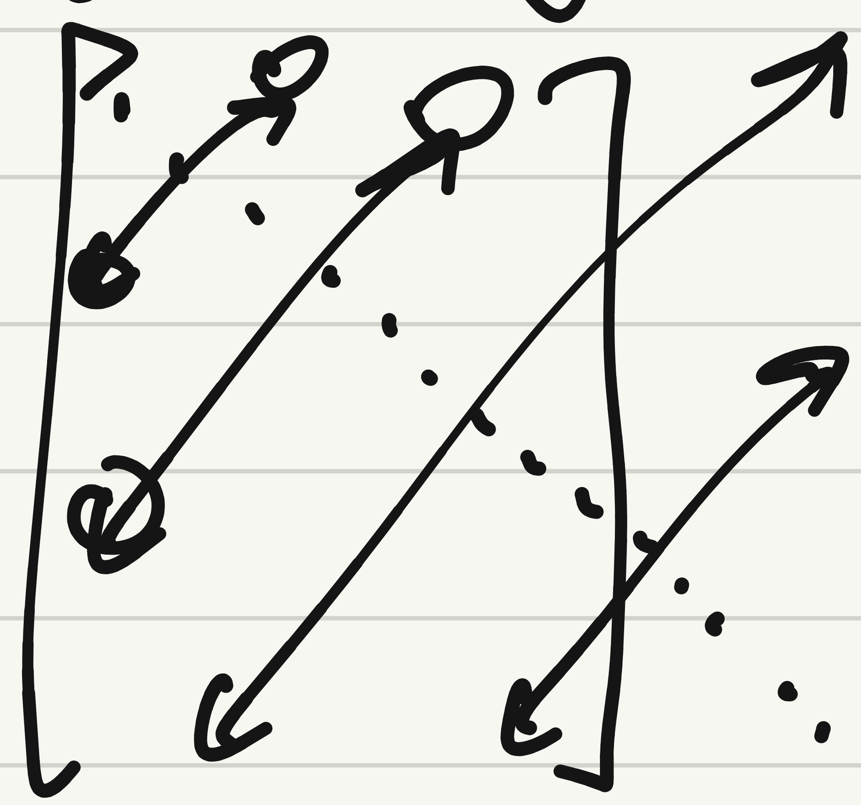
COL726: Orthogonality and norms

$A \in \mathbb{C}^{m \times n}$

$A =$ 

$A^T =$ 

$B = A^T$

$b_{ij} = a_{ji}$

$\boxed{\text{Transpose}}$

$(x + iy)^* = x - iy$

$\boxed{\text{adjoint}}$ A^* $\quad B = A^* \iff b_{ij} = a_{ji}^*$

$A^T = A$: $\boxed{\text{Symmetric}}$ $\qquad A = A^*$: $\boxed{\text{Hermitian}}$

- $(A+B)^* = A^* + B^*$
- $(aA)^* = a^* A^*$
- $(AB)^* = B^* A^*$
- $(A^*)^{-1} = (A^{-1})^* = A^{-*}$

$(AB)^{-1} = B^{-1} A^{-1}$

(similarly $(A^T)^{-1} = (A^{-1})^T = A^{-T}$)

$$\vec{x}\cdot\vec{y} = x_1y_1 + \cdots + x_my_m = \sum_{i=1}^{m} x_iy_i \qquad \vec{y},\vec{x}\in\mathbb{R}^m$$

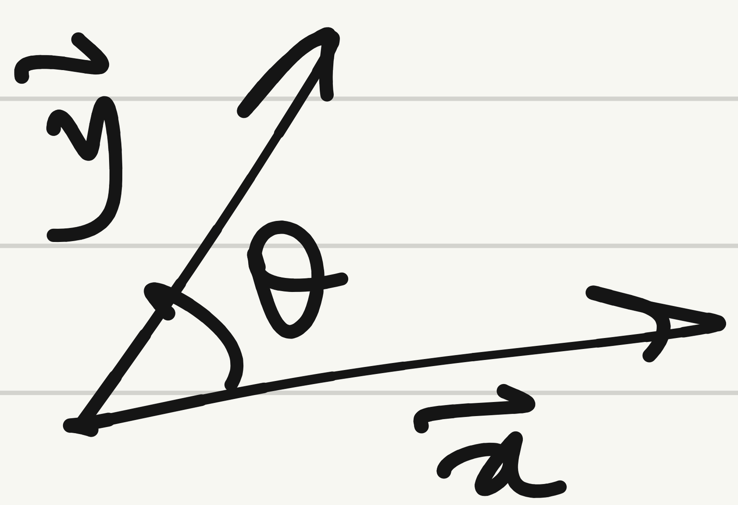
if complex: $x_1^*y_1 + \cdots + x_m^*y_m = \vec{x}^*\vec{y}$ $\boxed{\text{inner product}}$

$$\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} \qquad \vec{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}$$

(outer prod $\vec{x}\vec{y}^* = \begin{bmatrix} \cdot & \cdots & \cdot \\ \vdots & & \\ \cdot & & \end{bmatrix}$)

Euclidean length $\|\vec{x}\| = \sqrt{|x_1|^2 + \cdots + |x_m|^2} = \sqrt{\vec{x}^*\vec{x}}$

$\vec{x}$ is a $\boxed{\text{unit vector}}$ if $\|\vec{x}\| = 1$.



$$\cos\theta = \frac{\vec{x}^*\vec{y}}{\|\vec{x}\|\;\|\vec{y}\|}$$

$\vec{x},\vec{y}\in\mathbb{R}^m$

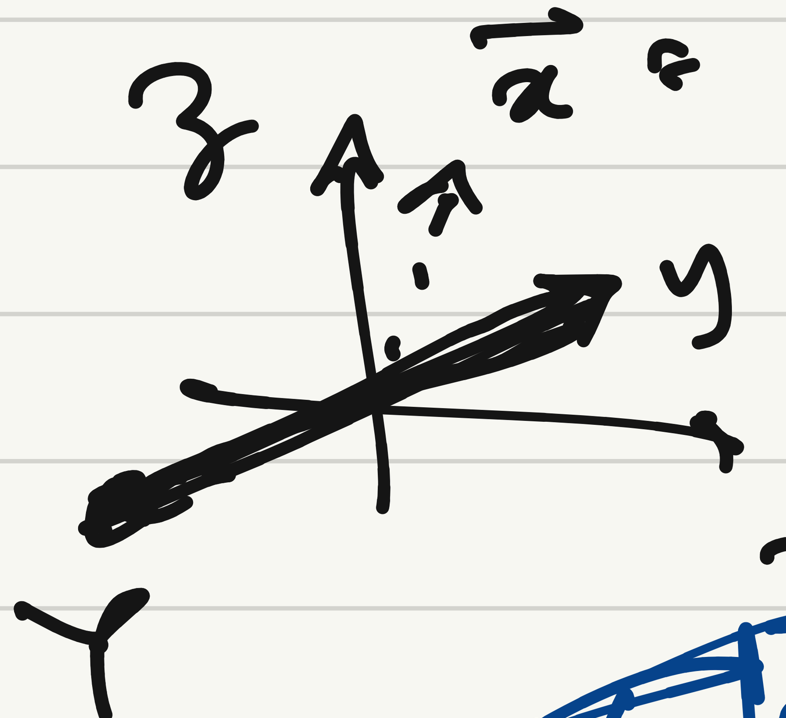
$\theta = 90^\circ \iff \vec{x}^*\vec{y} = 0$

$\vec{x},\vec{y}$ are $\boxed{\text{orthogonal}}$ if $\vec{x}^*\vec{y} = 0$

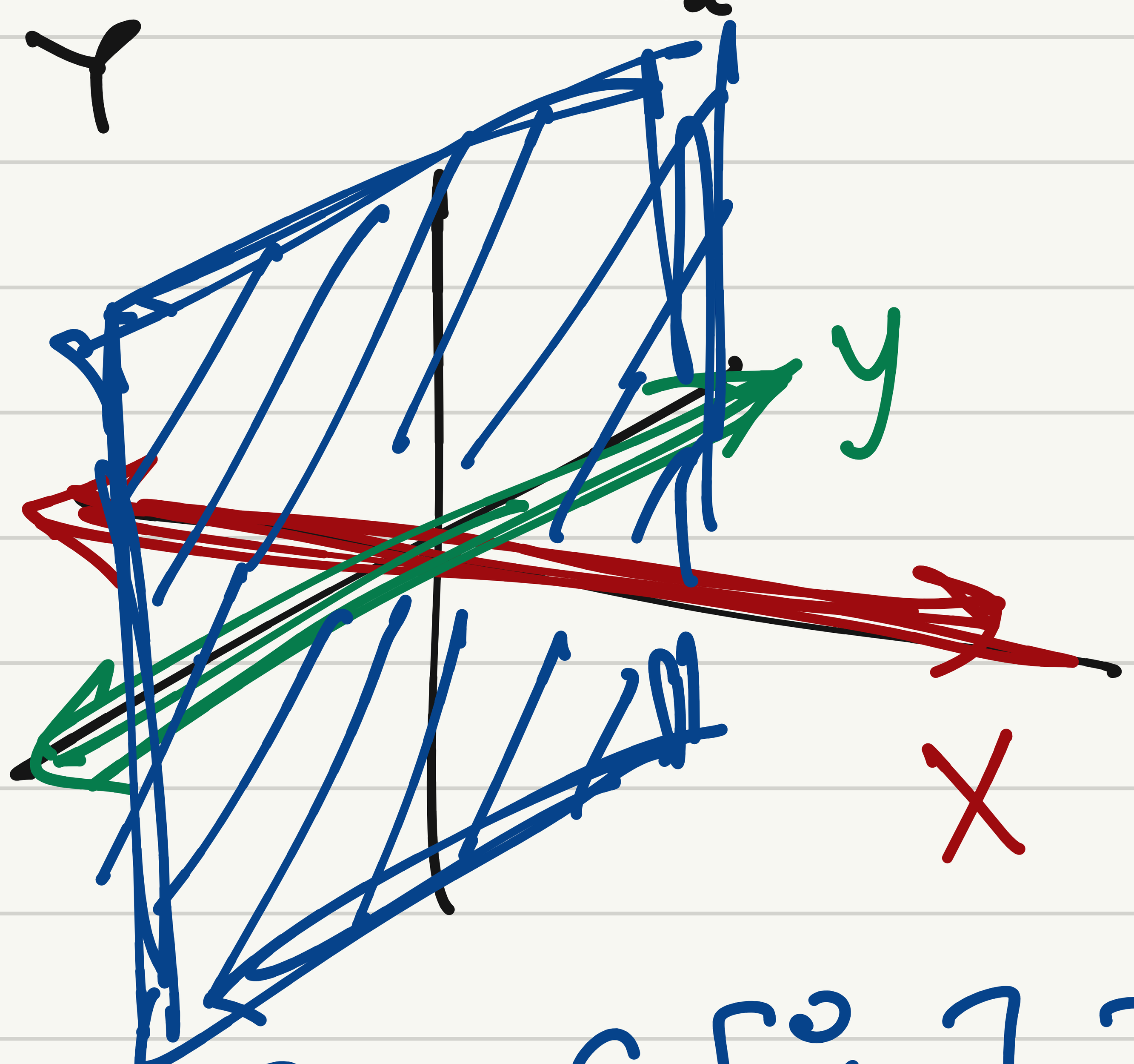
Two vectors $\vec{x}, \vec{y} \in \mathbb{C}^m$ are $\boxed{\text{orthogonal vectors}}$ if $\vec{x}^* \vec{y} = 0$

A vector $\vec{x} \in \mathbb{C}^m$ is $\boxed{\text{orthogonal to a set}}$ $Y \subseteq \mathbb{C}^m$ if $\vec{x}$ is ortho. to all $\vec{y} \in Y$.

$\vec{x} = \begin{bmatrix} x_1 \\ 0 \\ x_3 \end{bmatrix}$



Two sets $X, Y \subseteq \mathbb{C}^m$ are <u>orthogonal</u> to each other if every $\vec{x} \in X$ is ortho. to every $\vec{y} \in Y$.



$S = \left\{ \begin{bmatrix} 0 \\ x_2 \\ x_3 \end{bmatrix} \right\}$

A set $S \subseteq \mathbb{C}^m$ is an $\boxed{\text{orthogonal set}}$ if any $\vec{x} \in S$ is <u>nonzero</u> and any distinct $\vec{x}, \vec{y} \in S$ are ortho. to each other.

$S = \left\{ \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -3 \\ 0 \end{bmatrix} \right\}$

if S is orthogonal then it is also lin. indep.

$\Rightarrow |S| \le m$. if $|S| = m$ then it is a <u>basis</u>.

If S is orthogonal set and all $\vec{x} \in S$ are unit vectors
then S is $\boxed{\text{orthonormal}}$.

$\mathcal{B} = \{\vec{b}_1, \dots, \vec{b}_m\} \longrightarrow B = \left[\vec{b}_1 \middle| \cdots \middle| \vec{b}_m\right]$

$\|\vec{q}_i\| = 1$ for all i

$\vec{q}_i^{\,*}\vec{q}_j = 0$ for all $i \neq j$

$\vec{v} \in \mathbb{C}^m$, find $\vec{x}$ s.t. $B\vec{x} = \vec{v}$

If $\{\vec{q}_1, \dots, \vec{q}_m\}$ is orthonormal basis, $\quad Q = \left[\vec{q}_1 \middle| \cdots \middle| \vec{q}_m\right]$

$$\vec{v} = \vec{q}_1 x_1 + \dots + \vec{q}_m x_m$$

$$\vec{q}_1^{\,*}\vec{v} = \underbrace{\cancel{\vec{q}_1^{\,*}\vec{q}_1}}_{1} x_1 + \underbrace{\cancel{\vec{q}_1^{\,*}\vec{q}_2}}_{0} x_2 + \dots + \underbrace{\cancel{\vec{q}_1^{\,*}\vec{q}_m}}_{0} x_m$$

$$\vec{q}_1^{\,*}\vec{v} = x_1 \quad \cdots \quad \vec{q}_m^{\,*}\vec{v} = x_m$$

$$\vec{v} = (\vec{q}_1^{\,*}\vec{v})\vec{q}_1 + \dots + (\vec{q}_m^{\,*}\vec{v})\vec{q}_m$$

$$\vec{x} = \begin{bmatrix} \vec{q}_1^{\,*}\vec{v} \\ \vec{q}_2^{\,*}\vec{v} \\ \vdots \\ \vec{q}_m^{\,*}\vec{v} \end{bmatrix} = \begin{bmatrix} - \vec{q}_1^{\,*} - \\ - \vec{q}_2^{\,*} - \\ \vdots \\ - \vec{q}_m^{\,*} - \end{bmatrix} \begin{bmatrix} \vec{v} \end{bmatrix} = Q^*\vec{v}$$

$\underline{\vec{x} = Q^* \vec{v}}$ but $\underline{\vec{x} = Q^{-1} \vec{v}}$ $\qquad Q^* \vec{v} = Q^{-1} \vec{v}$ for all $\vec{v}$

$$Q^* = Q^{-1} \iff Q^* Q = I$$

$$\begin{bmatrix} \text{---} & \vec{q}_1^{\,*} & \text{---} \\ & \vdots & \\ \text{---} & \vec{q}_m^{\,*} & \text{---} \end{bmatrix} \begin{bmatrix} | & & | \\ q_1 & \cdots & q_m \\ | & & | \end{bmatrix}$$

A square matrix Q with orthonormal columns is called an $\boxed{\text{orthogonal}}$ $\rightarrow$ real

or $\boxed{\text{unitary}}$ matrix

complex

$$\iff \underline{Q^* Q = I} \iff \underline{Q^{-1} = Q^*}$$

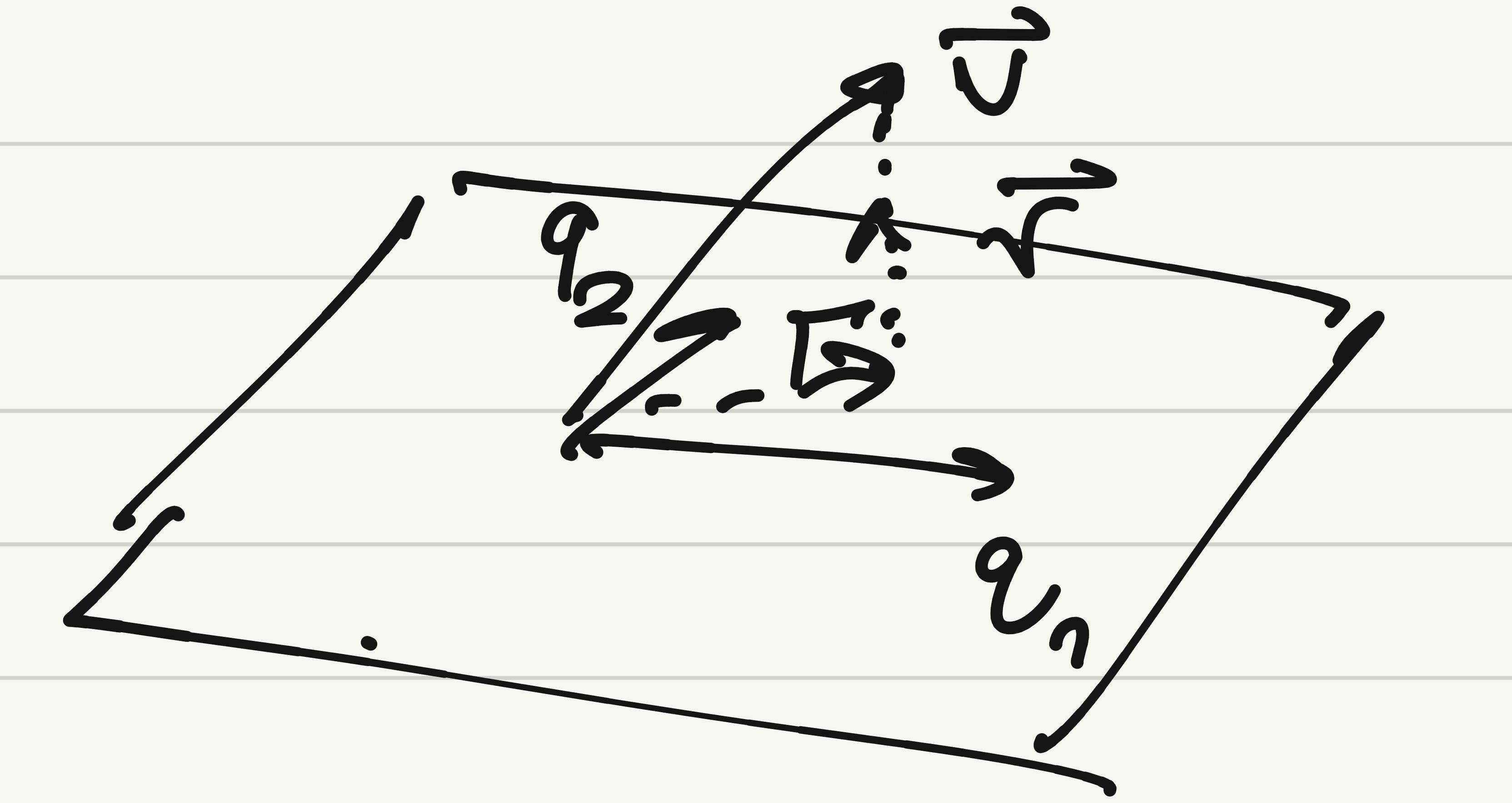
If $\vec{x}, \vec{y} \in \mathbb{C}^m \qquad \vec{x}^* \vec{y} \in \mathbb{C}$

$\hookrightarrow Q\vec{x},\ Q\vec{y} \qquad \underbrace{(Q\vec{x})^*}(Q\vec{y}) = \vec{x}^* \cancel{Q^* Q}^{\,I} \vec{y} = \vec{x}^* \vec{y}$

mult. with

$\|\vec{x}\| = \|Q\vec{x}\|$ $\qquad$ In $\mathbb{R}^m$, orthogonal matrix corresponds to rotation or reflection

If $\{\vec{q}_1, \dots, \vec{q}_n\} \subseteq \mathbb{C}^m$, $n \leq m$ then ...?



Norms

$$\|\vec{x}\|_2 = \underline{\sqrt{|x_1|^2 + \cdots + |x_m|^2}}$$

↳ 2-norm

$\|\cdot\| : \mathbb{C}^m \to \mathbb{R}$

1. $\|\vec{x}\| \geq 0$, $\|\vec{x}\| = 0 \iff \vec{x} = 0$
2. $\|a\vec{x}\| = |a| \, \|\vec{x}\|$
3. $\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$

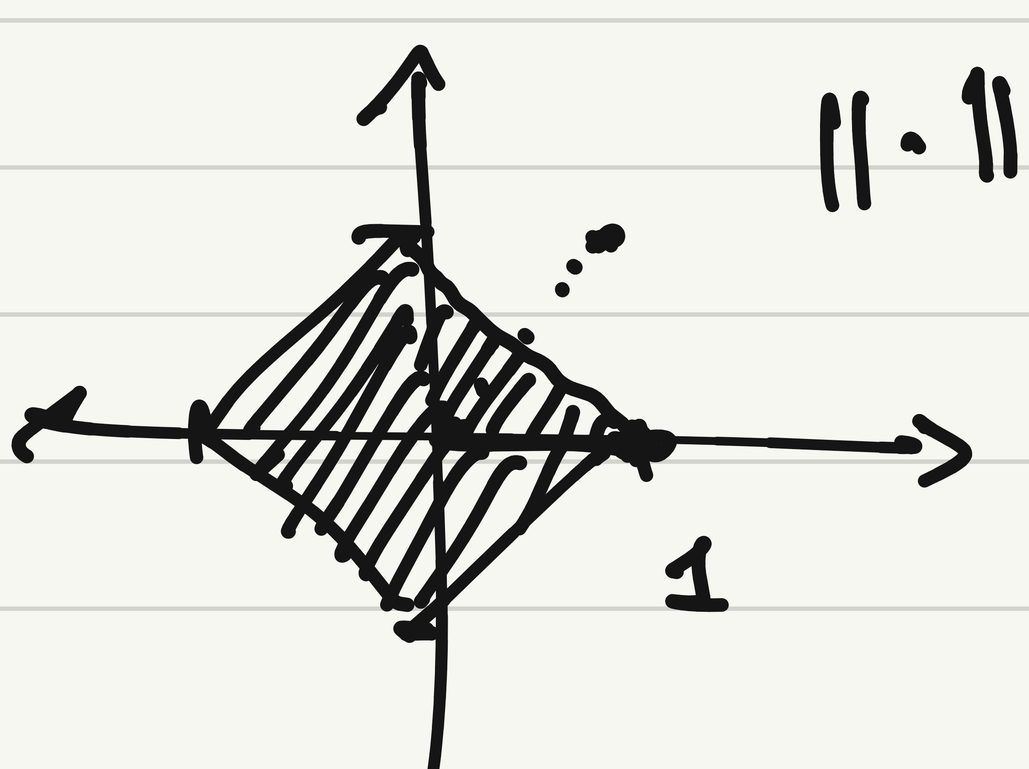
$\boxed{p\text{-norms}}$ (or $\boxed{L^p\text{-norms}}$)

$$\|\vec{x}\|_p = \left(|x_1|^p + \cdots + |x_m|^p\right)^{1/p}$$

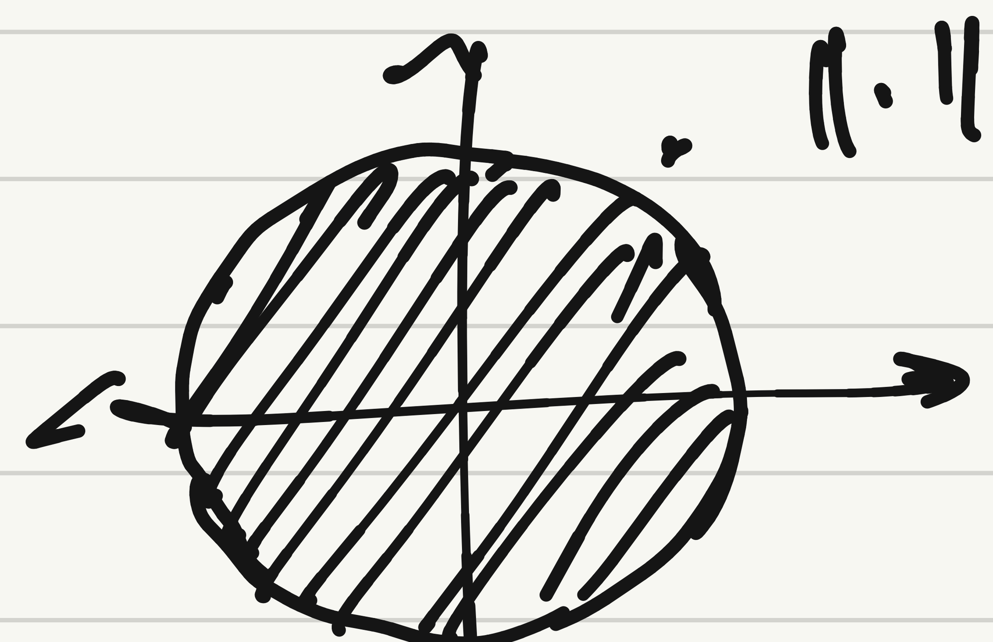
$$\|\vec{x}\|_1 = |x_1| + \cdots + |x_m|$$

$$\|\vec{x}\|_\infty = \lim_{p \to \infty} \|\vec{x}\|_p = \max_{j=1\dots m} |x_j|$$

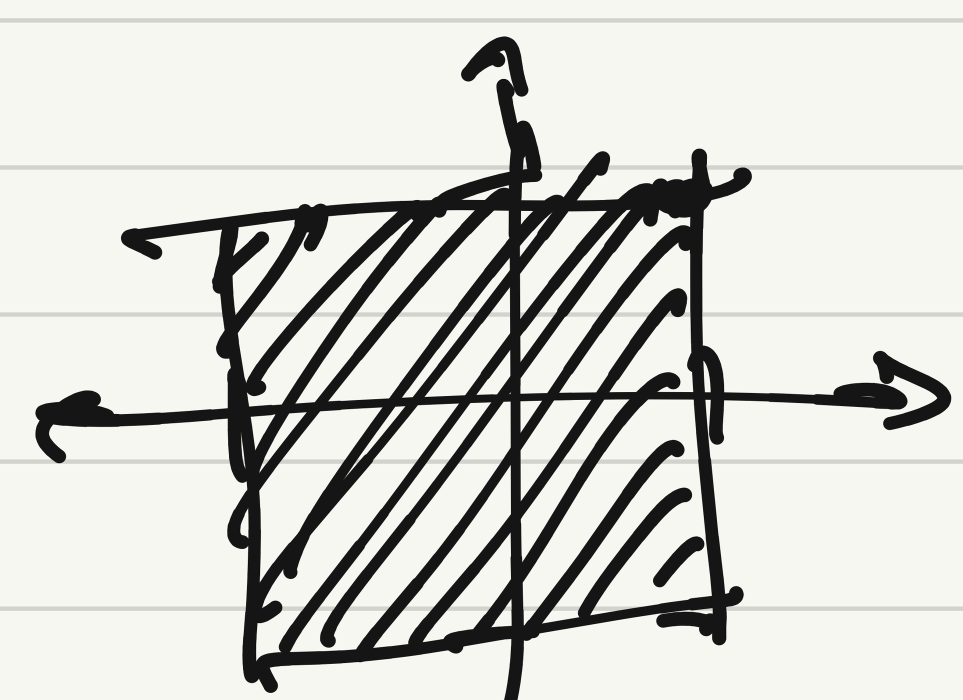
$\{\vec{x} \in \mathbb{R}^2 : \|\vec{x}\| \leq 1\}$ for any $p \geq 1$



$\|\cdot\|_1$



$\|\cdot\|_2$



$\|\cdot\|_\infty$

$$\|\cdot\| : \mathbb{C}^m \to \mathbb{R}$$

$$\vec{x} = \begin{bmatrix} \cdots\ m \\ \cdots\ kg \\ \cdots\ s \end{bmatrix} \longrightarrow \vec{v} \begin{bmatrix} \cdots\ km \\ \cdots\ g \\ \cdots\ hr \end{bmatrix}$$

$$\vec{v} = W^{-1}\vec{x}$$

$$\underbrace{\|\vec{v}\|'} = \|W^{-1}\vec{x}\|' = \underbrace{\|\vec{x}\|}$$

$$\|\cdot\| \longrightarrow \|\vec{x}\|' = \|W\vec{x}\|$$

$$\|\vec{v}\|' = \|W\vec{v}\|$$

$\boxed{\text{Cauchy - Schwarz}}$ inequality

कोशी श्वार्ट्स

$$|\vec{x}^*\vec{y}| \leq \|\vec{x}\|_2 \|\vec{y}\|_2$$

Hölder ineq. $|\vec{x}^*\vec{y}| \leq \|\vec{x}\|_p \|\vec{y}\|_q$ s.t. $\frac{1}{p} + \frac{1}{q} = 1$