COL726: Numerical Algorithms

Numerical analysis = "the study of algorithms for the problems of continuous mathematics"

Real numbers, continuous functions, vector spaces, ...

- cannot represent quantities exactly ←
- cannot solve most problems exactly

$x = e^{\pi}$

```
import math
print math.exp(math.pi)
```

$$\exp(\pi) = 1 + \pi + \frac{\pi^2}{2} + \frac{\pi^3}{3!} + \cdots$$

$\pi \approx \frac{22}{7}$ or 3.14159 or ...

$\underbrace{3.14159}\overbrace{26\ldots}$

$$\pi^2 \approx 3.14 \times 3.14 \approx \underbrace{9.\#\,\#}\overbrace{\#\,\#\,\#}^{\text{rounding error}}$$

$$\exp(\pi) = \underbrace{1 + \pi + \cdots + \frac{\pi^n}{n!}} + \underbrace{\frac{\pi^{n+1}}{(n+1)!} + \cdots}_{\text{truncation error}}$$

<u>Applications</u>

- Computational science & engineering
- Graphics & VR
- Vision, ML, Data mining
- Robotics, intelligent systems

<u>Course format</u>

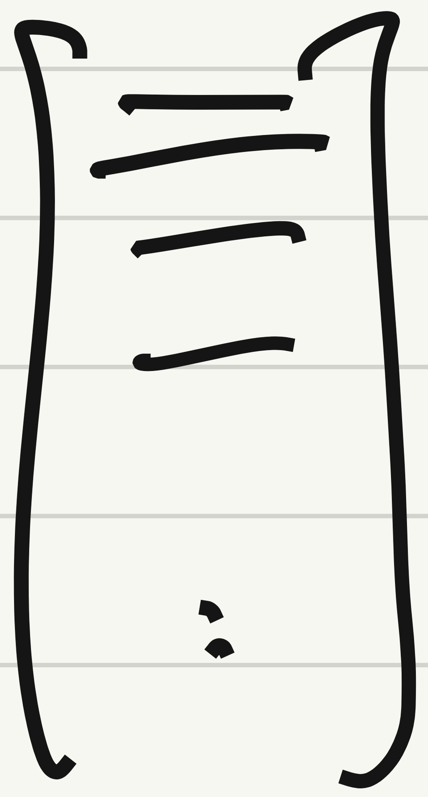
- 2 lectures M, Th
- Two TAs : col726ta @

- foundations

- linear algebra :

given A, b, find x s.t. $Ax = b$: solving linear sys.

given A, b. find x to minimize $\|Ax - b\|$: least sq. problems

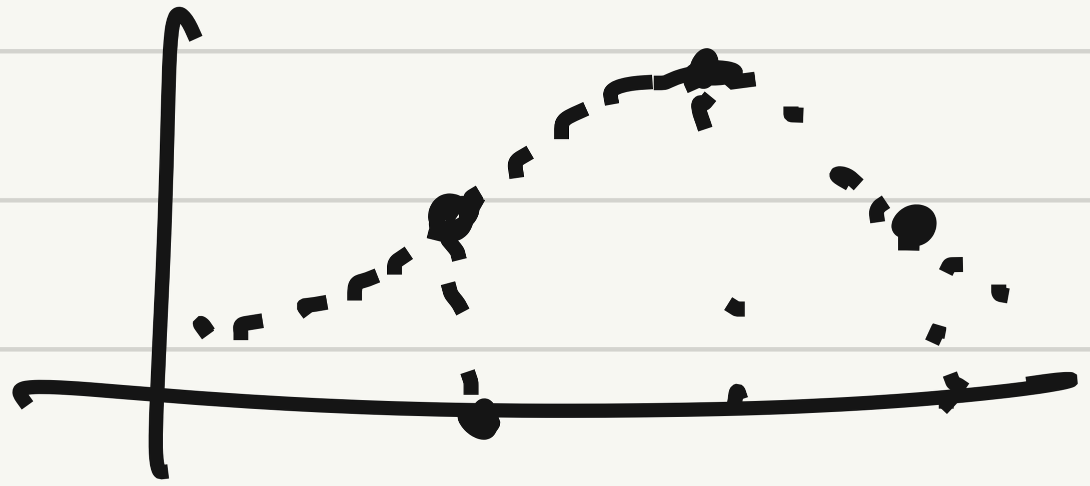


Given A, find x, λ s.t. $Ax = \lambda x$: eigenvalues eigenvectors

$x \neq 0$

iterative algs: $x_0 \to x_1 \to x_2 \to \cdots \quad \lim_{n \to \infty} x_n \to x^*$

- Nonlinear problems: Given $\to \boxed{f} \to$ find x s.t. $f(x) = y$



find $\int_a^b f(x)\,dx$ or $f'(x)$

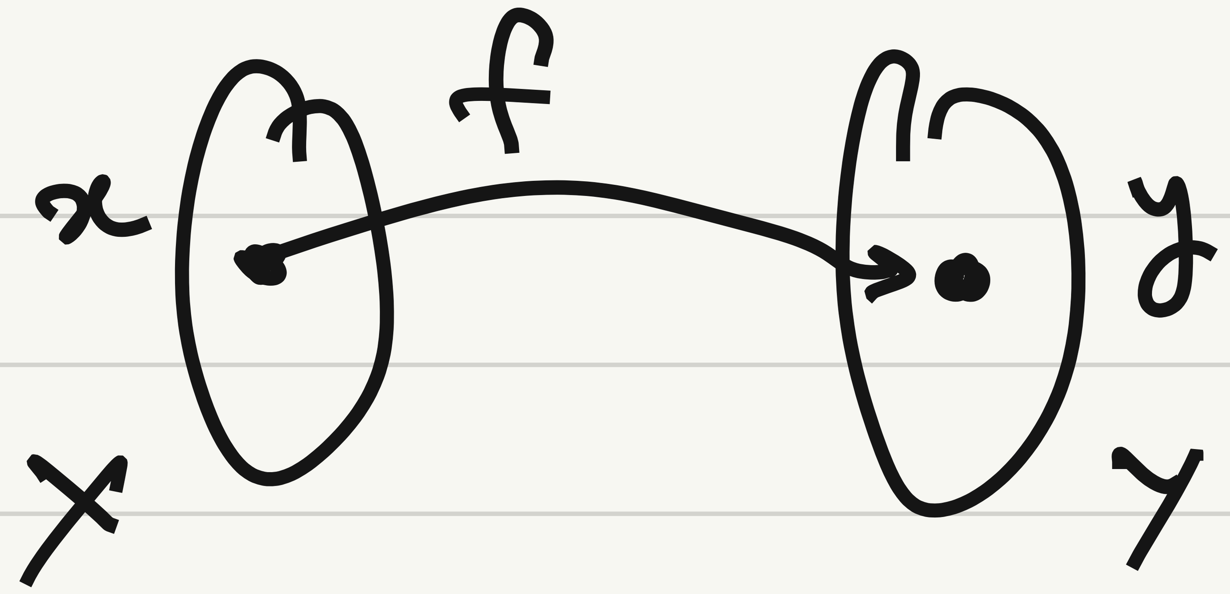
Optimization:

find x s.t.

$f(x) \le f(y)$

- Analyzing conditioning of problems, stability of algorithms
- Identify appropriate algs for num. problem
- Design new algs. for novel problems

$\boxed{\text{Conditioning}}$



Think of a problem as a <u>function</u> from some vector space X to some vector space Y.

eg. find $\sqrt{x}$: $\quad f: \mathbb{R}_{\geq 0} \to \mathbb{R}$ or $f: \mathbb{C} \to \mathbb{C}$ or ...

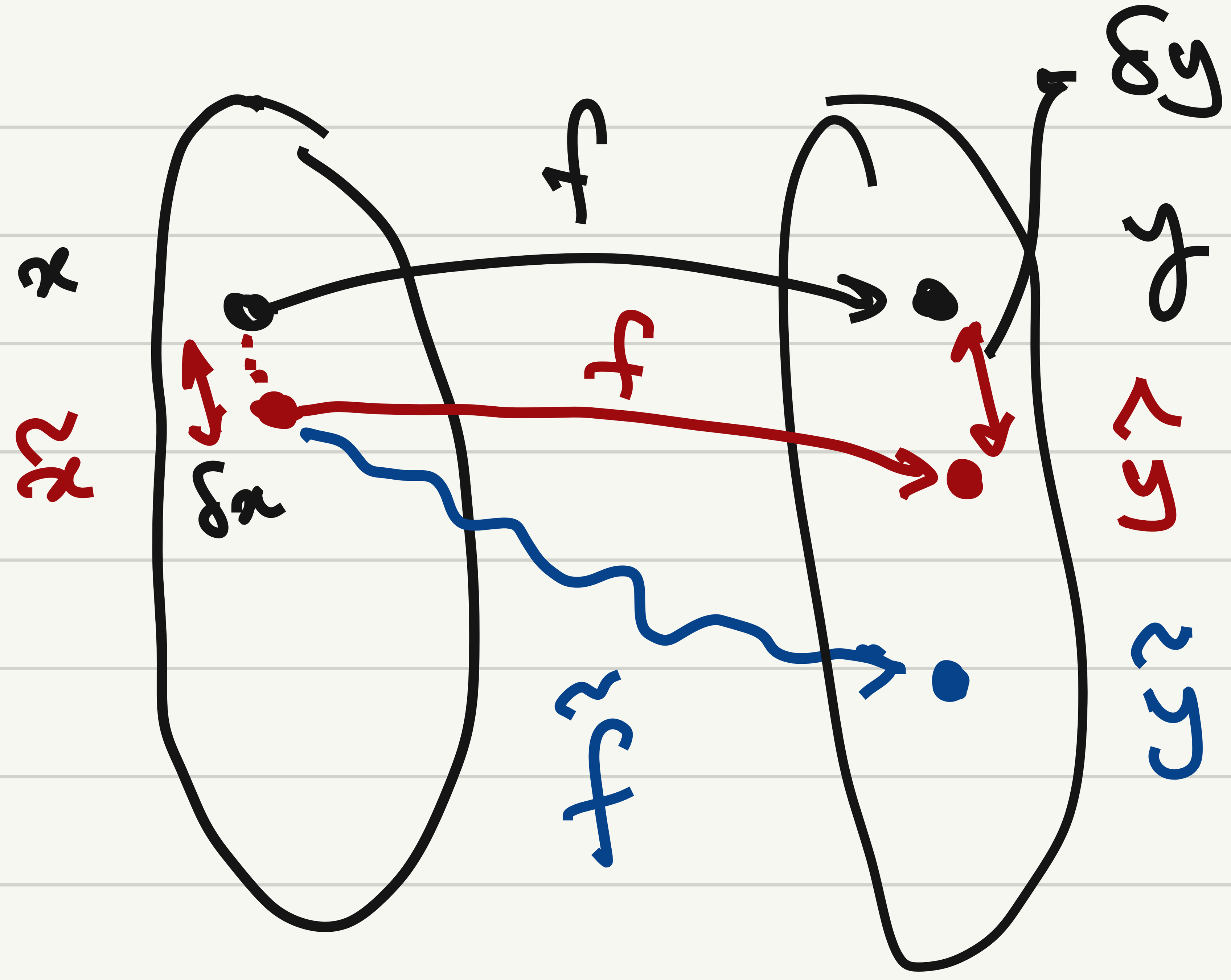
Given A, b, solve $Ax = b$ $\quad f: \mathbb{C}^{m \times m} \times \mathbb{C}^m \longrightarrow \mathbb{C}^m$

invertible $\qquad\qquad A \qquad b \qquad x$

find roots of polynomial

$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ $\quad f: \mathbb{C}^{n+1} \longrightarrow \mathbb{C}^n$

$\longrightarrow \quad x_1, x_2, \ldots, x_n$

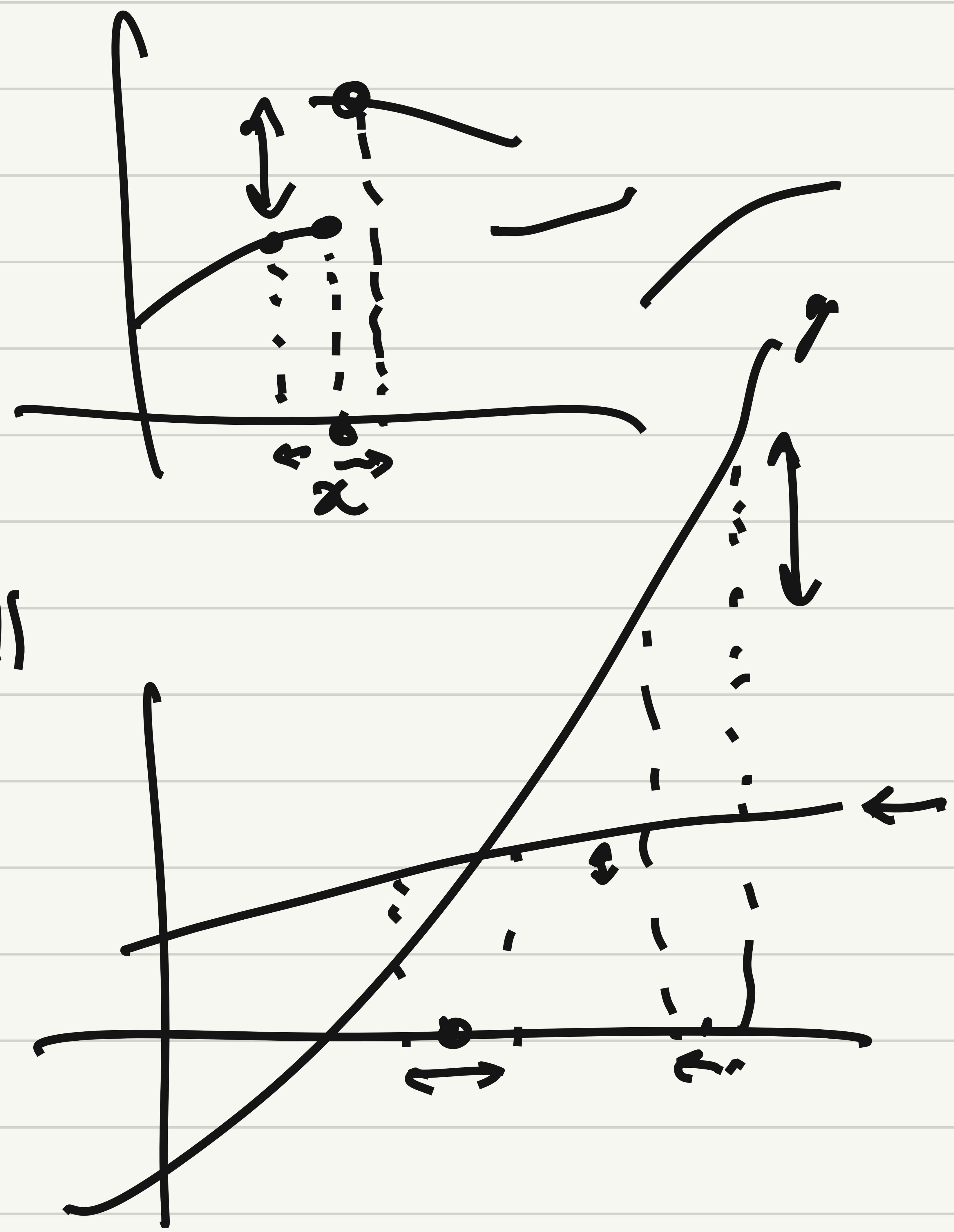


$x \longrightarrow \tilde{x} = x + \delta x$

How big is this error?

$\boxed{\text{Absolute error}}$: $\|\delta x\|$

$\boxed{\text{Relative error}}$: $\|\delta x\| / \|x\|$

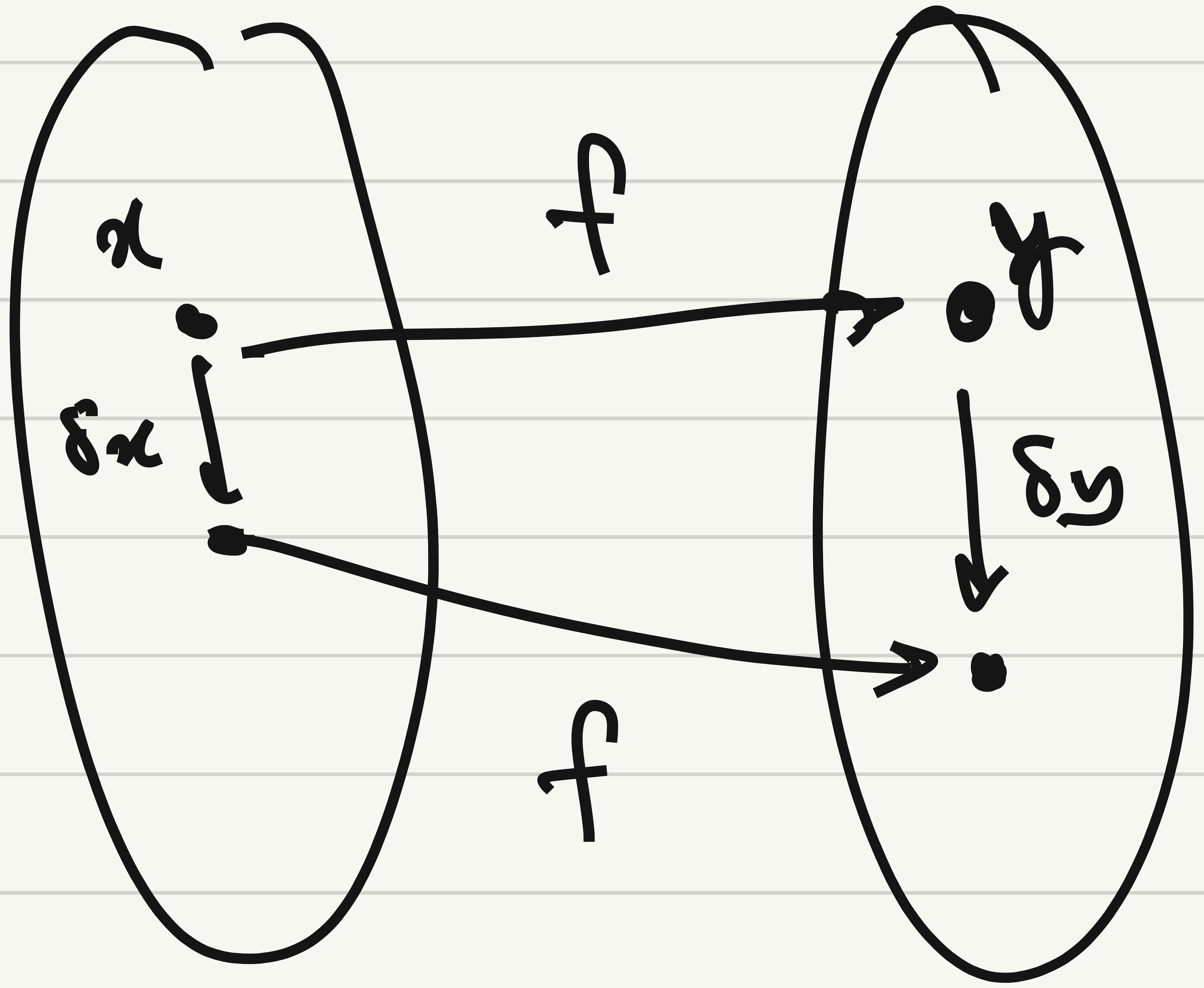


$\|\cdot\|$ denotes a <u>norm</u>: measures size of vector

e.g. $\|\vec{x}\|_2 = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$

$\|\vec{x}\|_\infty = \max(|x_1|, |x_2|, \ldots, |x_n|)$

$f(x) = y, \quad f(x + \delta x) = y + \delta y$

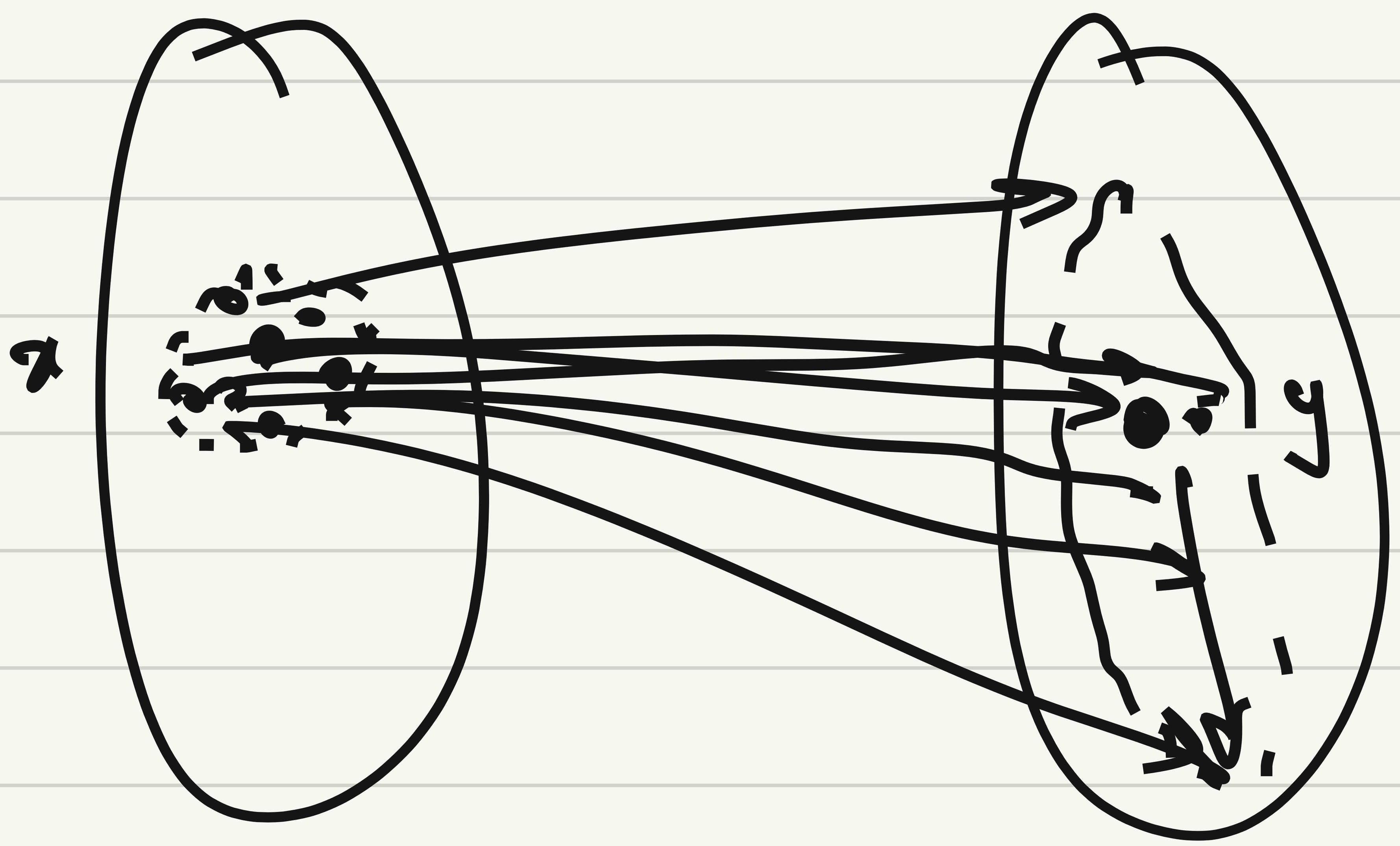


$f(x) = y$, $\quad f(x+\delta x) = y + \delta y$

$y = f(x) \qquad \delta y = f(x+\delta x) - y$

$\boxed{\text{Absolute condition number}}$ of f at x

(kappa) $\longrightarrow \hat{\kappa} = \lim_{\delta \to 0} \sup_{\|\delta x\| \le \delta} \dfrac{\|\delta y\|}{\|\delta x\|} = \sup_{\delta x} \dfrac{\|\delta y\|}{\|\delta x\|}$



$\boxed{\text{(Relative) condition number}}$; $= \sup_{\delta x} \dfrac{\|\delta y\| / \|\delta x\|}{\|y\| / \|x\|}$

$\kappa = \sup_{\delta x} \dfrac{\|\delta y\| / \|y\|}{\|\delta x\| / \|x\|} = \dfrac{\|x\|}{\|y\|} \sup_{\delta x} \dfrac{\|\delta y\|}{\|\delta x\|}$

example: $f : \mathbb{R} \to \mathbb{R}$ $\qquad \kappa = \sup_{\delta x} \dfrac{\|\delta f\| / \|f\|}{\|\delta x\| / \|x\|} = \dfrac{|x|}{|f|} \sup_{\delta x} \left| \dfrac{\delta f}{\delta x} \right|$

$$= \frac{|x|}{|f(x)|} \cdot |f'(x)|$$

example: $f(x) = x^2$

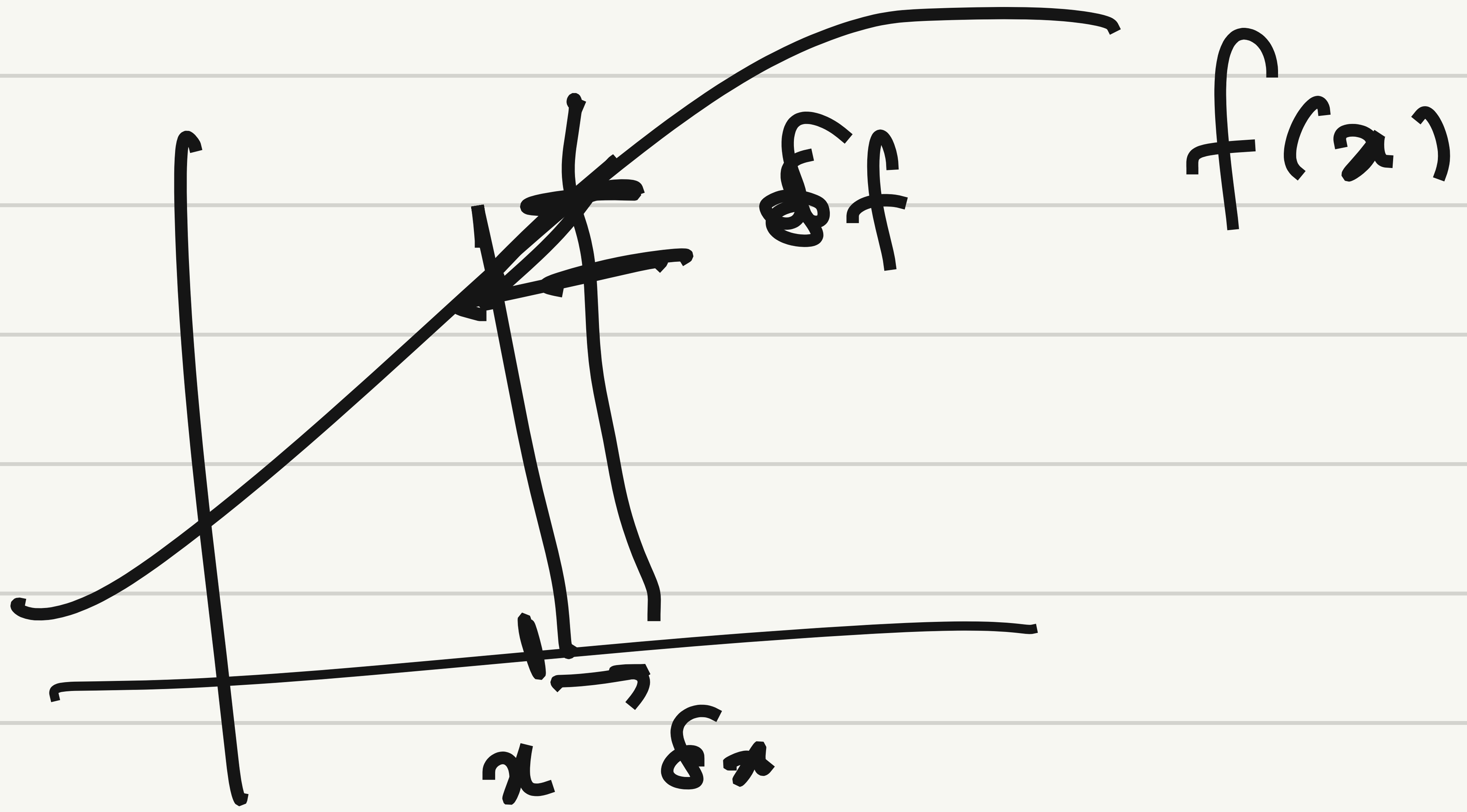
$$\kappa = \frac{|x|}{|x^2|} \cdot |2x| = 2$$

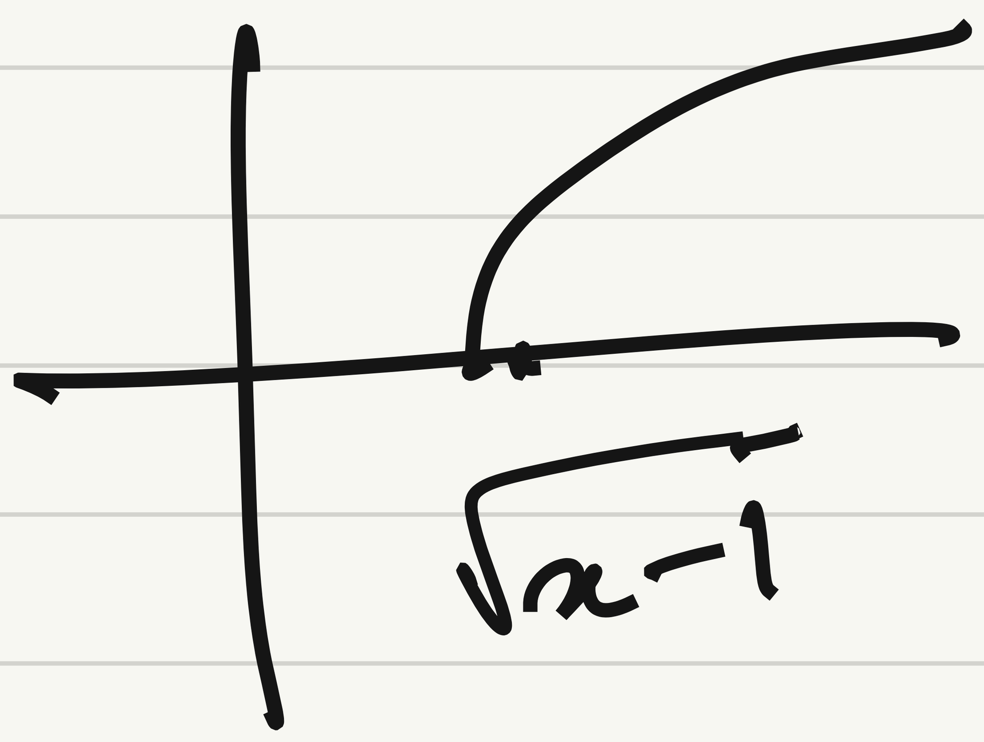
$f(x) = \sqrt{x}$: $\kappa = 1/2$

$\lim_{\delta \to 0} \sup_{|\delta x| \leq \delta} \ldots$

$$\frac{\delta f}{\delta x} \to f'(x)$$

$$\left| \frac{\delta f}{\delta x} \right| \to |f'(x)|$$





Example: $f(x_1, x_2) = x_1 - x_2$

$$f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = x_1 - x_2 \qquad : \mathbb{R}^2 \to \mathbb{R}$$

$$\kappa = \underbrace{\sup_{\delta x}}\ \frac{|\delta f| / |f|}{\|\delta x\| / \|x\|} \qquad \|x\| = \max(|x_1|, |x_2|)$$

$$f(x_1 + \delta x_1, x_2 + \delta x_2) = \underbrace{x_1 - x_2}_{y} + \underbrace{\delta x_1 - \delta x_2}_{\delta y}$$

$$= \lim_{\delta \to 0} \sup_{\|\delta x\| \le \delta} \frac{|\delta x_1 - \delta x_2| / |x_1 - x_2|}{\max(|\delta x_1|, |\delta x_2|) / \max(|x_1|, |x_2|)}$$

$$= \underbrace{\frac{\max(|x_1|, |x_2|)}{|x_1 - x_2|}}\ \underbrace{\lim_{\delta \to 0} \sup_{\|\delta x\| \le \delta} \frac{|\delta x_1 - \delta x_2|}{\max(|\delta x_1|, |\delta x_2|)}}$$

$\|\delta x\| \le \delta$

$|\delta x_1| \le \delta$ and $|\delta x_2| \le \delta$

$$K = (\ \ldots\ )\ \underbrace{\lim_{\delta \to 0}\ \sup_{\underbrace{\|\delta x\| \le \delta}} \frac{|\delta x_1 - \delta x_2|}{\max(|\delta x_1|, |\delta x_2|)}}_{} \longrightarrow 2$$

Sup is attained when $\delta x_1 = -\delta x_2$

$$\therefore K = \frac{\max(|x_1|, |x_2|)}{|x_1 - x_2|} \times 2$$

$$\begin{array}{r} \overbrace{12345} \\ -\ 12321 \\ \hline \underbrace{24} \end{array}$$