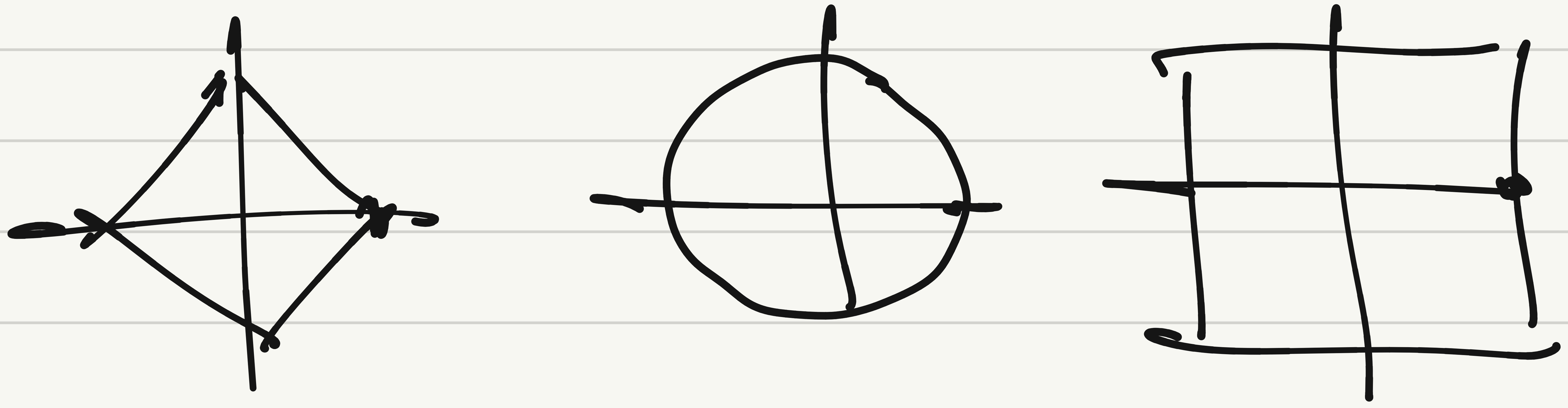
Matrix norms

$$\|\cdot\| : \mathbb{C}^{m\times n} \to \mathbb{R}$$

1. $\|A\| \geq 0$, $\|A\| = 0 \iff A = 0$
2. $\|sA\| = |s|\,\|A\|$
3. $\|A+B\| \leq \|A\| + \|B\|$

$$\frac{\|A\cdot(s\vec{x})\|}{\|s\vec{x}\|} = \frac{\cancel{|s|}\,\|A\vec{x}\|}{\cancel{|s|}\,\|\vec{x}\|}$$



$$\vec{x} \in \mathbb{C}^n \longrightarrow \vec{y} = Ax \in \mathbb{C}^m$$

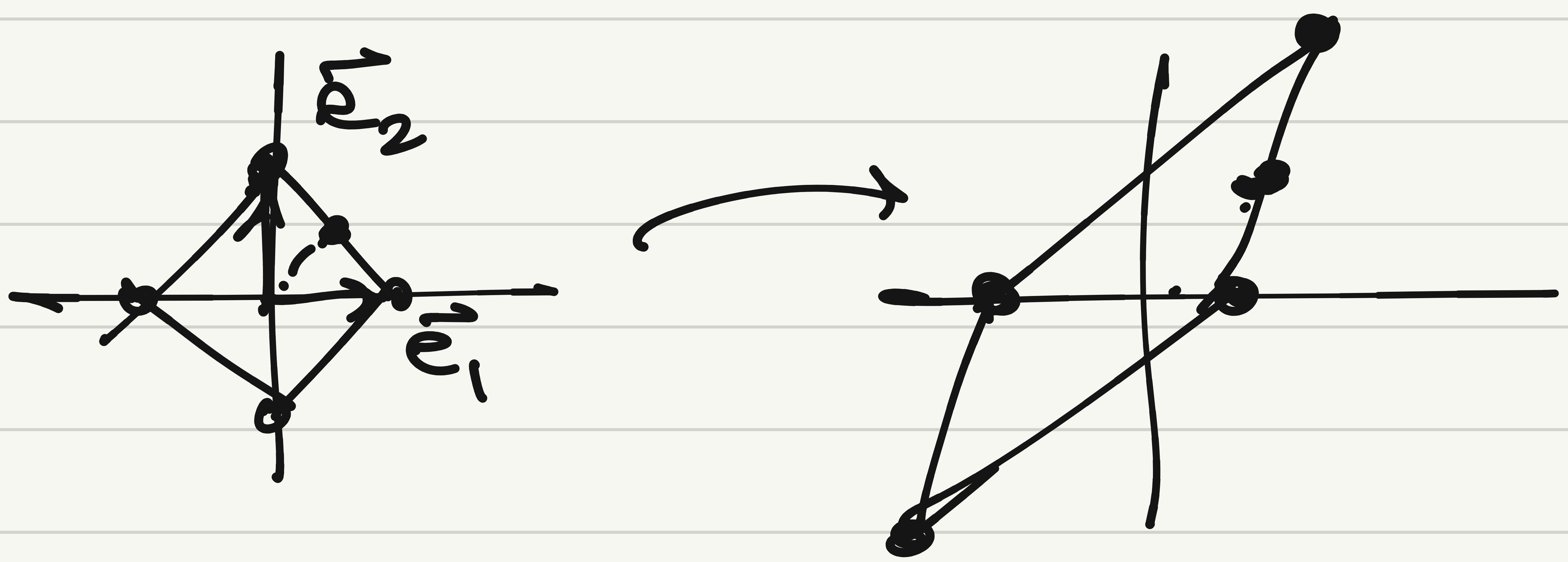
$$\|\vec{x}\|_{(n)} \xrightarrow{?} \|\vec{y}\|_{(m)}$$

$$\|A\|_{(m,n)} = \sup_{\vec{x}\neq\vec{0}} \frac{\|A\vec{x}\|_{(m)}}{\|\vec{x}\|_{(n)}}$$: $\boxed{\text{induced norm}}$ or $\boxed{\text{operator norm}}$

$$= \sup_{\|\vec{x}\|=1} \|A\vec{x}\|$$

$$\|A\|_p = \sup \frac{\|A\vec{x}\|_p}{\|\vec{x}\|_p}$$

$$\left\{\begin{aligned} \|A\|_1 &= \sup_{\|\vec{x}\|_1=1} \|A\vec{x}\|_1 \\ &= \max_j \|A\vec{e}_j\|_1 \\ &= \max_j \|\vec{a}_{\cdot j}\|_1 = \max_j \sum_i |a_{ij}| \end{aligned}\right.$$



$\|A\|_1$ = max abs. col. sum

$\|A\|_\infty$ = max abs. row sum $= \max_i \sum_j |a_{ij}|$

$\boxed{\text{Frobenius norm}}$

$\boxed{\text{Hilbert-Schmidt norm}}$

4. $\|AB\| \leq \|A\| \, \|B\|$: $\boxed{\text{submultiplicativity}}$

induced $\Rightarrow$ submult.

submult but not induced:

$$\|A\|_F = \sum_i \sum_j |a_{ij}|^2$$

$$\|A\|_F = \sqrt{\sum_i \sum_j |a_{ij}|^2} = \sqrt{\sum_j \|\vec{a}_j\|^2} = \sqrt{\operatorname{tr}(A^* A)} = \sqrt{\operatorname{tr}(AA^*)}$$

$$\operatorname{tr} M = \sum_i m_{ii}$$

$$\|AB\|_F \leq \|A\|_F \|B\|_F$$

$$\|A\vec{x}\|_2 \leq \|A\|_F \|\vec{x}\|_2$$

$$\|\vec{x}\|_2 = \sqrt{x^* x}$$

$\boxed{\text{unitarily invariant}}$: A, unitary Q, R

$$\|A\|_F = \|QA\|_F = \|AR\|_F$$

$$\|A\|_2 = \|QA\|_2 = \|AR\|_2$$

not true for 1-norm or ∞-norm

Condition number of mat-vec mult $= \sup_{\delta x} \dfrac{\|A\,\delta x\| / \|Ax\|}{\|\delta x\| / \|x\|} = \dfrac{\|x\|}{\|Ax\|} \sup_{\delta x \neq 0} \dfrac{\|A\,\delta x\|}{\|\delta x\|}$

$$= \frac{\|x\|}{\|Ax\|} \cdot \|A\|$$

Cond. num of matrix $\kappa(A) = \sup_{\vec{x} \neq 0} \dfrac{\|x\|}{\|Ax\|} \cdot \|A\| = \|A\| \sup_{x \neq 0} \dfrac{\|x\|}{\|Ax\|}$

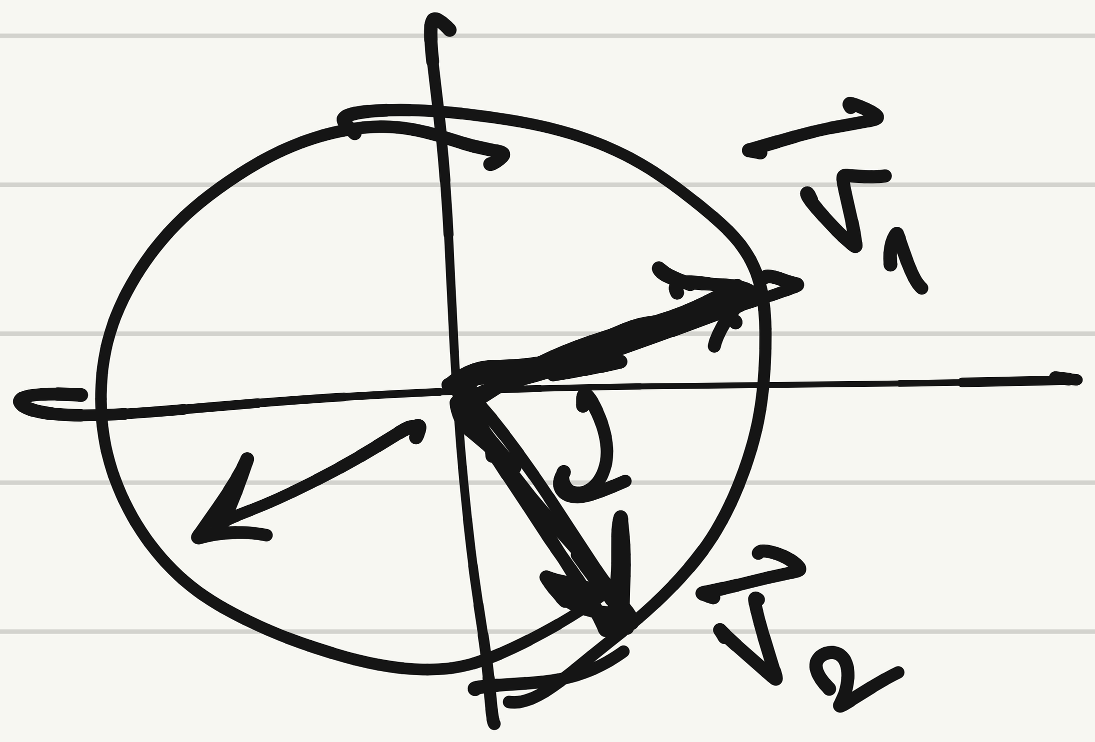
if A is singular, $\kappa(A) = \infty$

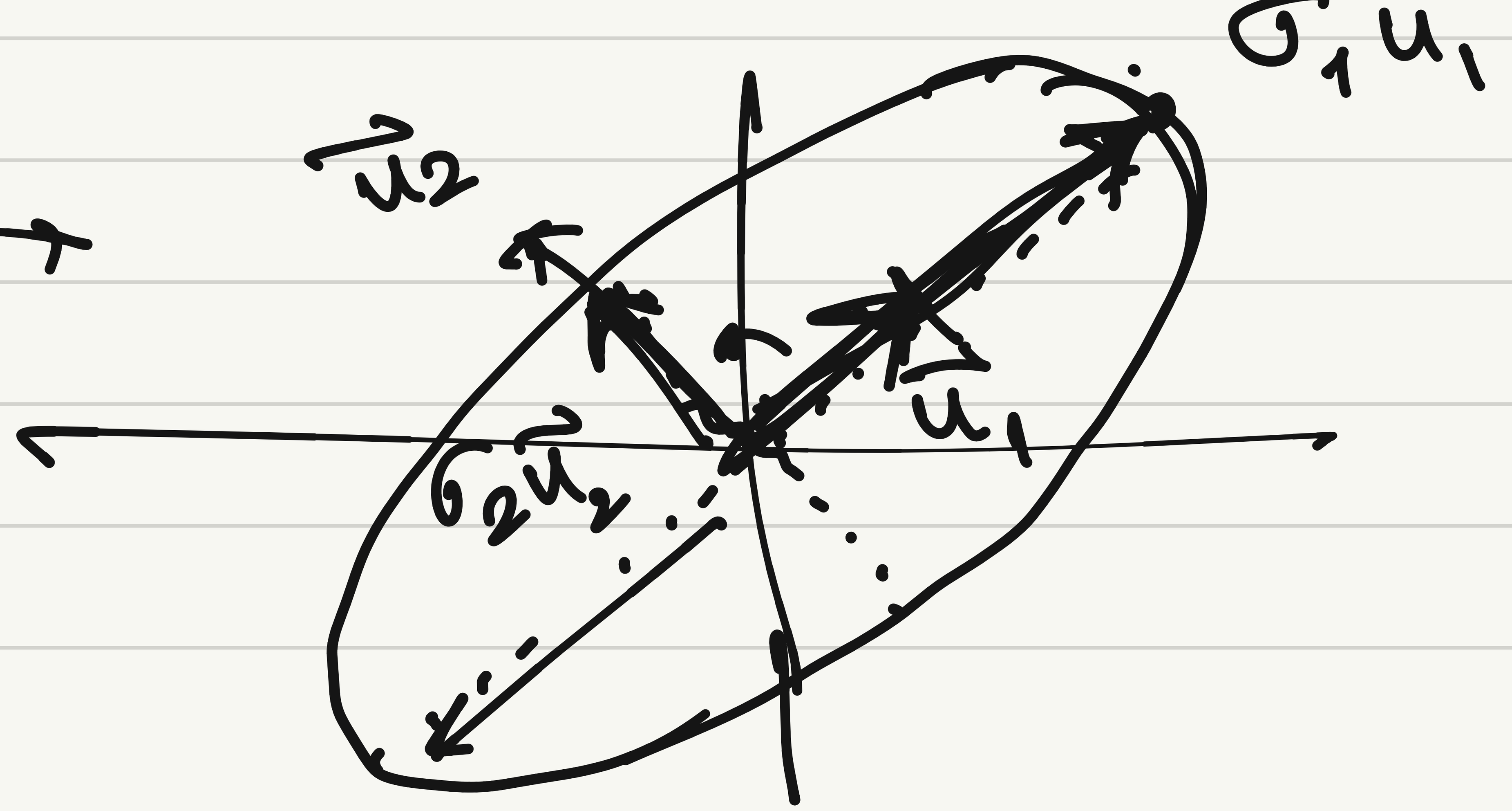
if A is invertible, let $\vec{x} = A^{-1}\vec{y} \longrightarrow \|A\| \sup_{\vec{y} \neq 0} \dfrac{\|A^{-1}\vec{y}\|}{\|\vec{y}\|}$

$$\kappa(A) = \|A\|_2 \|A^{-1}\|_2$$

<u>Singular Value Decomposition</u> (SVD)

$A \in \mathbb{R}^{2\times 2}$





$A\vec{v}_1 = \sigma_1 \vec{u}_1$

$A\vec{v}_2 = \sigma_2 \vec{u}_2$

$$A \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix} = \begin{bmatrix} \sigma_1\vec{u}_1 & \sigma_2\vec{u}_2 \end{bmatrix}$$

$$A\begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix} = \begin{bmatrix} \sigma_1 \vec{u}_1 & \sigma_2 \vec{u}_2 \end{bmatrix} = \begin{bmatrix} \vec{u}_1 & \vec{u}_2 \end{bmatrix} \begin{bmatrix} \sigma_1 & \\ & \sigma_2 \end{bmatrix}$$

$$\begin{array}{cccc} V & & U & \Sigma \end{array}$$

left singular vectors

singular values

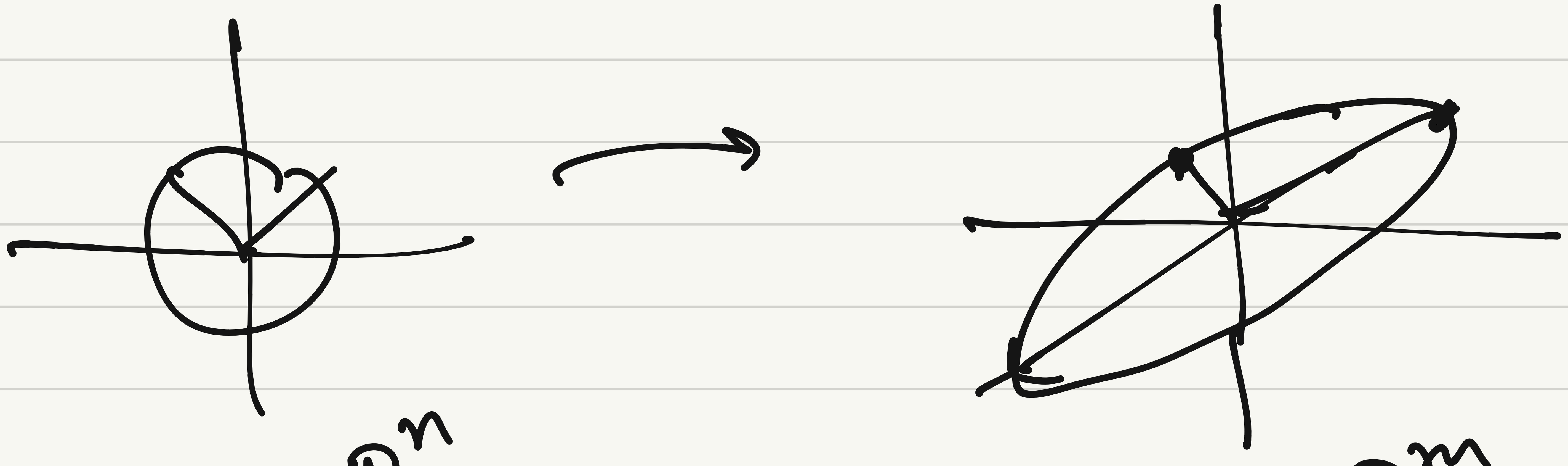
right singular vectors

orthonormal

orthonormal

+ve reals

$A^{m\times n}$ $\quad m \geq n$, $\quad \text{rank}(A) = n$



$\mathbb{R}^n$ $\qquad\qquad$ $\mathbb{R}^m$

$$\underbrace{\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_n \end{bmatrix}}_{V} = \underbrace{\begin{bmatrix} | & & | \\ \vec{u}_1 & \cdots & \vec{u}_n \\ | & & | \end{bmatrix}}_{\hat{U}} \cdot \underbrace{\begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \end{bmatrix}}_{\hat{\Sigma}}$$

square

orthogonal

$$AV = \hat{U}\hat{\Sigma} \Longrightarrow \boxed{A = \hat{U}\hat{\Sigma}V^*}$$

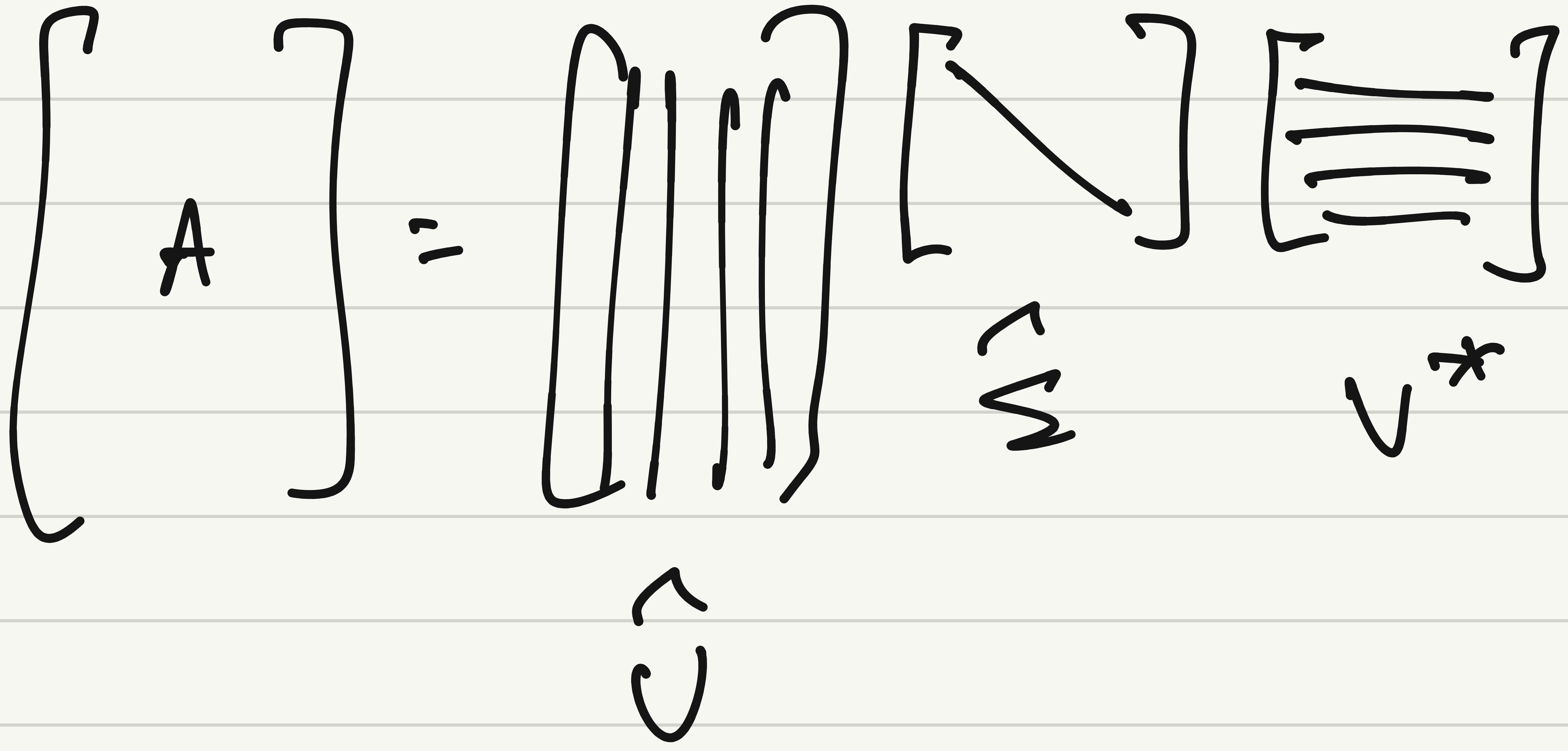
$A \in \mathbb{R}^{m\times n}$, $m \geq n$, full rank

$A = \hat{U}\hat{\Sigma}V^*$: $\boxed{\text{reduced SVD}}$

$\hat{U}$: $m\times n$, orthonormal cols.

$\hat{\Sigma}$: $n\times n$, diagonal

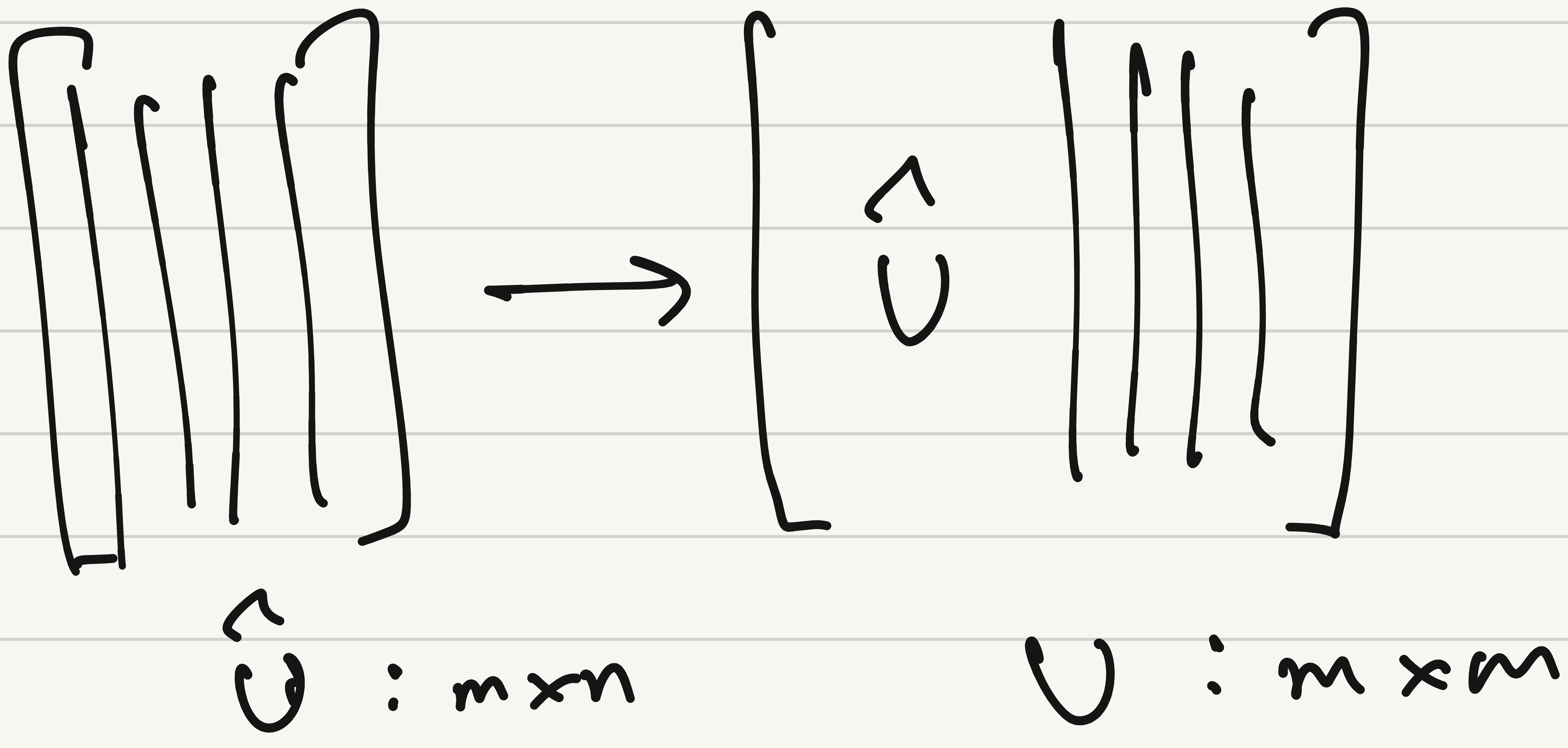
V : $n\times n$, orthagonal

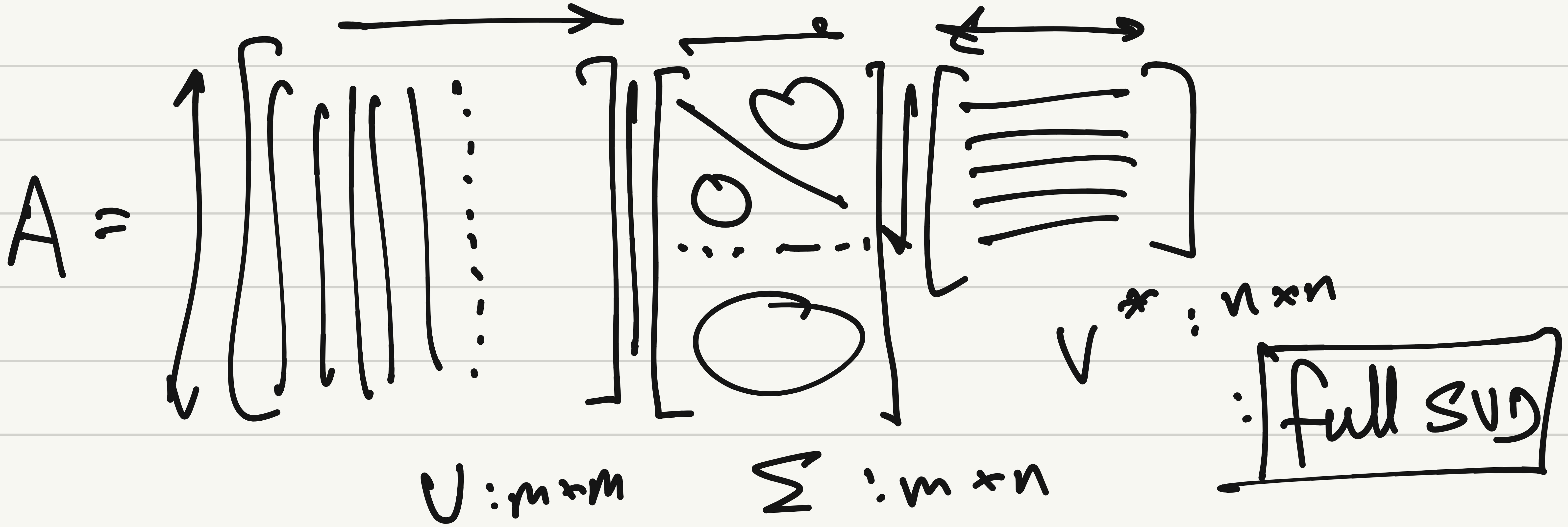


n orthonormal vectors in $\mathbb{R}^m$

$\Rightarrow \exists$ $m-n$ more vectors

$\rightarrow$ orthonormal basis





If $A \in \mathbb{C}^{m \times n}$, a $\boxed{\text{singular value decomp}}$ is $A = U \Sigma V^*$ s.t.

U : $m \times m$, unitary
Σ : $m \times n$, diagonal with nonnegative real entries
V : $n \times n$, unitary

Existence & uniqueness

Thm. (i) Every matrix $A \in \mathbb{C}^{m \times n}$ has an SVD,

(ii) $\{\sigma_1, \ldots, \sigma_p\}$ ($p = \min(m,n)$) is uniquely determined,

by convention $\sigma_1 \geq \sigma_2 \geq \sigma_3 \geq \cdots \geq \sigma_p$

(iii) If σ_j's are distinct, then $\{\vec{u}_1, \ldots, \vec{u}_p\}$, $\{\vec{v}_1, \ldots, \vec{v}_p\}$ are uniquely determined (up to complex sign).

complex scalar z with $|z| = 1$