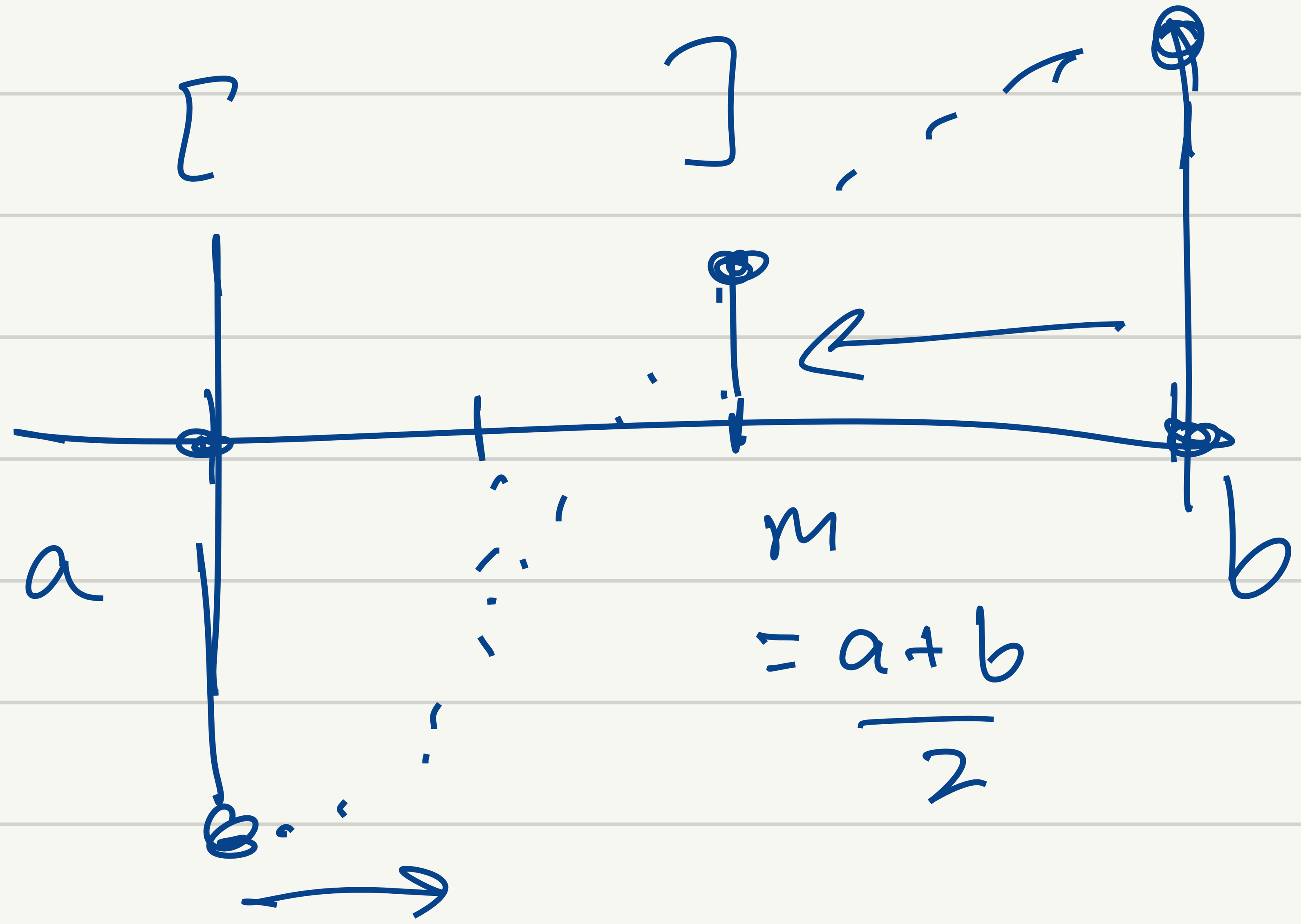
$$m = a + \frac{b-a}{2}$$

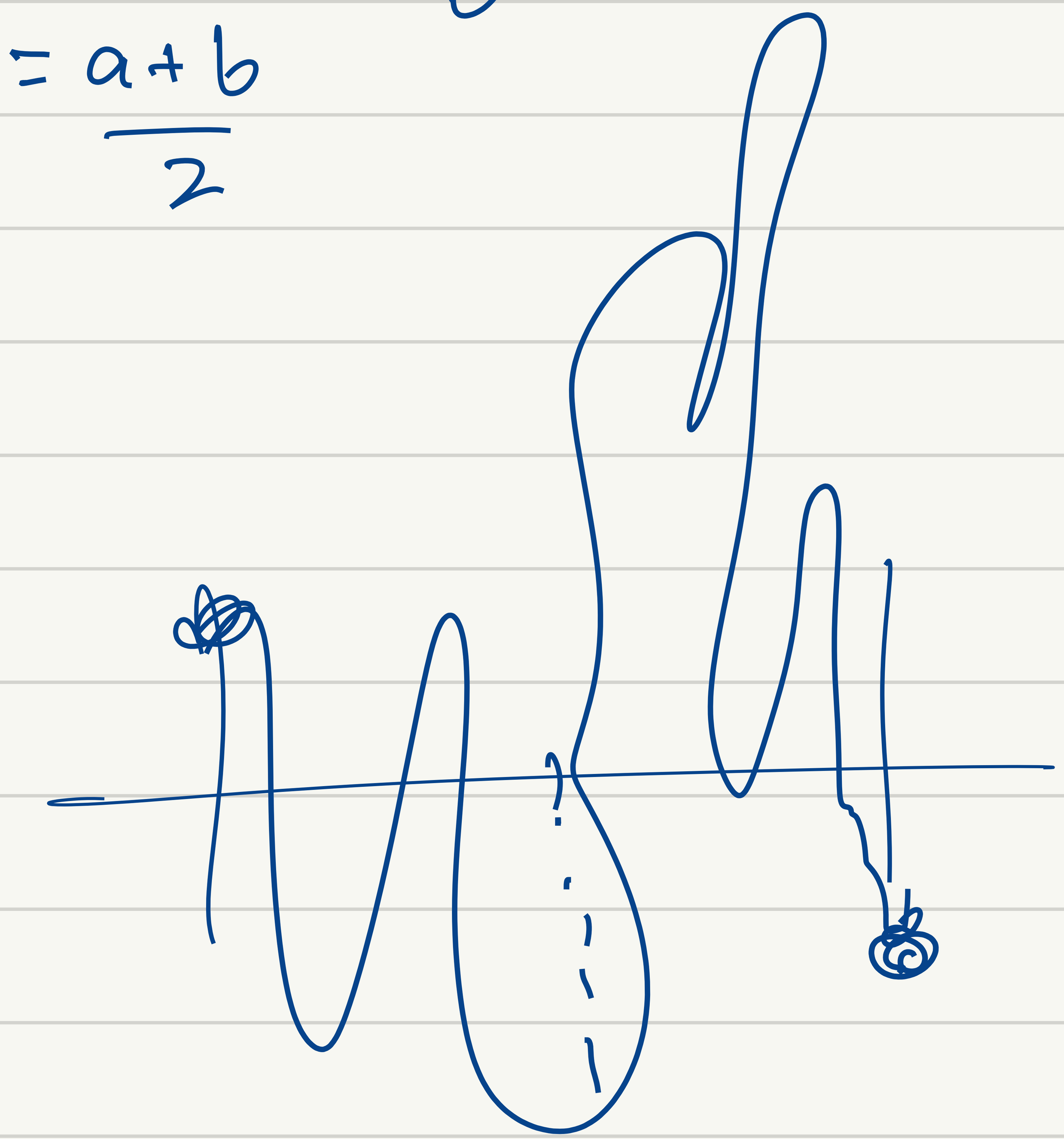
if $\operatorname{sign}(f(m)) = \operatorname{sign}(f(a))$

$a = m$

else:

$b = m$





At each step, error $\leq |b-a|$,

reduces by $1/2$ each time

$\Rightarrow$ linear convergence, $C = 1/2$

fixed point iteration

$x_0, \quad x_1 = g(x), \quad x_2 = g(g(x)), \ldots$

$f(x) = 0$

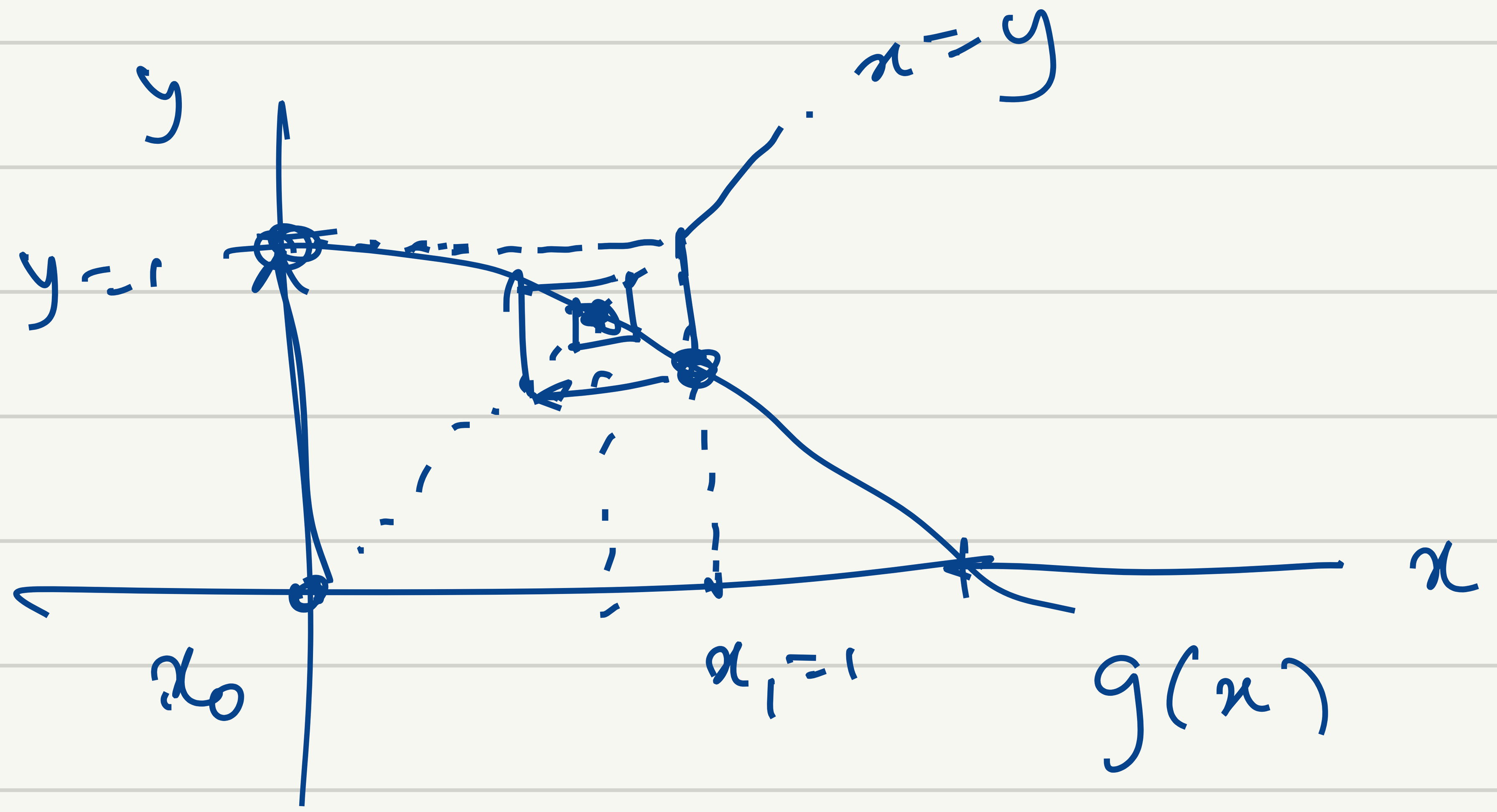
Suppose I have $g: \mathbb{R} \to \mathbb{R}$ s.t. $f(x) = 0 \iff x = g(x)$

FPI: $\underline{x_{k+1} = g(x_k)}$

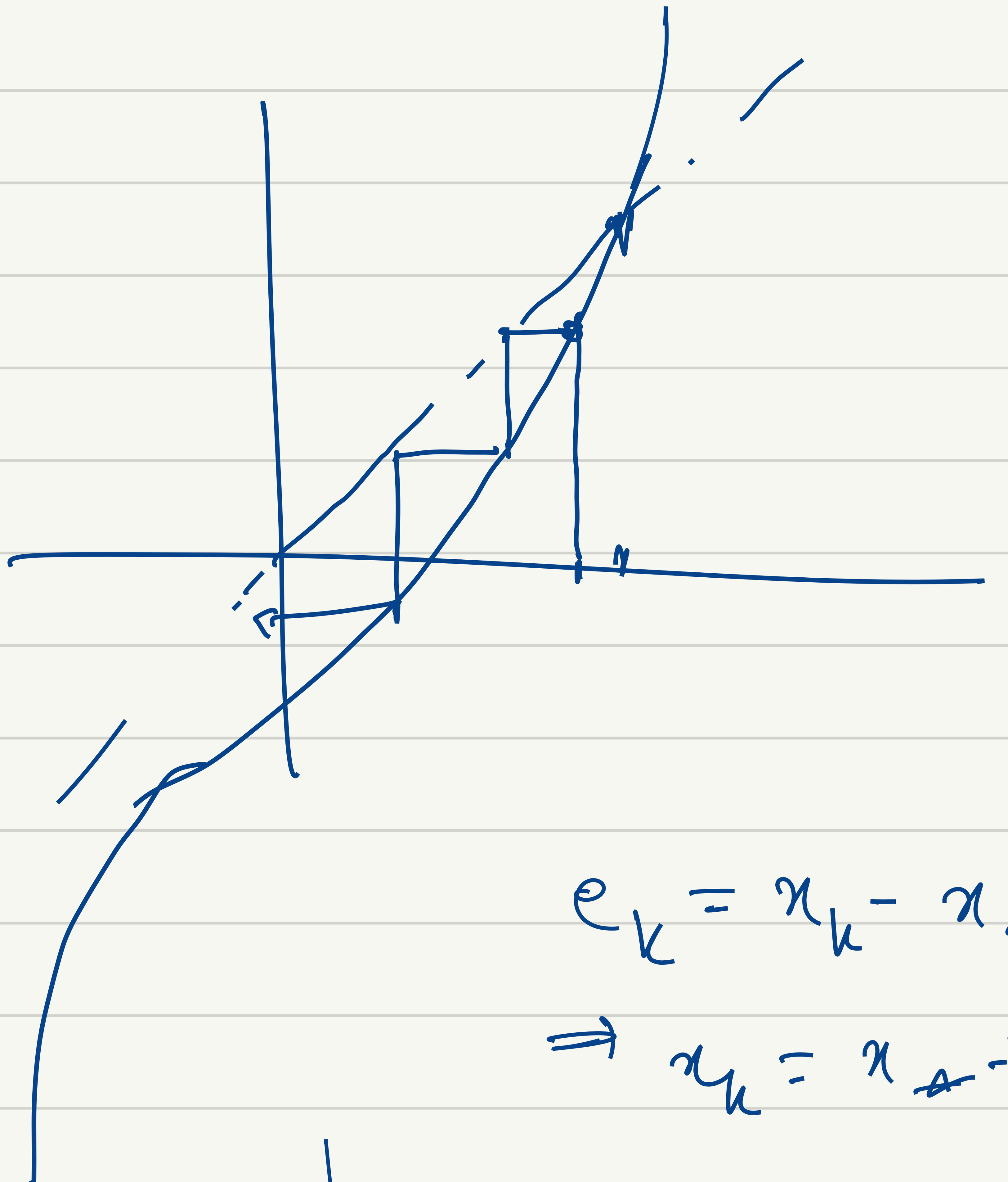
ex. $f(x) = x^2 - x - 2 = 0$: various choices of g:

$$\left\{\begin{array}{l} x = \overbrace{x^2 - 2}^{g(x)} \\ x^2 = x + 2 \Rightarrow x = \sqrt{x+2} \\ x - 1 - \frac{2}{x} \Rightarrow x = 1 + 2/x \\ x = (x^2 + 2)/(2x - 1) \end{array}\right.$$

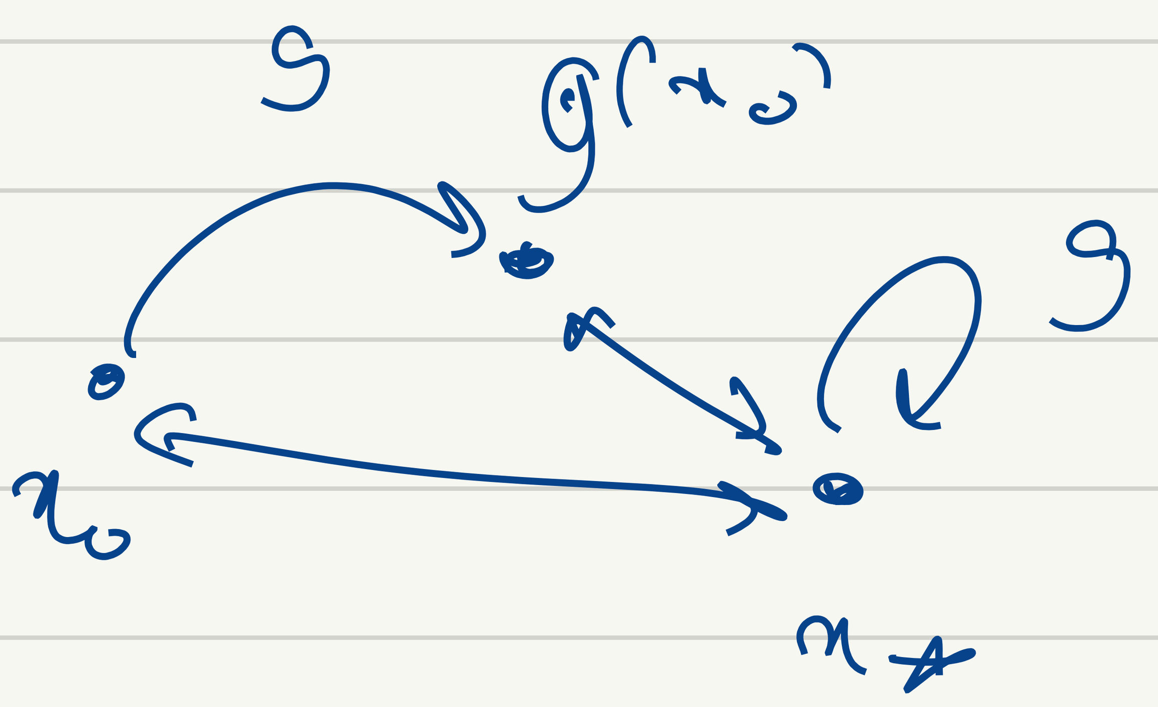
Any iterative alg. where x_{k+1} depends only on x_k is a fixed point iteration!



$x = \cos(x)$



If g is a <u>contraction</u> on some neighbourhood of x_* <u>and</u> $x_0 \in$ ⤴



Then: $|g(x_0) - g(x_*)| < |x_0 - x_*|$

$= |x_1 - x_*|$

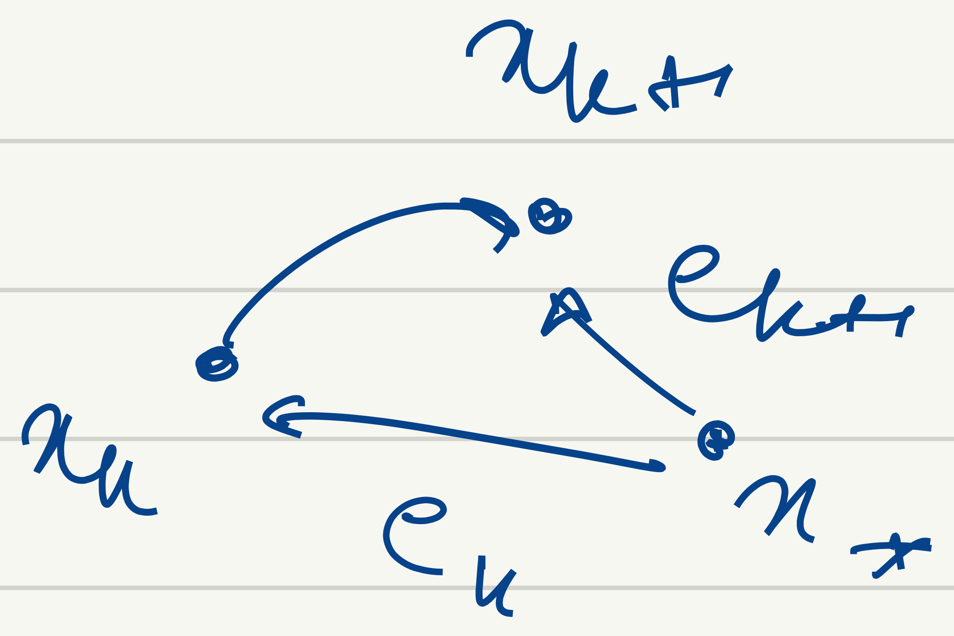
$e_k = x_k - x_*$

$\Rightarrow x_k = x_* + e_k$

$f(x + \delta x) = f(x) + f'(x)\,\delta x + \frac{1}{2} f''(x)\,\delta x^2 + \dots$

$e_{k+1} = x_{k+1} - x_* = \underbrace{g(x_k)} - x_* = g(x_* + e_k)$

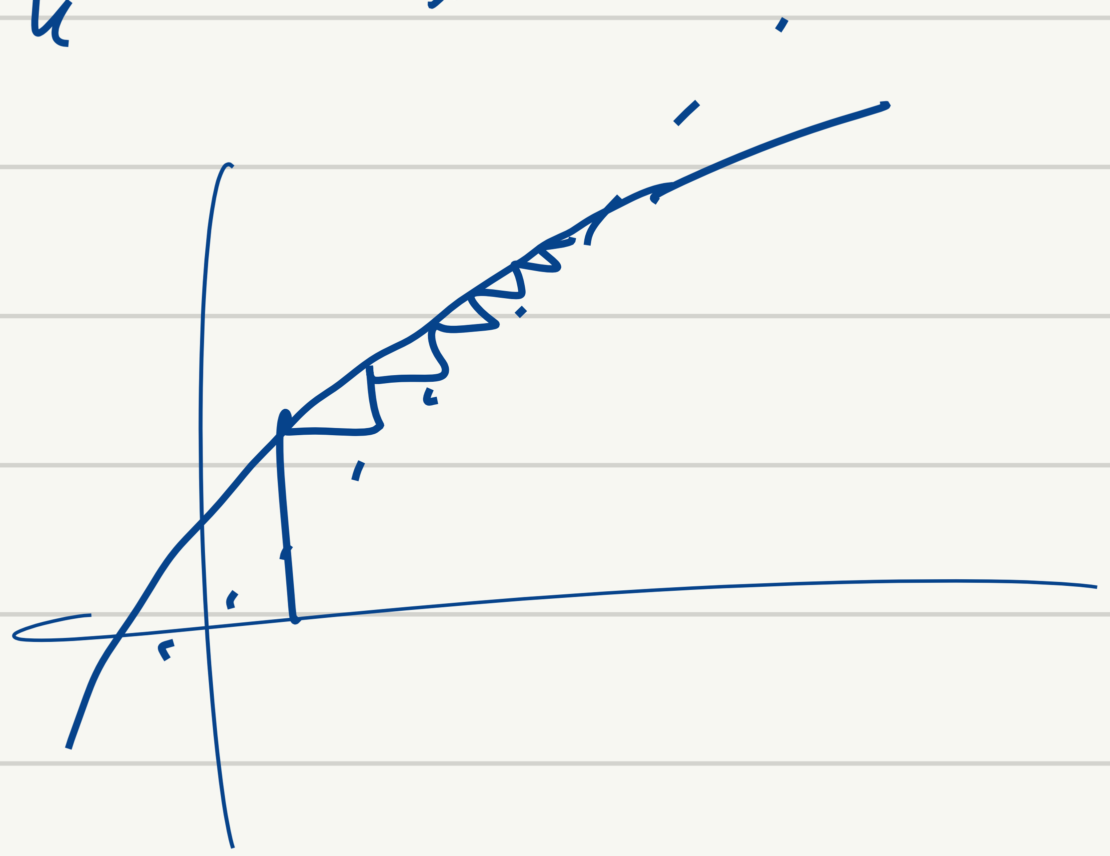
How much smaller is e_{k+1} compared to e_k?



$$e_{k+1} = g(x_* + e_k) - x_*$$

$$= \cancel{g(x_*)} + g'(x_*)\, e_k + O(|e_k|^2) - \cancel{x_*}$$

$$= g'(x_*)\, e_k + O(|e_k|^2)$$

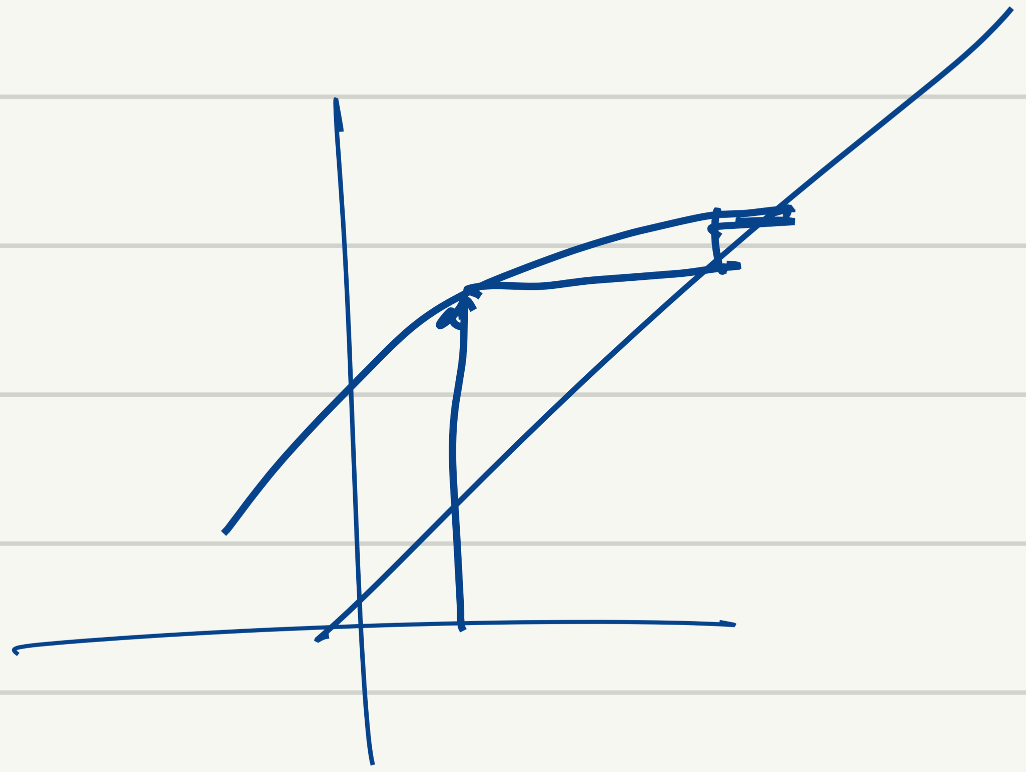


if $g'(x_*) \neq 0$ then

$$\lim_{k \to \infty} \frac{e_{k+1}}{e_k} = g'(x_*) \implies$$ Convergence rate is linear with rate constant $|g'(x_*)| < 1$

if $g'(x_*) = 0$: $e_{k+1} = O(|e_k|^2)$

then quadratic convergence!



Ex: find $g'(x_*)$ for 4 fns from before