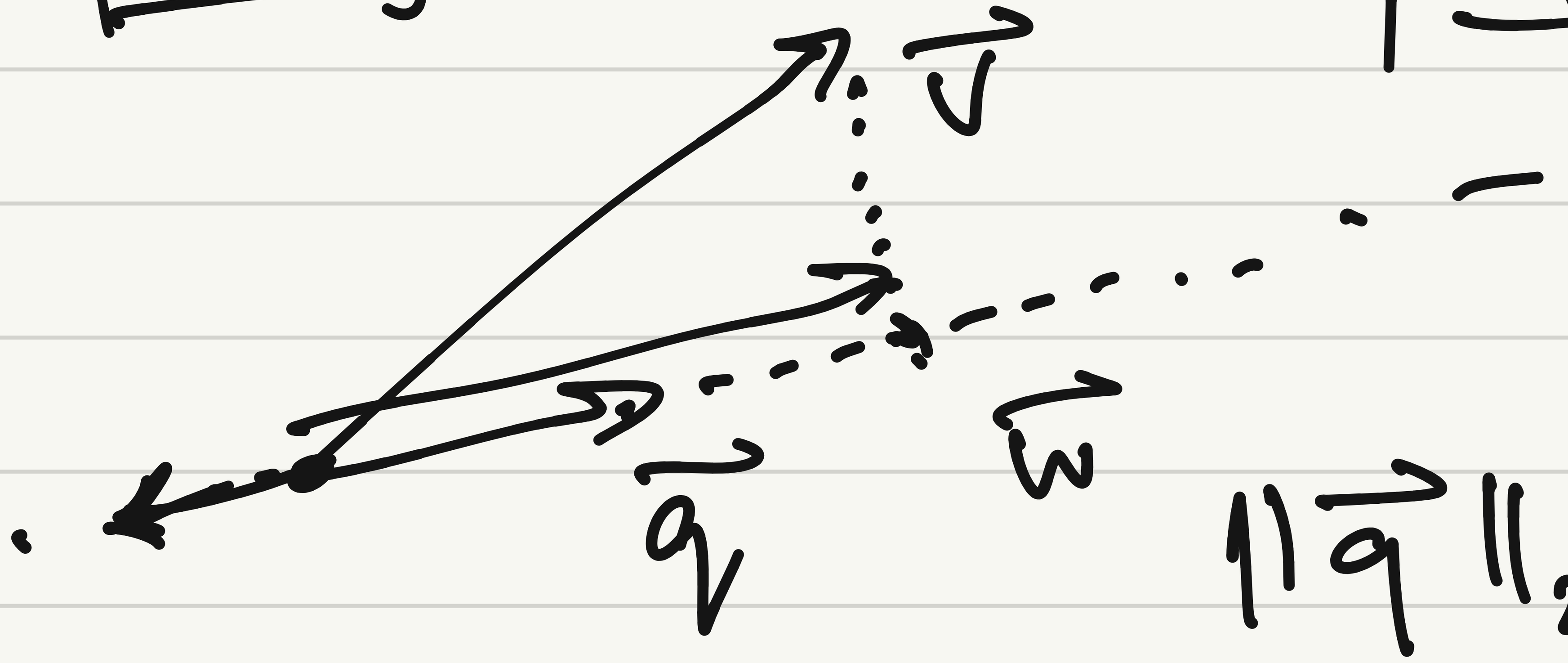
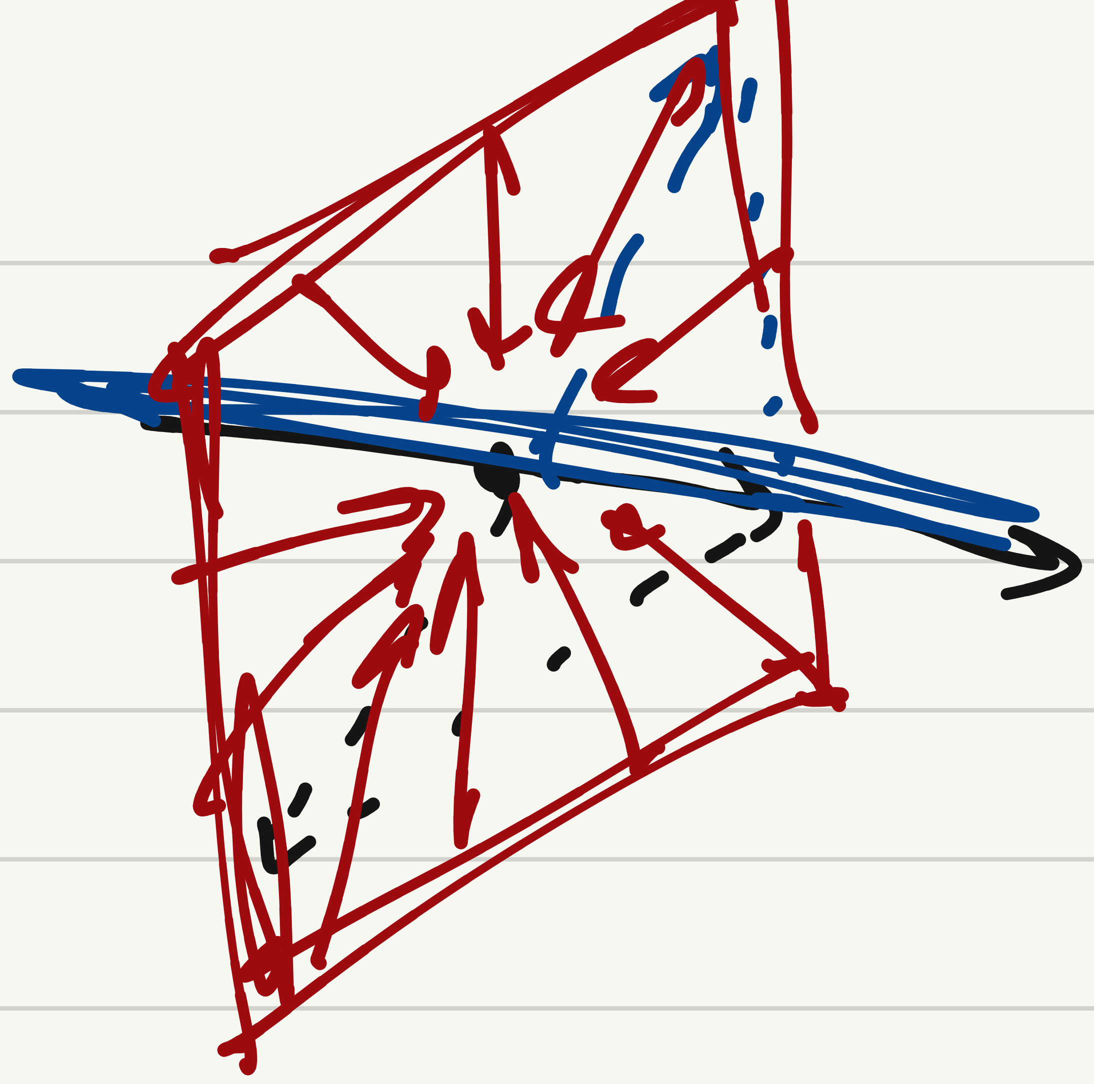
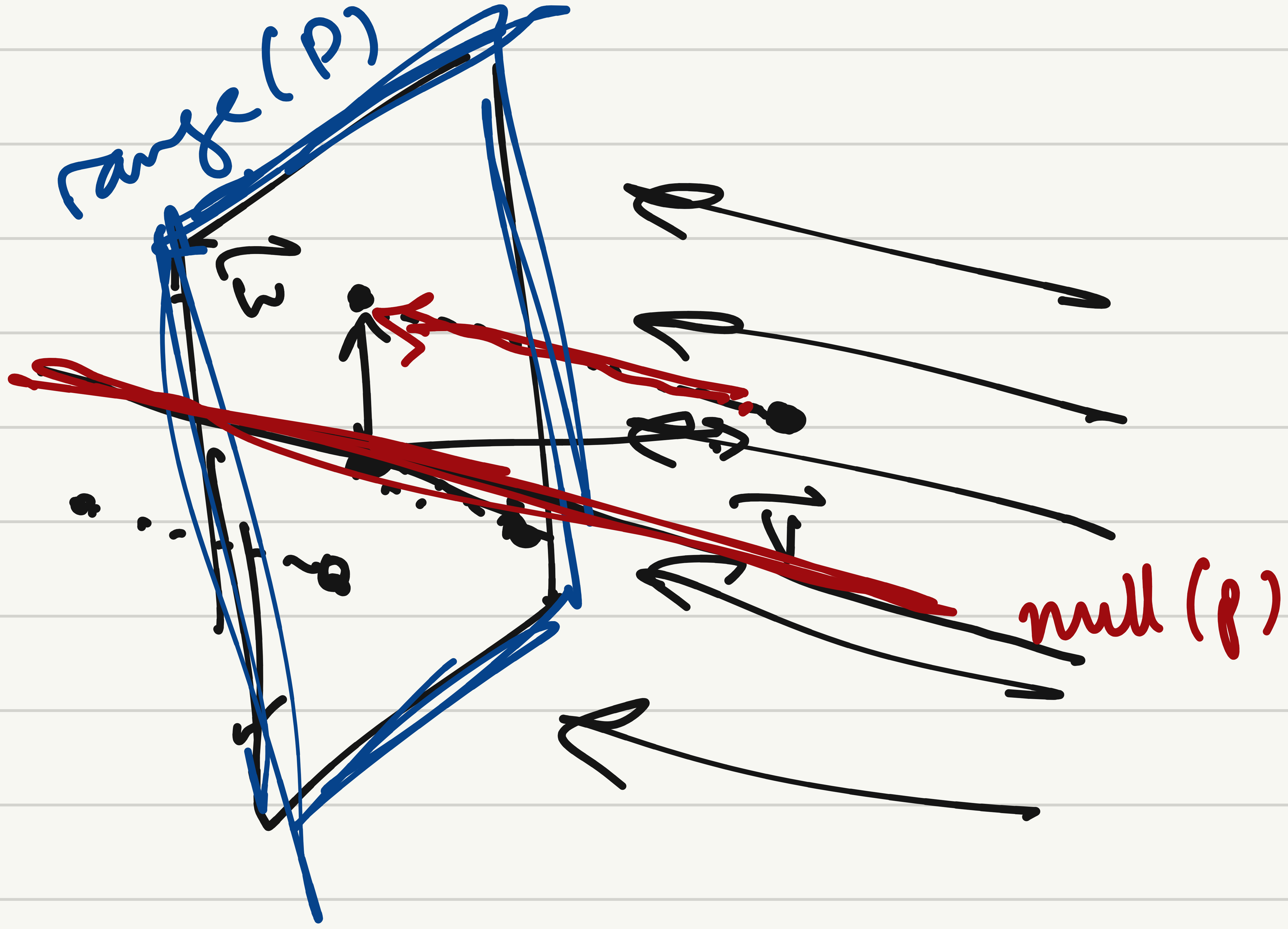
$\boxed{\text{Projectors}}$ or projection matrices



$$\vec{w} = \vec{q}\,(\vec{q}^{\,*}\vec{v}) = \underbrace{(\vec{q}\,\vec{q}^{\,*})}_{P}\,\vec{v}$$

$$\|\vec{q}\|_2 = 1$$





$$\underbrace{P\vec{v}}_{\vec{w}} \longrightarrow P\vec{w} = \vec{w}$$

$$P^2\vec{v} = P\vec{v} \quad \text{for all } \vec{v}$$

$$\Rightarrow \boxed{P^2 = P} \longleftarrow \text{alg. def. of } \boxed{\text{projector}}$$

set of vectors which don't change $= \boxed{\text{range}(P)}$

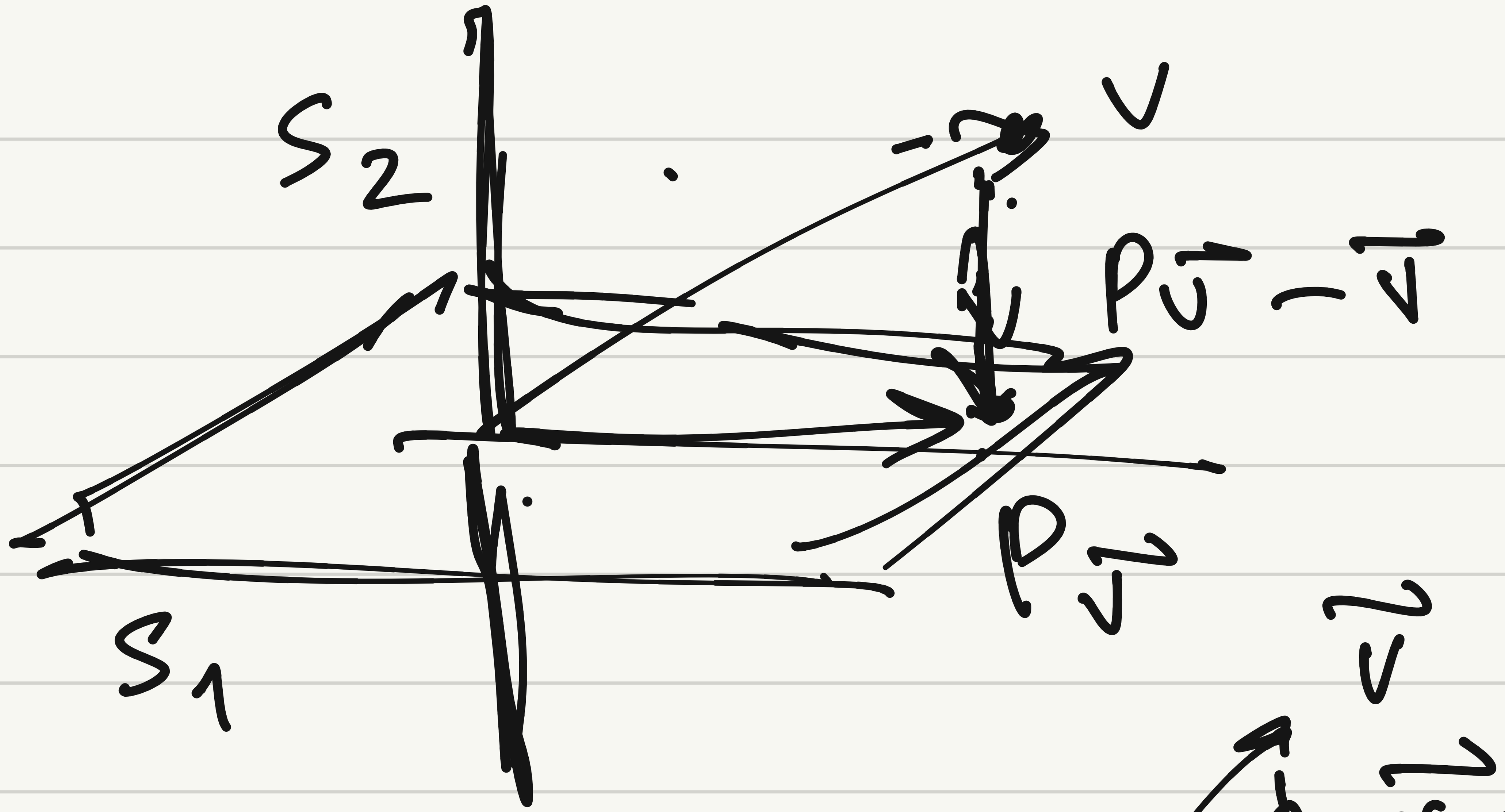
set of changes $\{\underline{P\vec{v} - \vec{v}} : \vec{v} \in \mathbb{C}^m\} = \text{range}(P - I)$

$$P(P\vec{v} - \vec{v}) = P^2\vec{v} - P\vec{v} = \cancel{(P^2 - P)}\,\vec{v} = 0$$

$= \boxed{\text{null}(P)}$

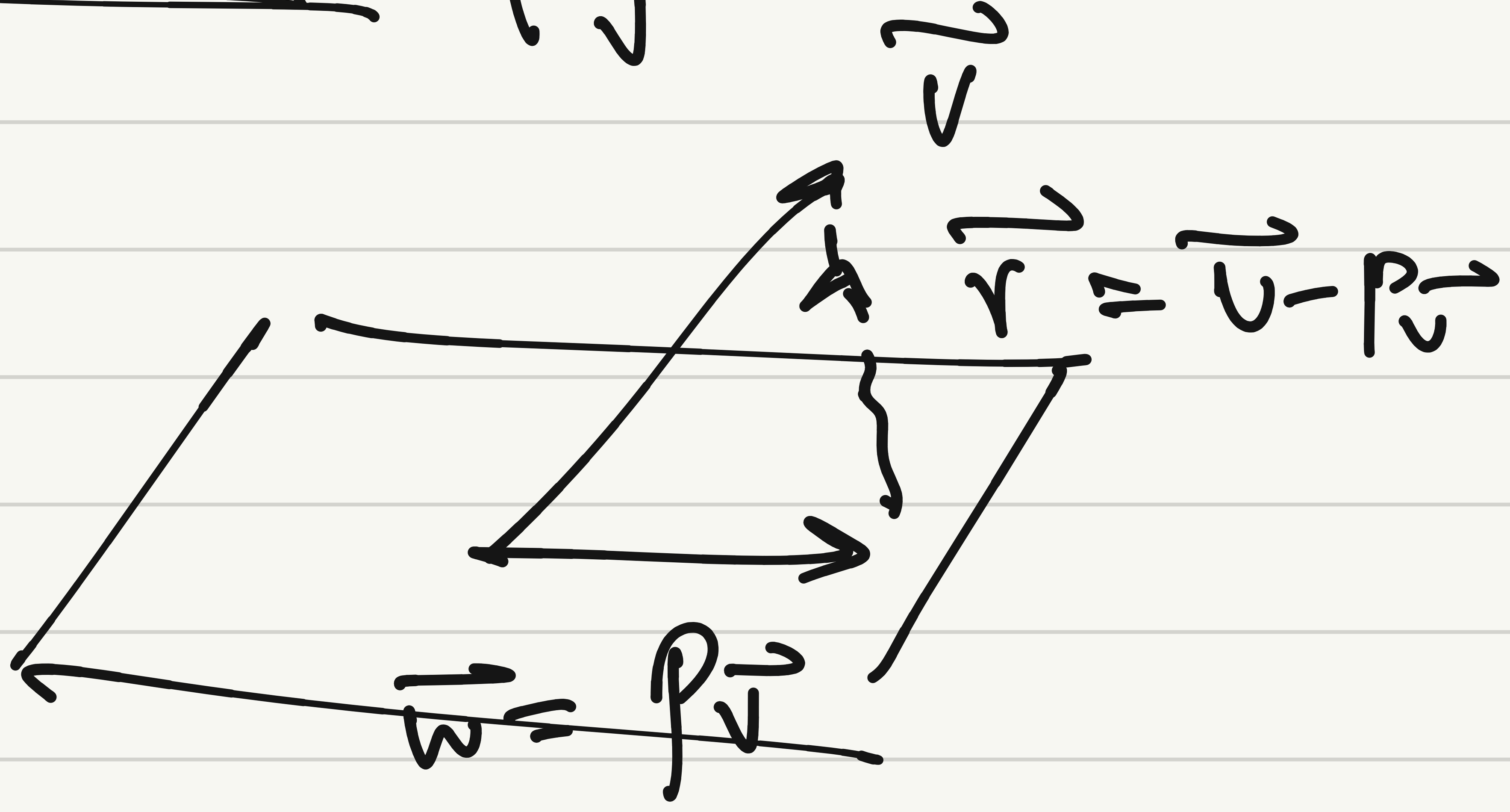
$\vec{v} \in \mathbb{C}^m$: $\quad P\vec{v} \in \text{range}(P)$

$\vec{v} - P\vec{v} \in \text{null}(P)$



$\vec{y} \in \text{null}(P) \quad \exists\, \vec{v} : P\vec{v} - \vec{v} = \vec{y}$

$\vec{r} = \vec{v} - P\vec{v} = \underbrace{(I-P)}\vec{v}$ ⟶ also a projection!



$\boxed{\text{complementary projector}}$

$(I-P)^2 = I - 2P + P^2$

$= I - P$

$\text{range}(I-P) = \text{null}(P)$

$\text{null}(I-P) = \text{range}(P)$

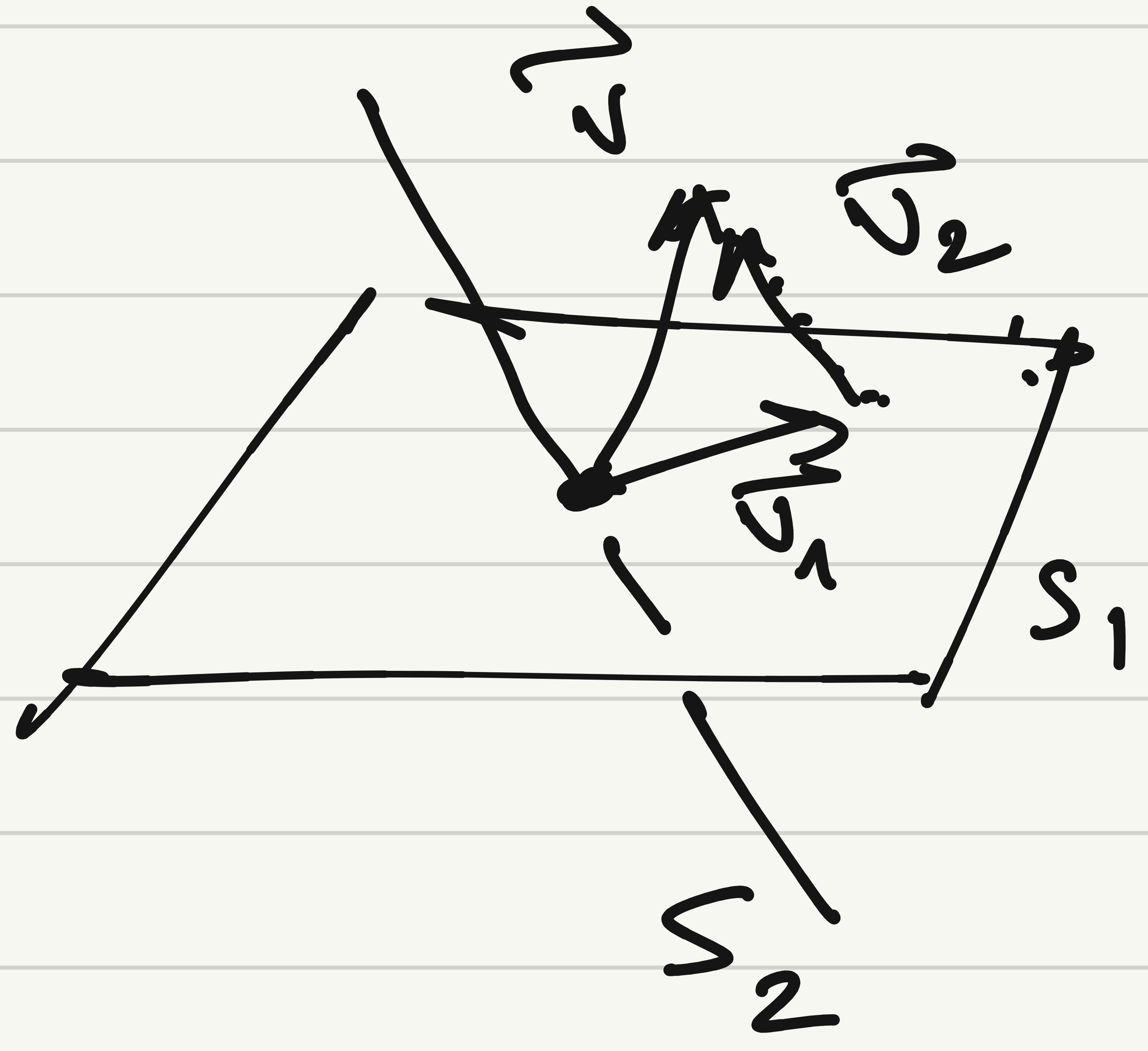
$$\begin{tikzcd} & S_1 & \\ P \arrow[ur, "\text{range}"] \arrow[dr, "\text{null}"'] & & I-P \arrow[ul, "\text{null}"'] \arrow[dl, "\text{range}"] \\ & S_2 & \end{tikzcd}$$

$$\vec{v} = \underset{\in S_1}{P\vec{v}} + \underset{\in S_2}{(\vec{v} - P\vec{v})}$$

S_1, S_2 subspaces : for any $\vec{v}$, $\exists$ unique $\vec{v}_1 \in S_1$, $\vec{v}_2 \in S_2$
s.t. $\vec{v} = \vec{v}_1 + \vec{v}_2$

$\underbrace{S_1, S_2}$ $\to$ $\boxed{\text{complementary subspaces}}$ $\Longleftrightarrow$ $S_1 \cap S_2 = \{\vec{0}\}$

and $\underbrace{S_1 + S_2}_{\{\vec{v}_1 + \vec{v}_2 \,:\, \vec{v}_1 \in S_1,\ \vec{v}_2 \in S_2\}} = \mathbb{C}^m$

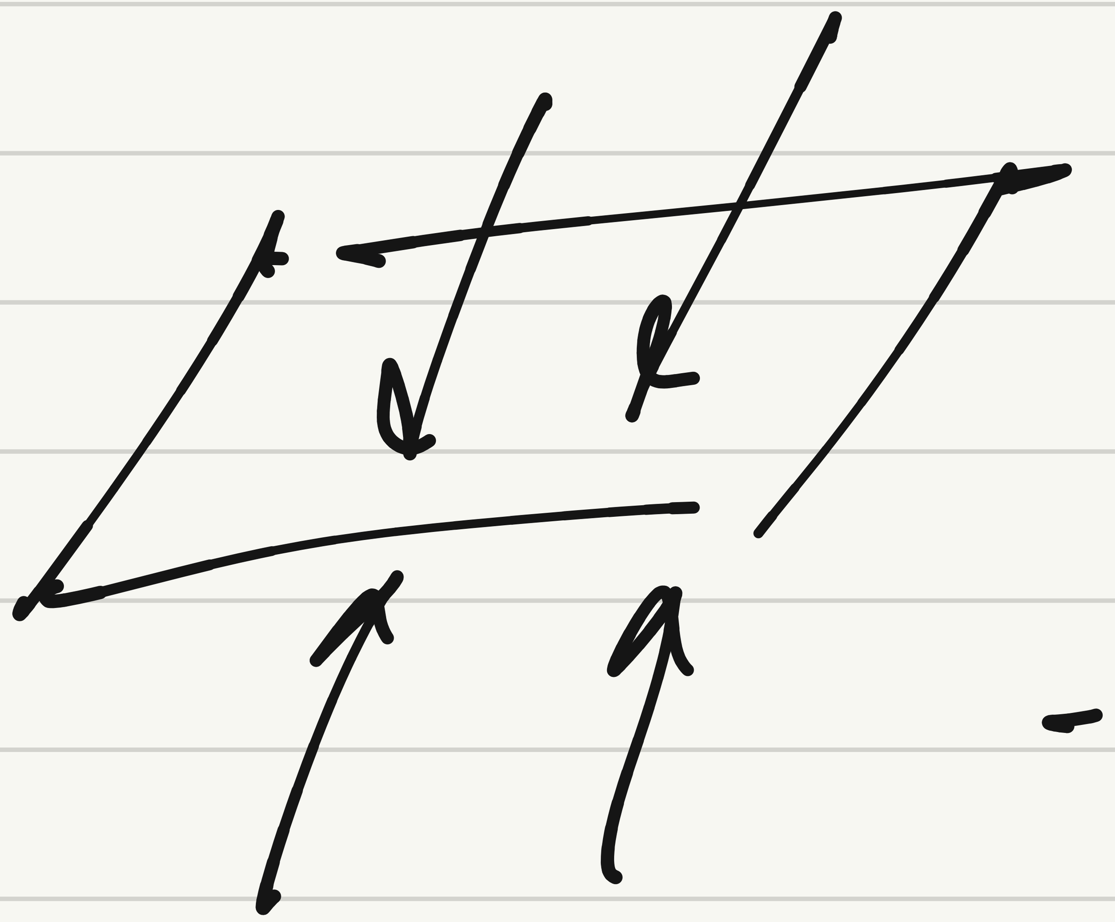


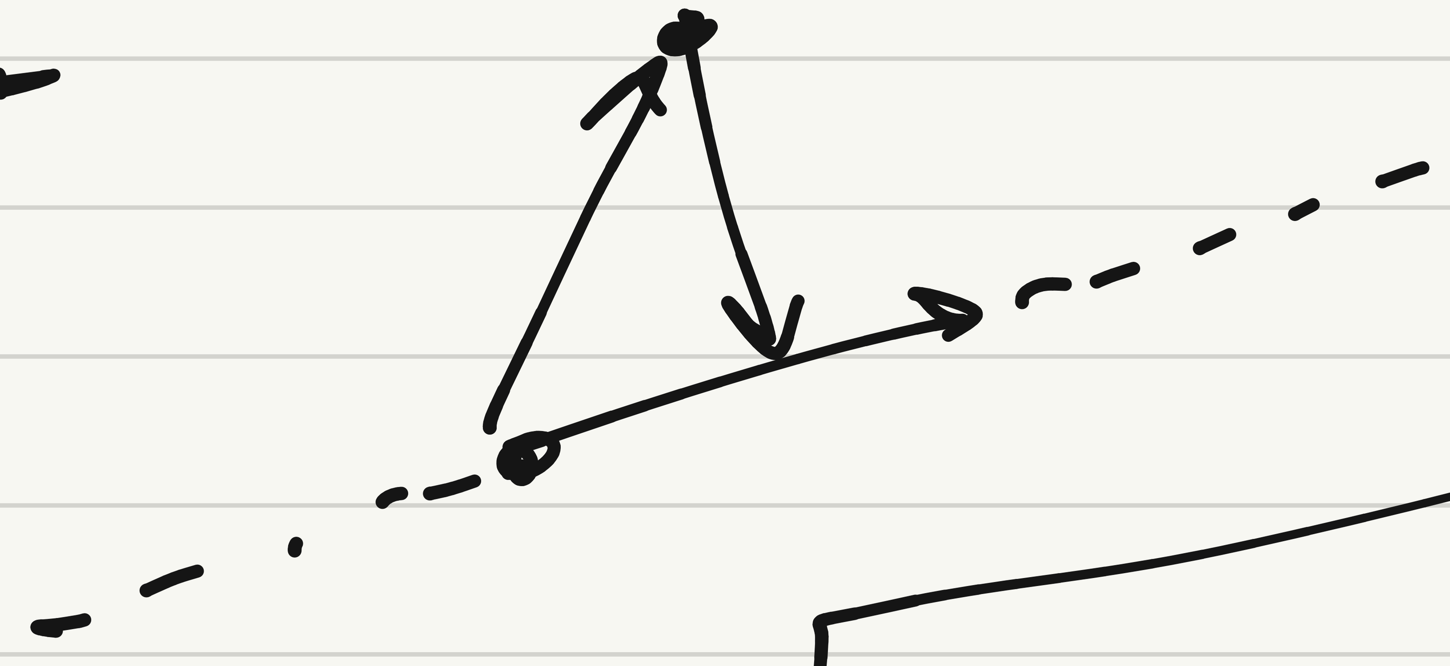
if $P^2 = P$

then range(P) and null(P) are complementary subspaces

If S_1, S_2 are compl. then $\exists$ matrix P s.t. $P^2 = P$,
range$(P) = S_1$, null$(P) = S_2$

P projects vectors $\boxed{\text{onto } S_1}$ $\boxed{\text{along } S_2}$





for any $\vec{v}$, $P\vec{v} \perp (\vec{v} - P\vec{v})$

: $\boxed{\text{orthogonal projection}}$

$\Leftrightarrow$ $\text{range}(P) \perp \text{null}(P)$

$\Leftrightarrow$ range, null are $\boxed{\text{orthogonal complements}}$

$$S^{\perp} = \{\vec{v} : \vec{v}^{*}\vec{w} = 0 \text{ for all } \vec{w} \in S\}$$

$$S_1 = S_2^{\perp}$$

$$S_2 = S_1^{\perp}$$

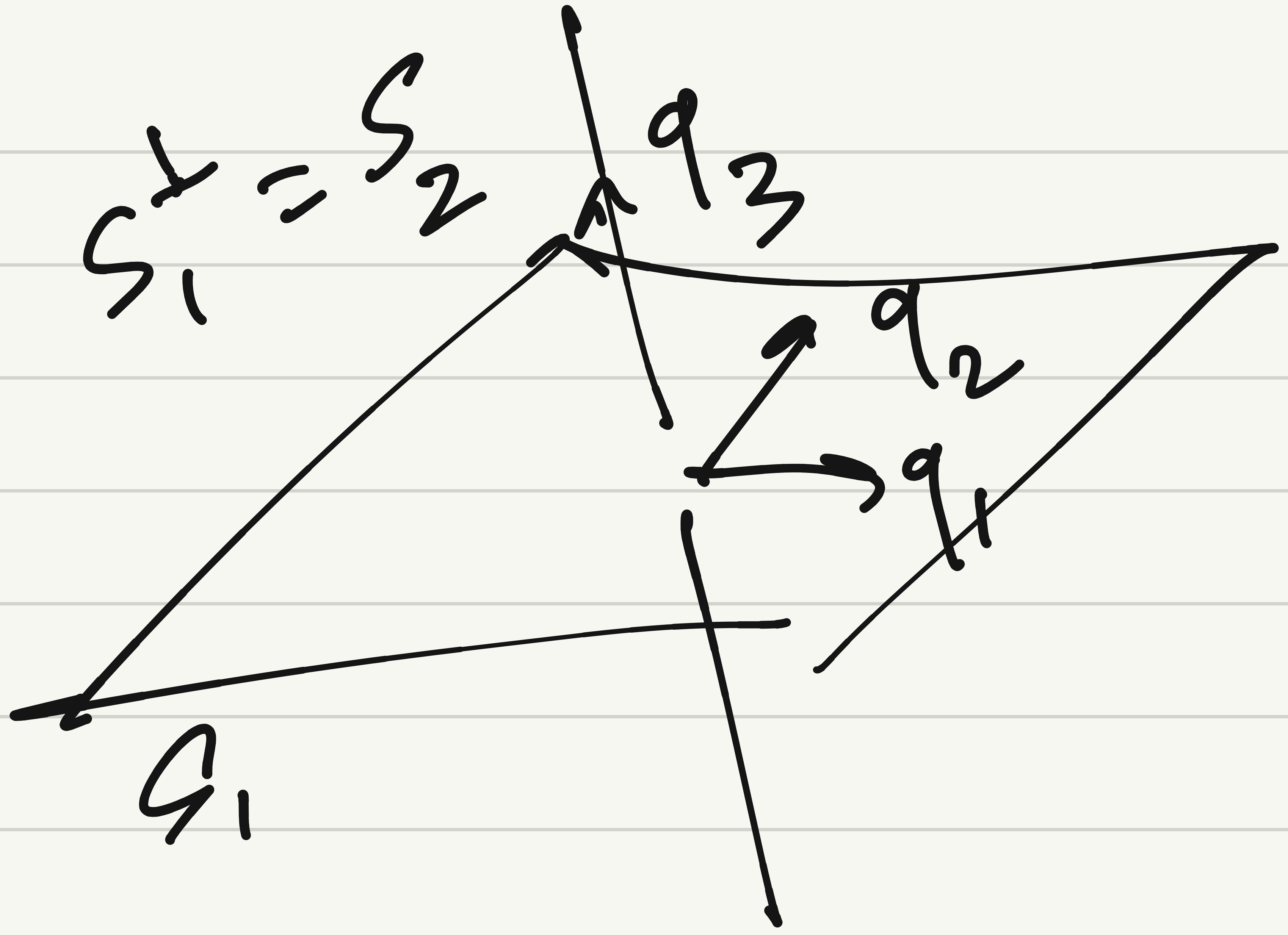
$$P^2 = P$$

and $P^* = P$

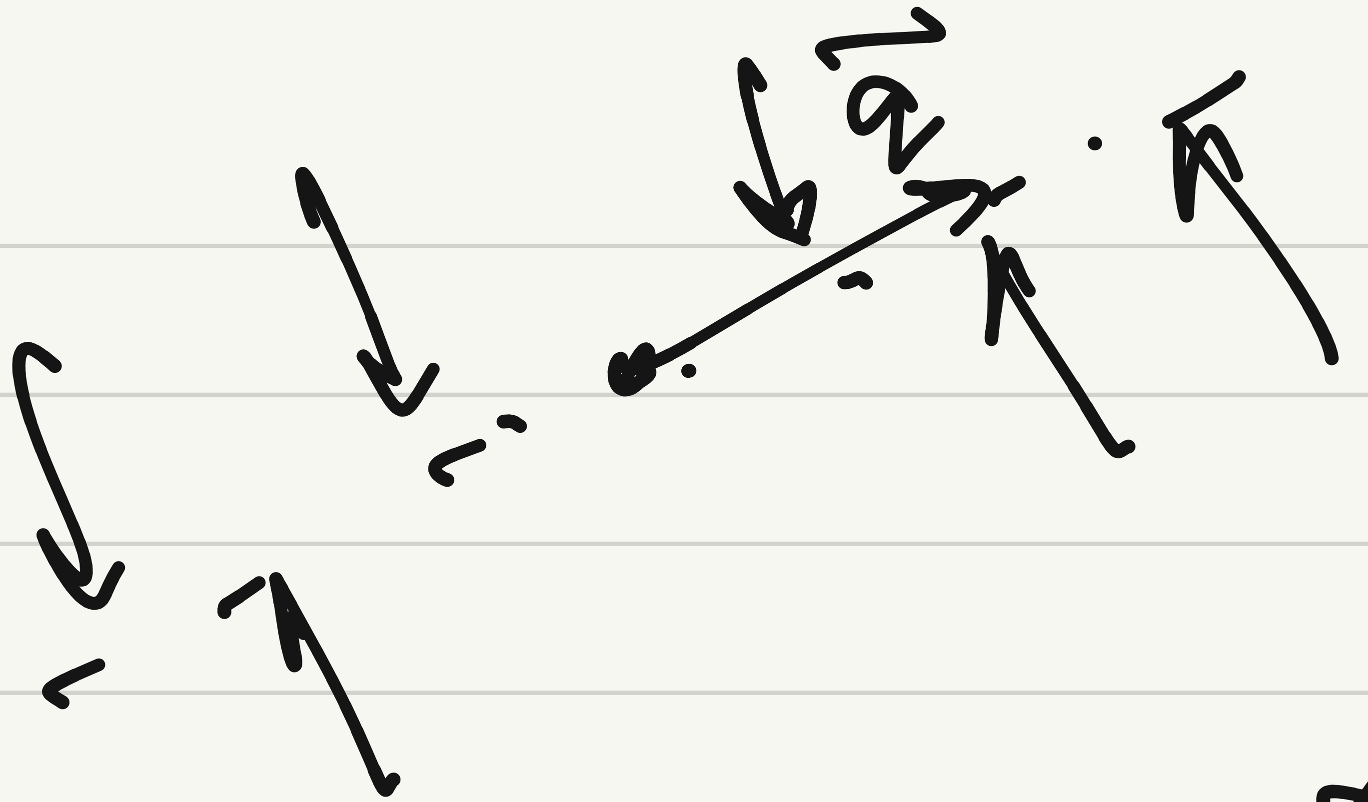
$$\text{range}(P) = S_1 = \langle \vec{q}_1, \ldots, \vec{q}_n \rangle \rightarrow \underbrace{\begin{bmatrix} \vec{q}_1 & \cdots & \vec{q}_n \end{bmatrix}}_{Q_1}$$

$$\text{null}(P) = S_2 \rightarrow Q_2$$

$$Q = \begin{bmatrix} Q_1 & \big| & Q_2 \end{bmatrix}$$



$$Q = \left[\begin{array}{c|c} Q_1 & Q_2 \end{array}\right]$$



$$PQ = \left[\begin{array}{c|c} Q_1 & 0 \end{array}\right]$$

$$Q^* P Q = \left[\begin{array}{c} Q_1^* \\ \hline Q_2^* \end{array}\right] \left[\begin{array}{c|c} Q_1 & 0 \end{array}\right]$$

$$= \left[\begin{array}{c|c} Q_1^* Q_1 & \cancel{Q_1^* 0} \to 0 \\ \hline \cancel{Q_2^* Q_1} \to 0 & \cancel{Q_2^* 0} \to 0 \end{array}\right]$$

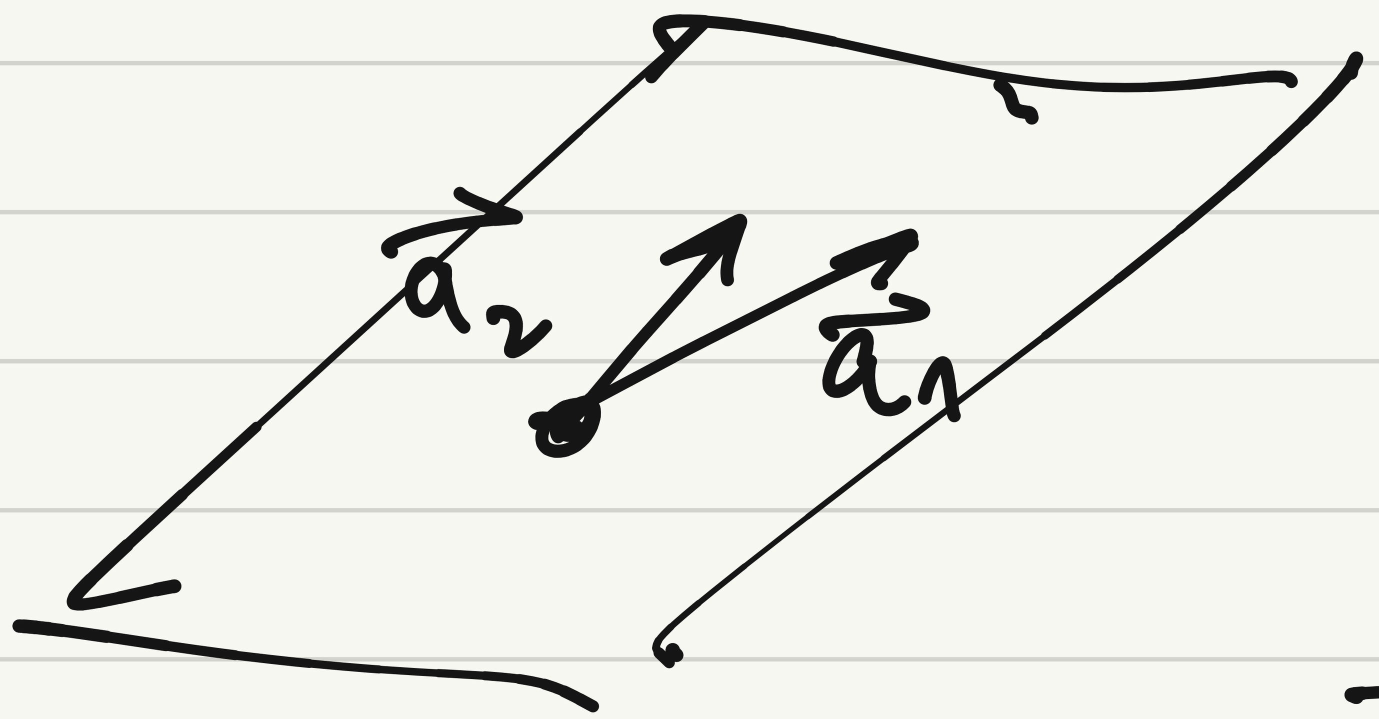
$$= \left[\begin{array}{c|c} I & 0 \\ \hline 0 & 0 \end{array}\right]$$

$$P = Q \left[\begin{array}{cc} I & 0 \\ 0 & 0 \end{array}\right] Q^* = Q_1 Q_1^*$$

unit diag unit

orthogonal projector onto S_1

$$P_q = qq^* \qquad P_{\perp q} = I - qq^*$$



$\boxed{\text{QR decomposition}}$ / $\boxed{\text{factorization}}$

$$\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n \in \mathbb{C}^m \qquad n \le m$$

$$S = \langle \vec{a}_1, \dots, \vec{a}_n \rangle \longrightarrow \langle \vec{q}_1, \dots \vec{q}_n \rangle \text{ orthonormal}$$

$$\langle \vec{a}_1 \rangle = \langle \vec{q}_1 \rangle$$

$$\langle \vec{a}_1, \vec{a}_2 \rangle = \langle \vec{q}_1, \vec{q}_2 \rangle$$

$$\langle \vec{a}_1, \dots, \vec{a}_j \rangle = \langle \vec{q}_1, \dots, \vec{q}_j \rangle$$

$$\langle \vec{a}_1, \dots, \vec{a}_n \rangle = \langle \vec{q}_1, \dots, \vec{q}_n \rangle$$

$$\vec{a}_j \in \langle \vec{q}_1, \dots \vec{q}_j \rangle$$

$$= \vec{q}_1 r_{1j} + \vec{q}_2 r_{2j} + \dots + \vec{q}_j r_{jj}$$

$$\begin{bmatrix} \vec{a}_j \end{bmatrix} = \underbrace{\begin{bmatrix} \vec{q}_1 & \dots & \vec{q}_j & \dots & \vec{q}_n \end{bmatrix}}_{Q} \begin{bmatrix} r_{1j} \\ \vdots \\ r_{jj} \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \end{bmatrix} = \begin{bmatrix} \vec{q}_1 & \vec{q}_2 & \cdots & \vec{q}_n \end{bmatrix} \cdot \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ & r_{22} & \cdots & r_{2n} \\ & & \ddots & \vdots \\ & & & r_{nn} \end{bmatrix}$$

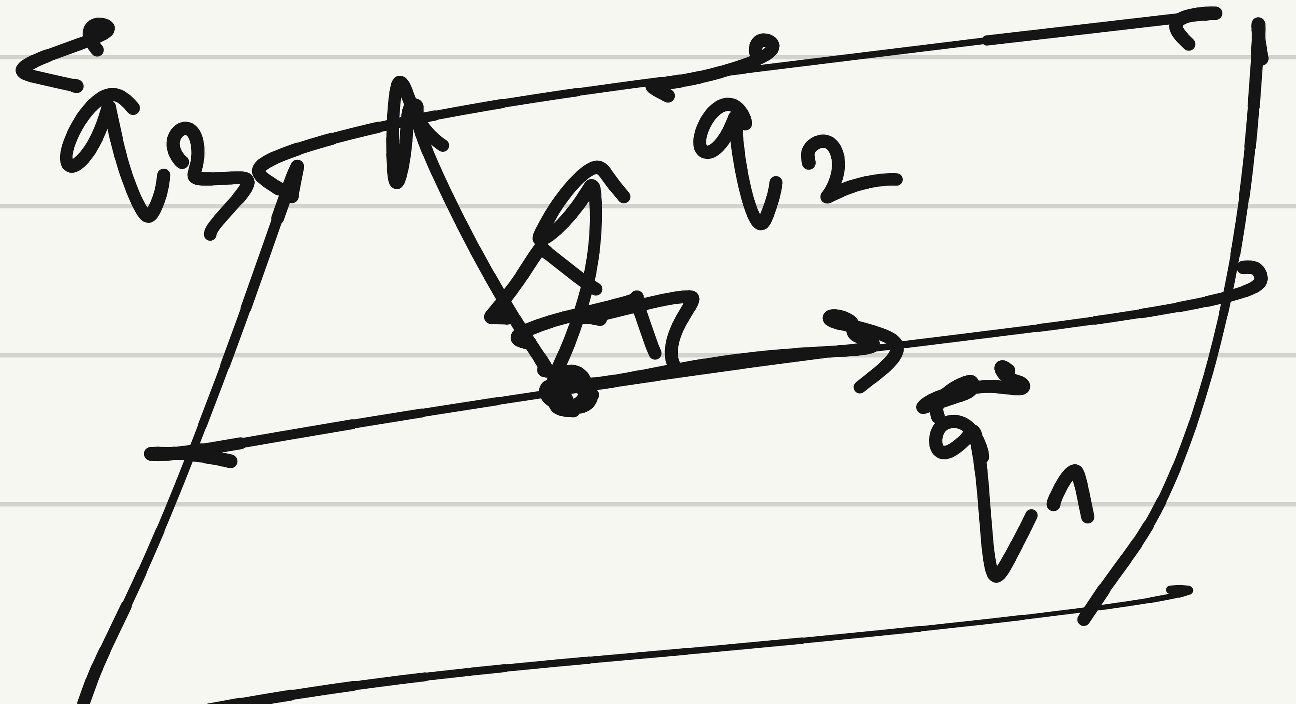
$A \in \mathbb{C}^{m \times n}$ $\qquad$ $\hat{Q} \in \mathbb{C}^{m \times n}$ $\qquad$ $\hat{R} \in \mathbb{C}^{n \times n}$

orthonormal columns. $\qquad$ upper triangular

$$\begin{bmatrix} A \end{bmatrix} = \overbrace{\begin{bmatrix} \hat{Q} & \Big| \Big| \Big| \end{bmatrix}}^{Q} \overbrace{\begin{bmatrix} \hat{R} \\ 0 \end{bmatrix}}^{R}$$

$\boxed{\text{reduced QR}}$ factorization

$\boxed{\text{full QR}}$



$$A = \hat{Q}\hat{R} \quad \text{square}$$

$$= QR$$

square

if A is full rank

$$\begin{bmatrix} \text{range}(A) = \text{range}(\hat{Q}) = \langle \vec{q}_1, \cdots, \vec{q}_n \rangle \\ \text{range}(A)^{\perp} = \langle \vec{q}_{n+1}, \cdots, \vec{q}_m \rangle \end{bmatrix}$$

$$A = QR$$

square

$$A\vec{x} = \vec{b}$$

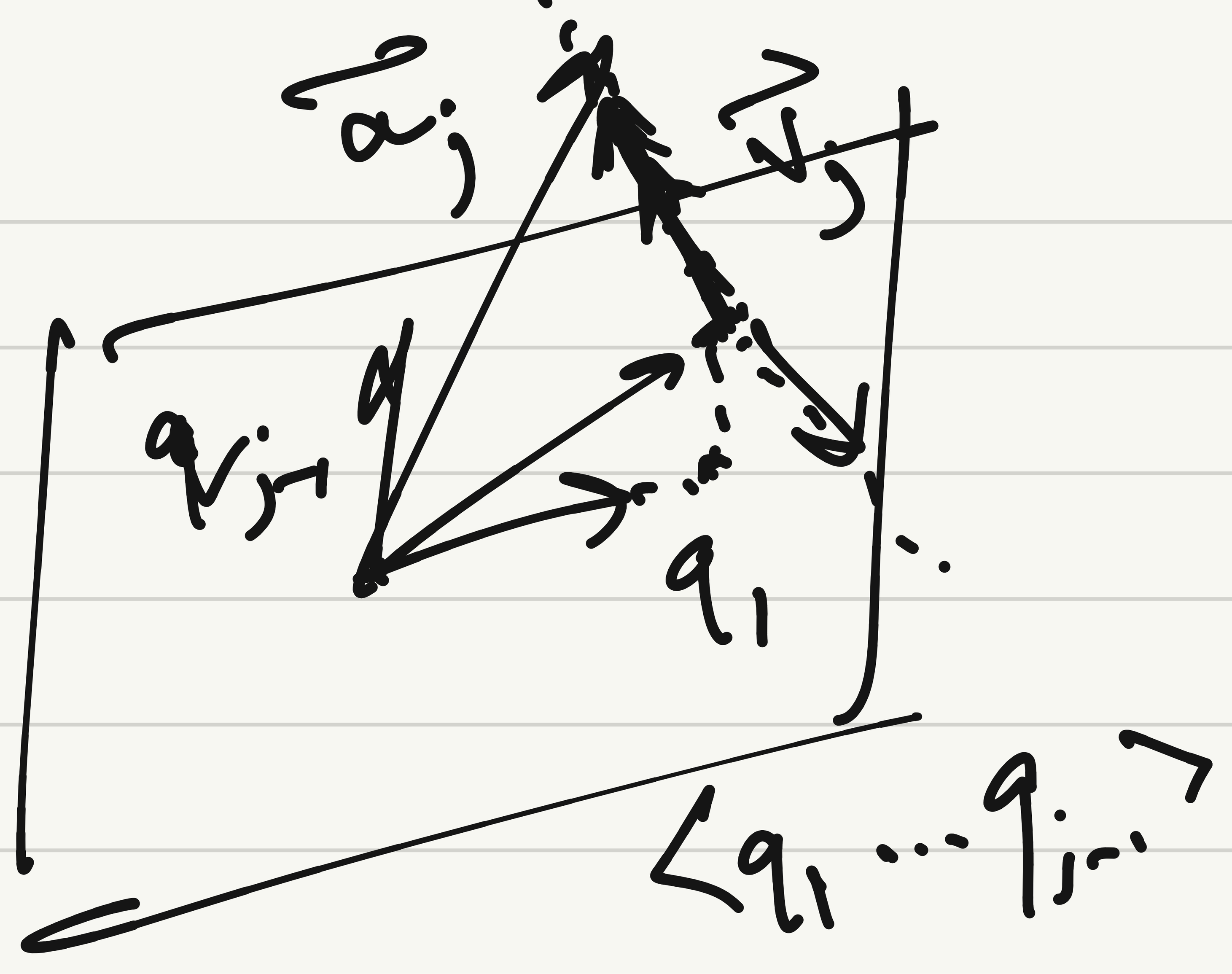
$$\underline{Q}\,\underline{R}\,\vec{x} = \vec{b}$$

$$R\vec{x} = \underbrace{Q^* \vec{b}}_{\vec{y}} \quad \rightarrow \quad R\vec{x} = \vec{y}$$

$$\begin{bmatrix} r_{11} & & r_{1n} \\ & \ddots & \vdots \\ & r_{n-1,n-1} & r_{n-1,n} \\ & & \boxed{r_{nn}} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$$

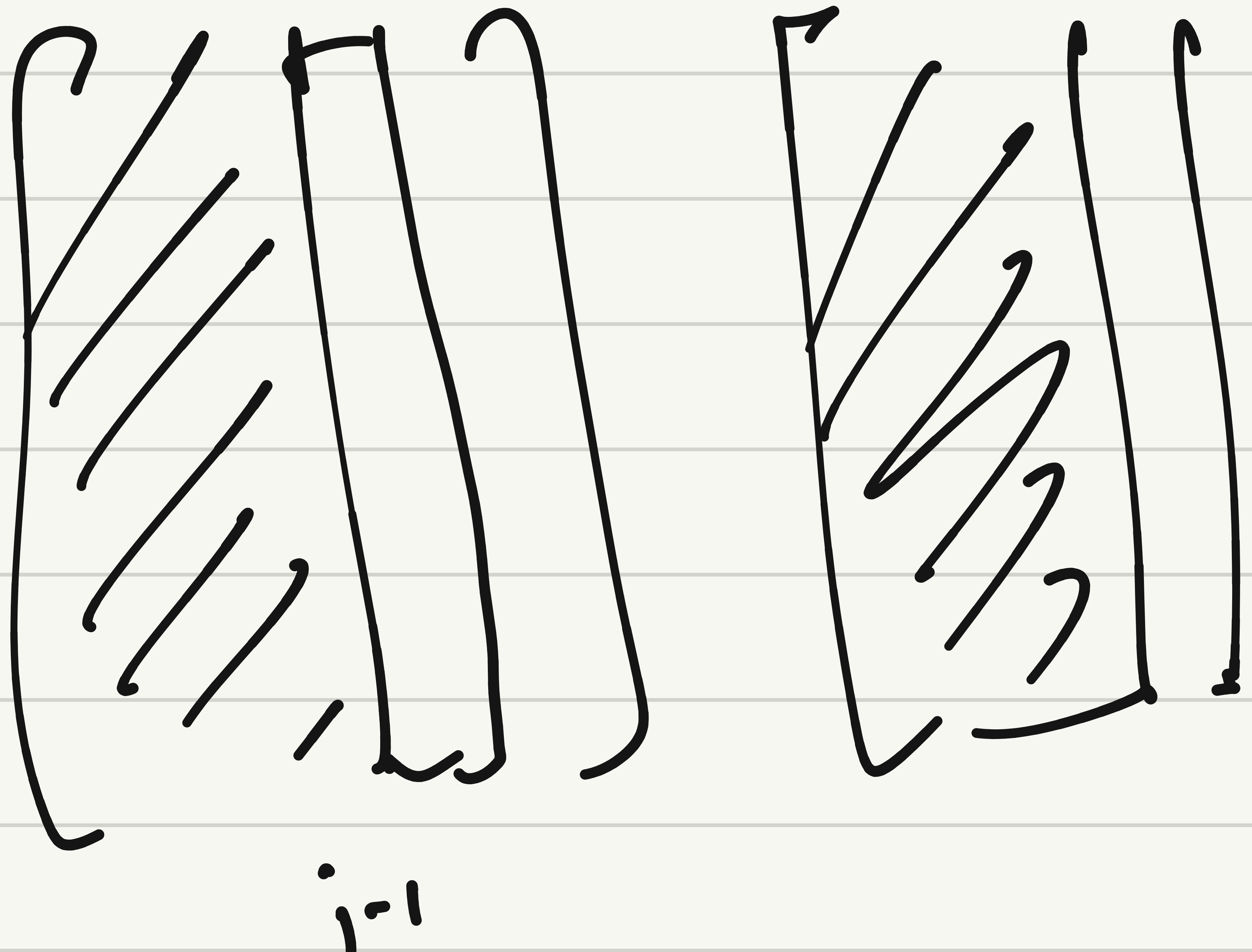
$\boxed{\text{backsubstitution}}$

$$A = \begin{bmatrix} \vec{a}_1 & \cdots & \vec{a}_n \end{bmatrix} \stackrel{?}{=} \underbrace{\begin{bmatrix} \vec{q}_1 & \cdots & \vec{q}_n \end{bmatrix}}_{\hat{Q}} \underbrace{\begin{bmatrix} r_{11} & \cdots & r_{1n} \\ & \ddots & \vdots \\ & & r_{nn} \end{bmatrix}}_{\hat{R}}$$



$$\langle \vec{a}_1, \ldots, \vec{a}_j \rangle = \langle \vec{q}_1, \ldots, \vec{q}_j \rangle$$

$$\vec{a}_j = \vec{q}_1 r_{1j} + \cdots + \vec{q}_{j-1} r_{j-1,j} + \vec{q}_j r_{jj}$$



$$r_{1j} = \vec{q}_1^{\,*} \vec{a}_j$$

$$\vdots$$

$$r_{j-1,j} = \vec{q}_{j-1}^{\,*} \vec{a}_j$$

$$\underbrace{\vec{a}_j - (\vec{q}_1 r_{1j} + \cdots)}_{\vec{v}_j} = \underbrace{\vec{q}_j r_{jj}}$$

$$\|\vec{v}_j\| = \|\vec{q}_j\| \cdot |r_{jj}| \qquad 1$$

$$\to \quad r_{jj} = \pm \|\vec{v}_j\|$$

$$\vec{a}_j \longrightarrow \vec{v}_j = \vec{a}_j - \sum_{i=1}^{j-1} \vec{q}_i \underbrace{\vec{q}_i^{\,*} \vec{a}_j}_{r_{ij}} \longrightarrow |r_{jj}| = \|\vec{v}_j\|$$

$$\vec{q}_j = \frac{\vec{v}_j}{r_{jj}}$$

for $j = 1$ to n

  $\vec{v}_j = \vec{a}_j$

  for $i = 1$ to $j-1$

    $r_{ij} = \vec{q}_i^{\,*} \vec{a}_j$

    $\vec{v}_j = \vec{v}_j - \vec{q}_i r_{ij}$

  $r_{jj} = \|\vec{v}_j\|_2$

  $\vec{q}_j = \vec{v}_j / r_{jj}$

$\boxed{\text{classical Gram-Schmidt process}}$

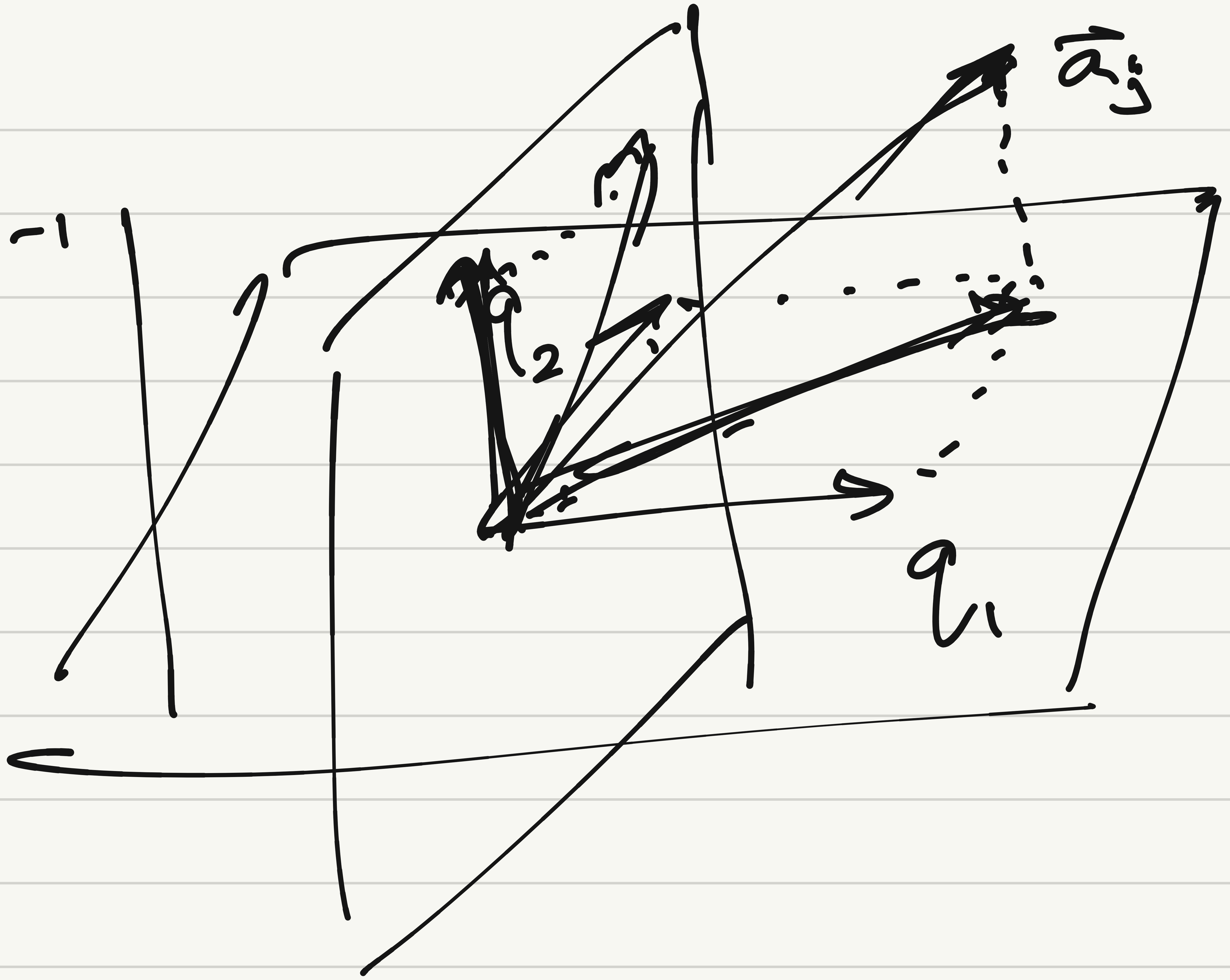
very unstable!

$$\vec{a}_j \to \vec{v}_j \perp \overbrace{\langle \vec{q}_1, \vec{q}_2, \dots, \vec{q}_{j-1} \rangle}^{S_{j-1}}$$

$$\vec{v}_j = P_j \vec{a}_j$$

P_j : ortho. proj. onto $S_{j-1}^{\perp}$

$$\vec{v}_j = P_j \vec{a}_j \qquad \vec{q}_1, \vec{q}_2, \dots, \vec{q}_{j-1}$$



$$= \underbrace{P_{\perp \vec{q}_{j-1}} \cdots P_{\perp \vec{q}_2} P_{\perp \vec{q}_1}} \vec{a}_j$$

$$P_{\perp \vec{q}_i} = I - \vec{q}_i \vec{q}_i^*$$

$$\vec{v}_1 = a_1$$

$$\vec{v}_2 = P_{\perp \vec{q}_1} \vec{a}_2 \qquad \vec{v}_j \longrightarrow P_{\perp \vec{q}_i} \vec{v}_j = \vec{v}_j - \vec{q}_i \underbrace{\vec{q}_i^* \vec{v}_j}_{r_{ij}}$$

$$\vec{v}_3 = P_{\perp \vec{q}_2} P_{\perp \vec{q}_1} \vec{a}_3$$

$$\vdots$$

$$\vec{v}_j = P_{\perp \vec{q}_{j-1}} \cdots P_{\perp \vec{q}_2} P_{\perp \vec{q}_1} \vec{a}_j$$

$$\vdots$$

$$\vec{v}_n = P_{\perp \vec{q}_{n-1}} \cdots P_{\perp \vec{q}_2} P_{\perp \vec{q}_1} \vec{a}_n$$

$\vec{q}_j \longrightarrow$ apply $P_{\perp \vec{q}_j}$ to all $\vec{v}_i$, $i > j$

also fill in r_{ij}

Modified Gram-Schmidt

for $i = 1 \ldots n$

$\vec{v}_i = \vec{a}_i$

for $i = 1 \ldots n$

($\vec{v}_i$ is already $P_i \vec{a}_i$)

$r_{ii} = \|\vec{v}_i\|$

$\vec{q}_i = \vec{v}_i / r_{ii}$

for $j = i+1 \ldots n$

$r_{ij} = \vec{q}_i^{\,*} \vec{v}_j$

$\vec{v}_j = \vec{v}_j - \vec{q}_i r_{ij}$