LU factorization (LU decomposition)

$A\vec{x} = \vec{b}$: $\underline{\underline{A = QR}}$, $R\vec{x} = Q^*\vec{b}$

Gaussian elimination

$$\begin{bmatrix} 2 & 1 & 1 \\ 4 & 3 & 3 \\ 8 & 7 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 5 \end{bmatrix} \longrightarrow \underbrace{\begin{bmatrix} 2 & 1 & 1 \\ 4 & 3 & 3 \\ 8 & 7 & 9 \end{bmatrix}}_{A} \xrightarrow{L_1} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 3 & 5 \end{bmatrix} \xrightarrow{L_2} \underbrace{\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}}_{U}$$

What matrix operation transforms

A to $\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 3 & 5 \end{bmatrix}$?

$$A \longrightarrow \underset{L_1}{\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}} A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 3 & 5 \end{bmatrix}$$

$$A \to L_1A \to L_2L_1A \to \cdots$$

$$\cdots \to L_{m-1}\cdots L_2L_1A = U$$

$$L_{m-1}\cdots L_2 L_1 A = U \quad \Rightarrow \quad A = \underbrace{(L_{m-1}\cdots L_2 L_1)^{-1}}\, U = LU$$

$$L = L_1^{-1} L_2^{-1} \cdots L_{m-1}^{-1}$$

$$L_1 = \begin{bmatrix} 1 & & & & \\ l_{21} & 1 & & & \\ l_{31} & & 1 & & \\ \vdots & & & \ddots & \\ l_{m1} & & & & 1 \end{bmatrix}$$

$$L_1^{-1} = \begin{bmatrix} 1 & & & & \\ -l_{21} & 1 & & & \\ -l_{31} & & 1 & & \\ \vdots & & & \ddots & \\ -l_{m1} & & & & 1 \end{bmatrix}$$

$$L_1^{-1} = \begin{bmatrix} 1 & & & & \\ | & 1 & & & \\ l_1 & & 1 & & \\ | & & & \ddots & \\ | & & & & 1 \end{bmatrix}$$

$$L_2^{-1} = \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & | & 1 & & \\ & l_2 & & \ddots & \\ & | & & & 1 \end{bmatrix}$$

$$L_1^{-1} L_2^{-1} = \begin{bmatrix} 1 & & & & \\ | & 1 & & & \\ l_1 & | & 1 & & \\ | & l_2 & & \ddots & \\ | & | & & & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & \cdots & & 1 \end{bmatrix}_L$$

Given A

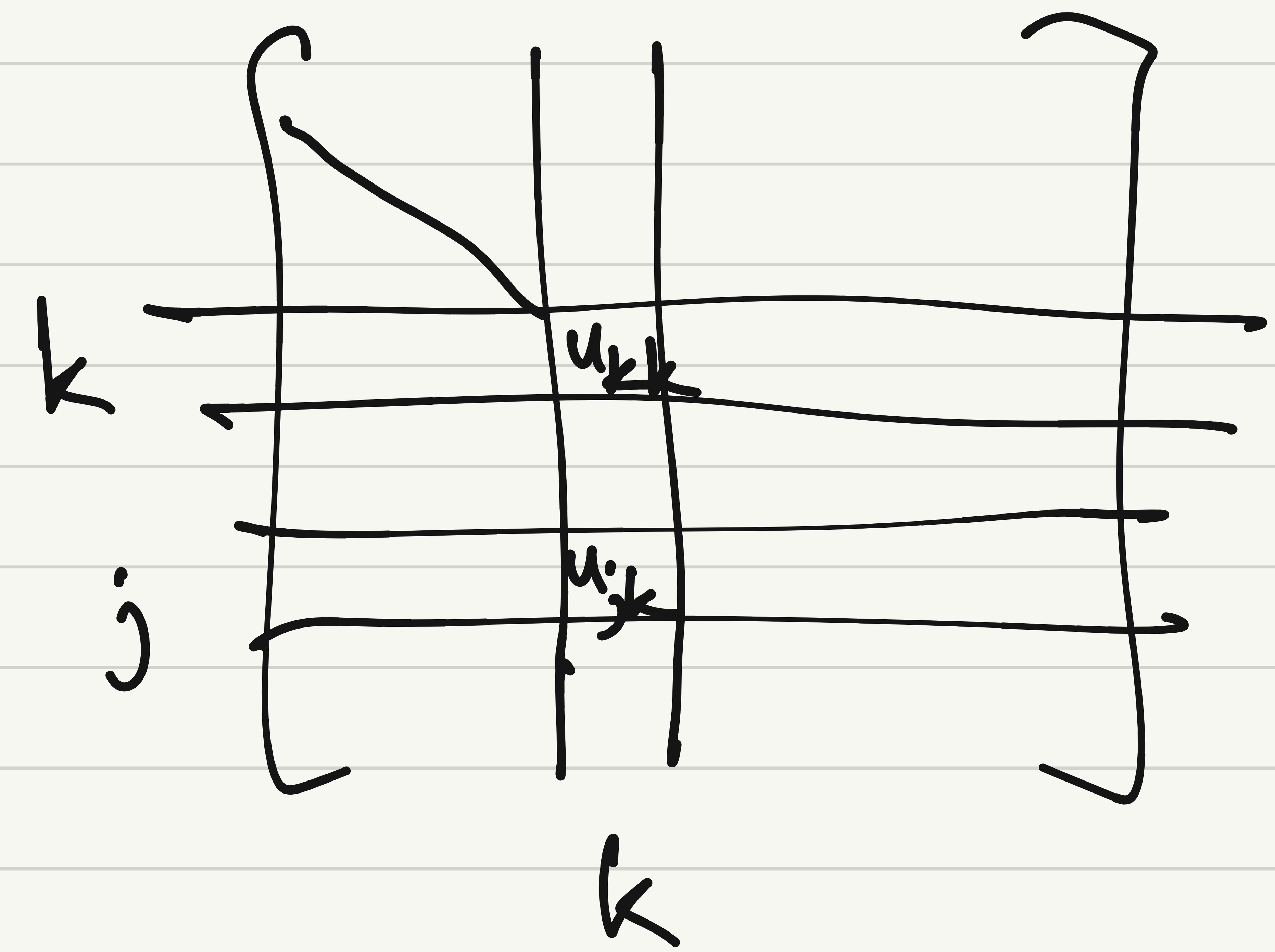
initalize $U = A,\ L = I$

for rows $k = 1, 2, \ldots, m-1$:

for rows $j = k+1, \ldots, m$:

$$l_{jk} = u_{jk} / u_{kk}$$

$$\vec{u}_j = \vec{u}_j - l_{jk} \cdot \vec{u}_k$$



op count: $\boxed{\sim \frac{2}{3} m^3 \text{ flops}}$

$A\vec{x} = \vec{b}$

$\hookrightarrow A = LU$

$O(m^3)$ flops

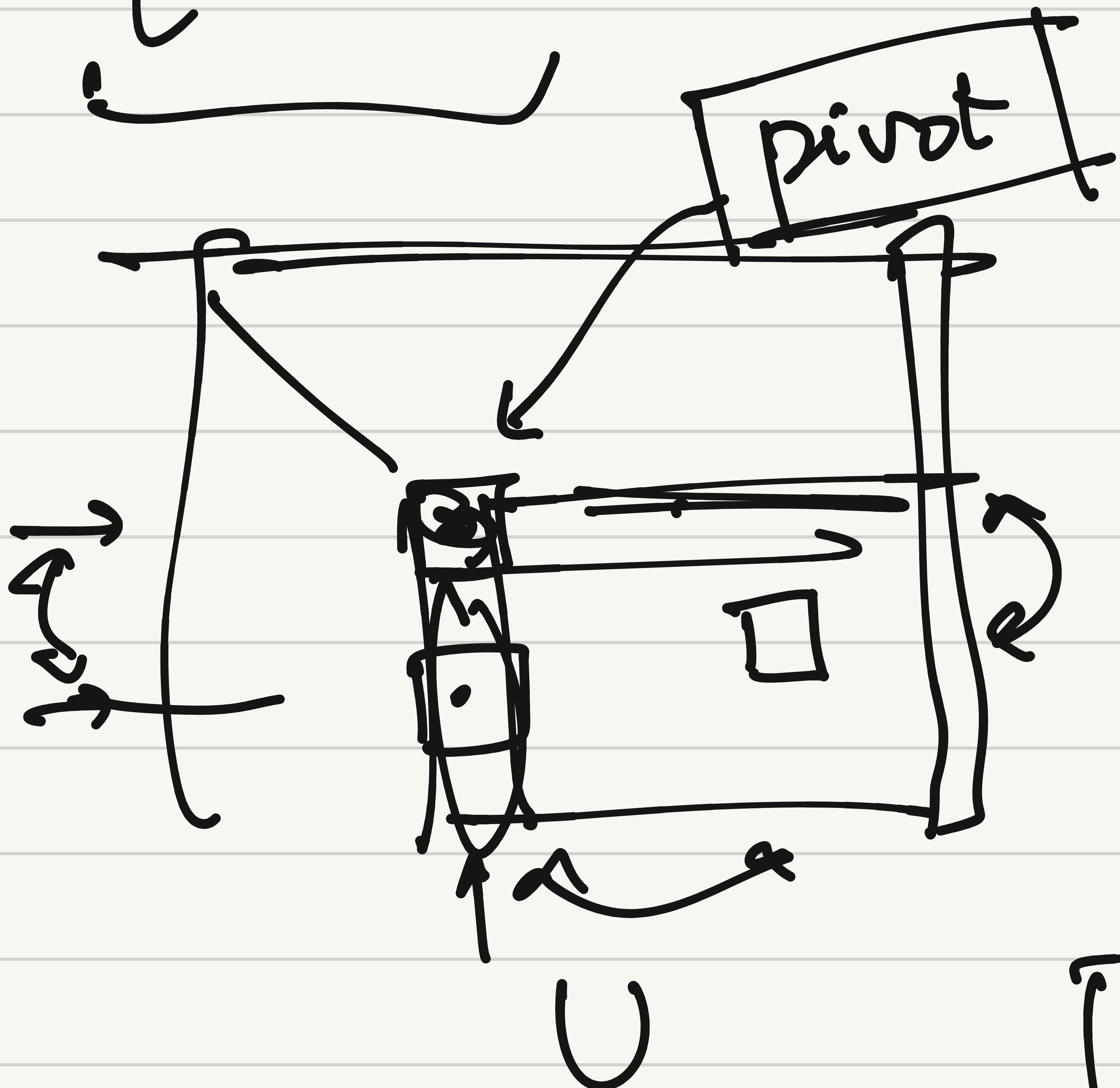
$L\underbrace{U\vec{x}}_{\vec{w}} = \vec{b}$

solve $L\vec{w} = \vec{b}$ $\quad O(m^2)$ flops

solve $U\vec{x} = \vec{w}$ $\quad O(m^2)$ flops

$$\underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}}$$

$$\begin{bmatrix} 10^{-20} & 1 \\ 1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 10^{-20} & 1 \\ 0 & -10^{20}+1 \end{bmatrix}$$



look in $\vec{u}_{k:m,k}$, find largest abs. entry u_{ik}

swap rows i, k

pivoting

$$A \to P_1 A \to L_1 P_1 A \to \cdots$$

partial pivoting

$$L_{m-1} P_{m-1} \cdots L_2 P_2 L_1 P_1 A = U$$

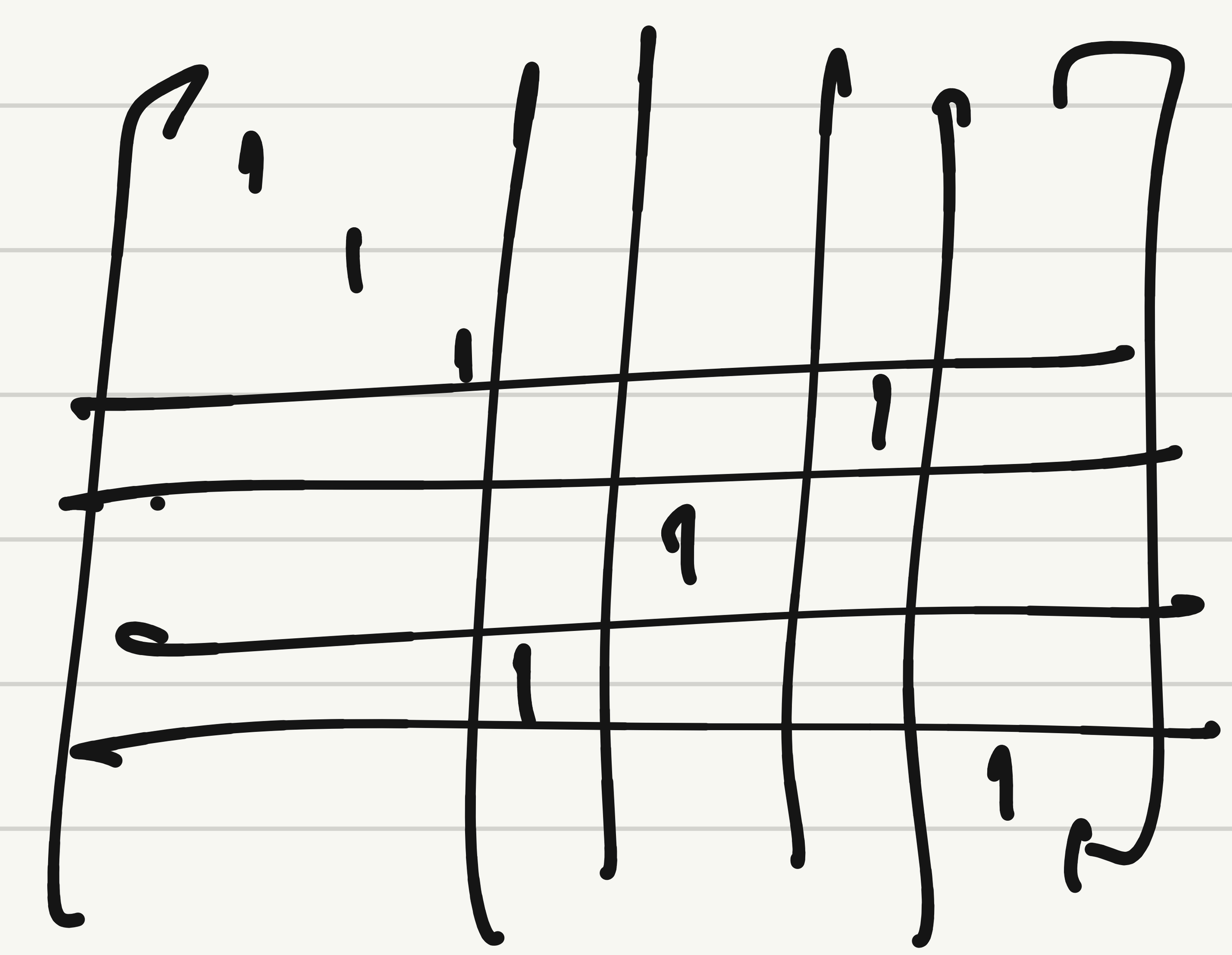
Complete pivoting

$$\cdots L_2 P_2 L_1 P_1 A Q_1 Q_2 \cdots$$

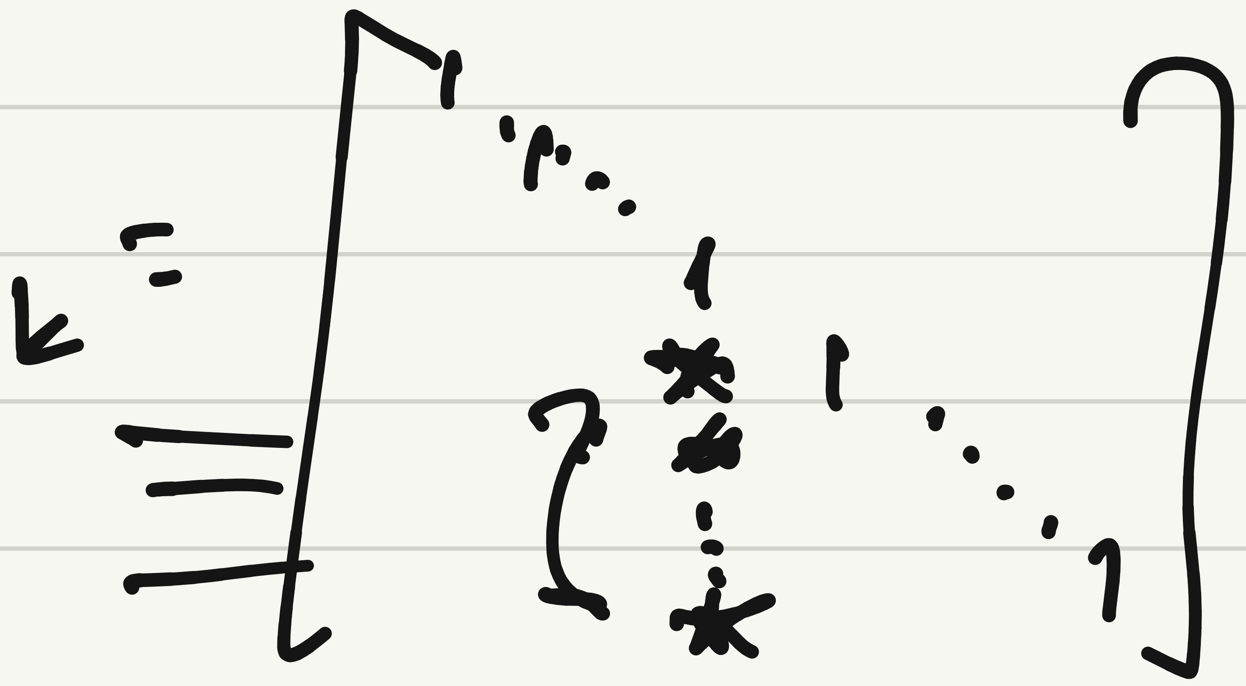
P_k

$k \to$

$i \to$

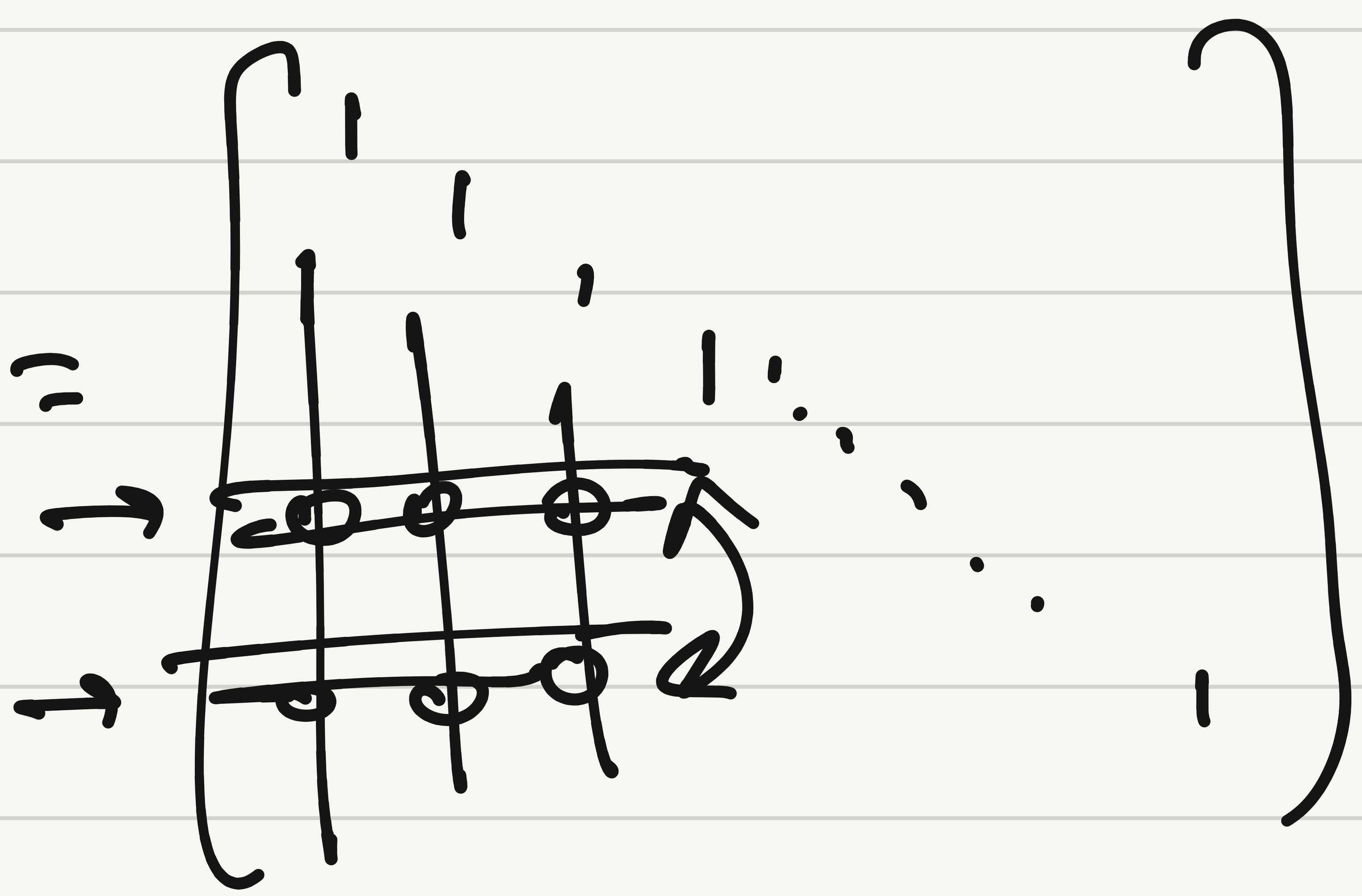


$$L_{m-1}P_{m-1}\cdots L_2P_2L_1P_1A = U$$

$L_k =$ 

$$L'_{m-1}\cdots L'_2L'_1\,P_{m-1}\cdots P_2P_1A = U \quad\longrightarrow\quad \underline{PA = LU}$$

P : $\boxed{\text{permutation matrix}}$: one 1 in each row & col.

$L =$ 

LU with pivoting:

Init $U = A$, $L = I$, $P = I$

for each row $k = 1, \ldots, m-1$:

pick row i with max $|u_{ik}|$

swap rows i, k of U, P

subdiagonal entries of rows i, k of L

for rows $j = k+1 \ldots m$

invariant: $PA = LU$

$\underline{|l_{jk}| \leq 1}$

cost $\sim \frac{2}{3}m^3$

Stability of LU

Given $A \in \mathbb{C}^{m \times m}$, computed $\tilde{L}, \tilde{U}$ satisfy $\tilde{L}\tilde{U} = A + \delta A$

with $\dfrac{\|\delta A\|}{\|L\|\,\|U\|} = O(\varepsilon_{machine})$

$$\frac{\|\delta A\|}{\|A\|} = O(\varepsilon_m)$$

LU is backward stable if $\underline{\|L\|\,\|U\|} = O(\|A\|)$

with partial pivoting, $\|L\| = O(1)$

$\|U\| = O(2^{m-1})$

$$\underbrace{\begin{bmatrix} 1 & & & & \\ | & 1 & & & \\ | & | & 1 & & \\ | & | & | & \ddots & \\ | & | & | & \cdots - & 1 \quad 1 \end{bmatrix}}_{|l_{jk}| \le 1}
\qquad
\begin{bmatrix} 1 & & & & & & 1 \\ -1 & 1 & & & & & 1 \\ -1 & -1 & 1 & & & & 1 \\ -1 & -1 & -1 & 1 & & & 1 \\ \vdots & & \vdots & & \ddots & & \vdots \\ & & & & & 1 & 1 \\ -1 & -1 & -1 & \cdots & -1 & -1 & 1 \end{bmatrix}
\longrightarrow
\begin{bmatrix} 1 & & & & & & 1 \\ & 1 & & & & & 2 \\ & & 1 & & & & 4 \\ & & & \ddots & & & 8 \\ & & & & 1 & & \vdots \\ & & & & & 1 & \vdots \\ & & & & & & 2^{m-1} \end{bmatrix}$$

matrices which cause exponential growth in $\|U\|$ are exponentially rare!

In practice LU with partial pivoting is extremely stable

$A \to U$

$$\rho = \frac{\max |u_{ij}|}{\max |a_{ij}|} \qquad \|U\| = O(\rho \|A\|)$$

$\rho \sim \sqrt{m}$ $\qquad \rho = 2^{m-1}$

Smoothed analysis

Cholesky factorization

$A = LL^*$

lower triangular $\in \mathbb{C}^{m \times m}$

$A \in \mathbb{C}^{m \times m}$

$A = A^*$. positive definite

A is Hermitian positive definite if $\vec{x}^* A \vec{x} > 0$ for all $\vec{x} \neq 0$

$\vec{x} \mapsto A\vec{x} \quad : \mathbb{C}^m \to \mathbb{C}^m$

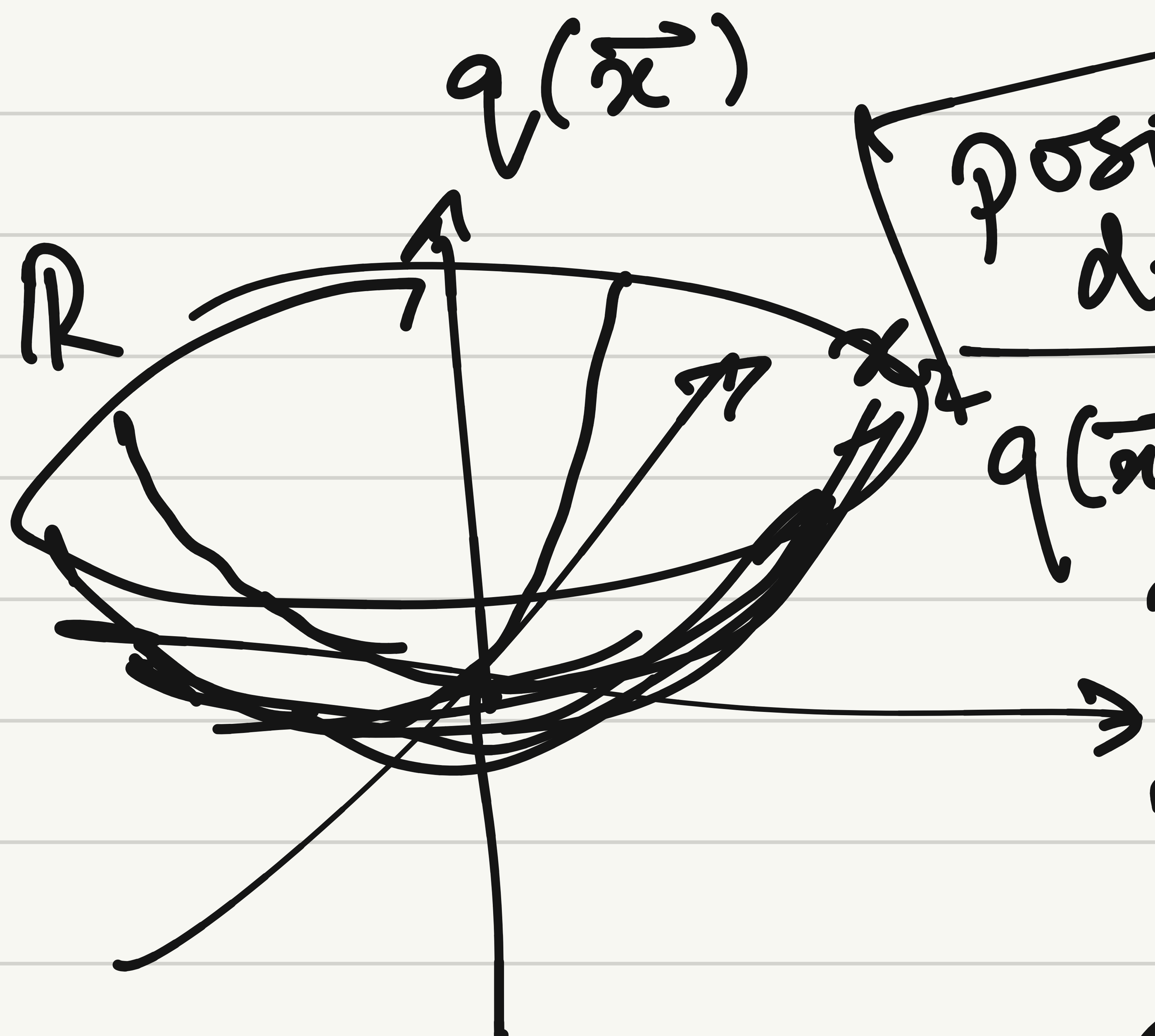
$\vec{x} \mapsto \vec{x}^* A\vec{x} \quad : \mathbb{C}^m \to \mathbb{R}$

<u>quadratic form</u>

$q : \mathbb{C}^m \to \mathbb{R}$

$q(\vec{x}) = \underline{\vec{x}^* A \vec{x}}$ $\longleftrightarrow$ $q(\vec{x}) = a_{11}|x_1|^2 + 2a_{12}x_1^* x_2 + a_{22}x_2^2 + \cdots$

$\mathbb{R}^2 \to \mathbb{R}$



$\boxed{\text{positive def.:}}$

$q(\vec{x}) > 0$ for $\vec{x} \neq 0$

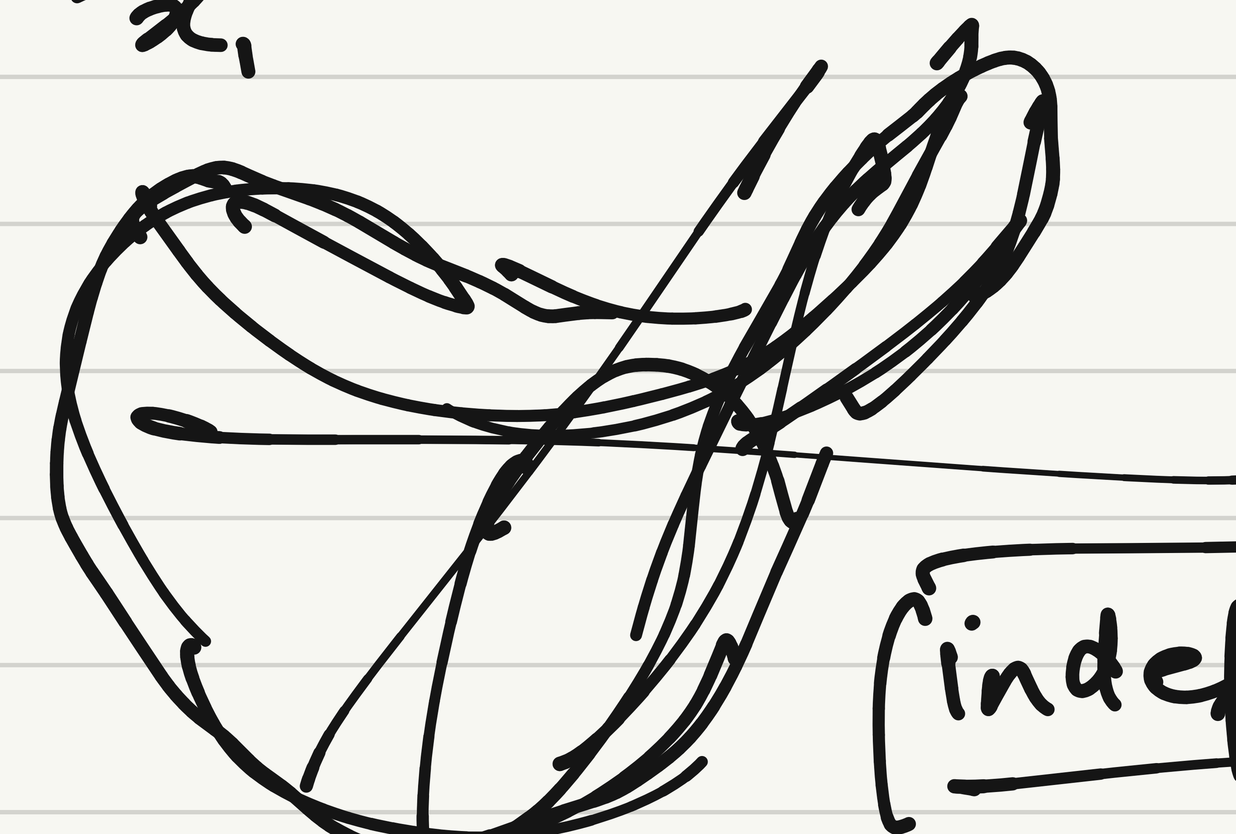
$A = I$

$A = \begin{bmatrix} 1 & \\ & 0 \end{bmatrix}$

$\boxed{\text{positive semidefinite}}$

$q(\vec{x}) \geq 0$ for $\vec{x} \neq 0$

$A = \begin{bmatrix} 1 & \\ & -1 \end{bmatrix}$



$\boxed{\text{indefinite}}$

$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$