COL726 : Recap

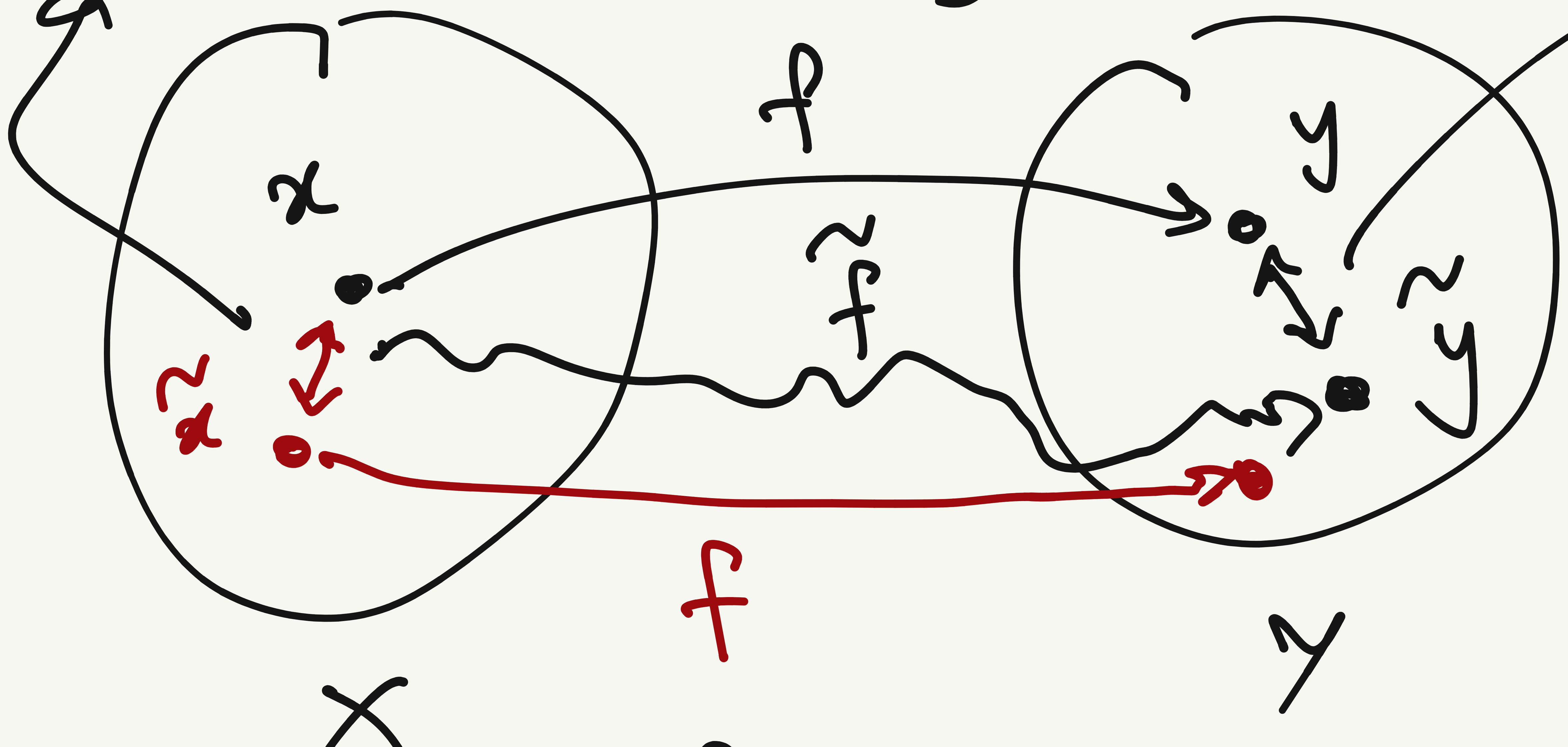
floating point numbers $\quad x \in M \times \beta^{E}$

$M \to \pm(d_0 . d_1 d_2 \dots d_{n-1})$

$E \to \mathbb{Z} \qquad \{E_{min}, \dots, E_{max}\}$

1. for any $x \in \mathbb{R}$, $fl(x) = x(1+\varepsilon)$ for some $|\varepsilon| \le \varepsilon_m$
2. for any $x, y \in \mathbb{F}$, $x \circledast y = (x * y)(1+\varepsilon)$ for some $|\varepsilon| \le \varepsilon_m$

backward stability



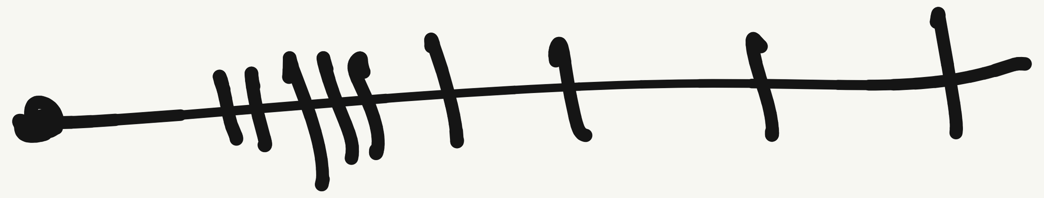
accuracy

$f: X \to Y$

$$\text{r.f.e.} \le \left(\kappa(x) + O(1)\right) \text{r.b.e.}$$

Example 1 : $f: \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n$

$$f(\vec{x}) = \frac{\vec{x}}{\|\vec{x}\|}$$



Linear algebra

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n \text{ or } \mathbb{C}^n$$

$\boxed{\text{vector space}}$ V over a field $\mathbb{K}$ is a set with operations

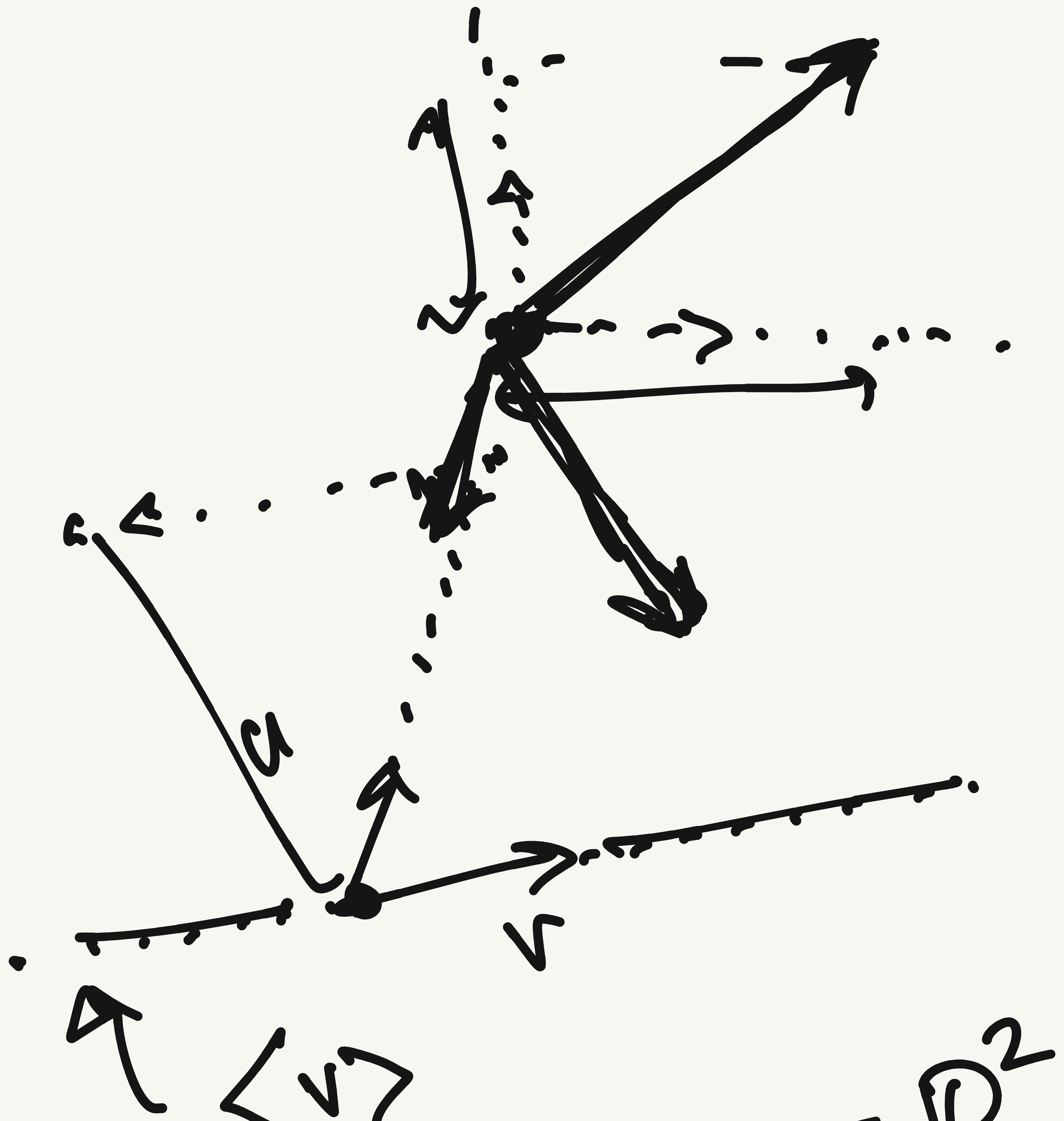
$+ : V \times V \to V$

$\cdot : \mathbb{K} \times V \to V$

($\mathbb{R}$ or $\mathbb{C}$)

such that...

$\{v_1, v_2, \ldots, v_n\} \in V$



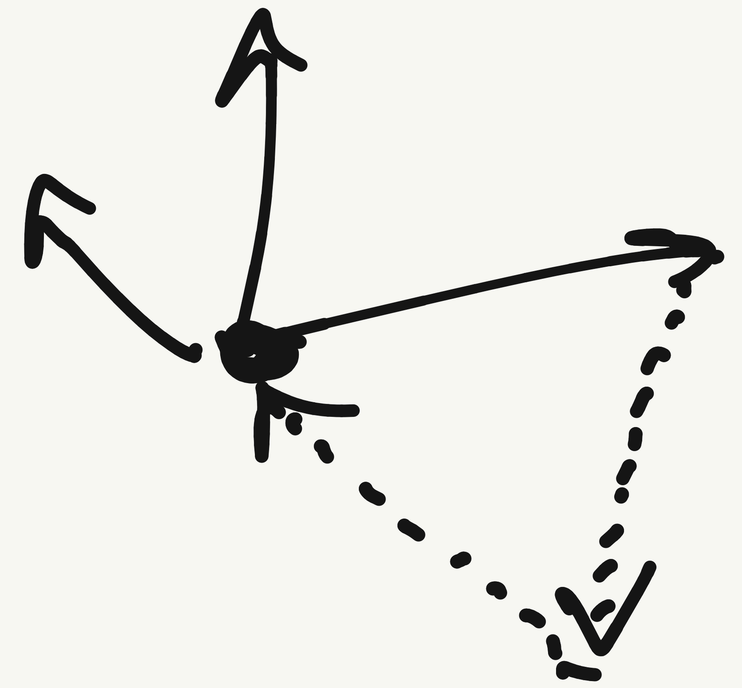
$\langle u, v \rangle = \mathbb{R}^2$

$\to v = \underline{s_1 v_1 + s_2 v_2 + \cdots + s_n v_n}$ for some $s_1, \ldots \in \mathbb{K}$

$\boxed{\text{linear combination}}$

$\boxed{\text{span}}$ = set of all l.c.'s = $\langle v_1, v_2, \ldots, v_n \rangle \subseteq V$

$\{v_1, \ldots, v_n\}$ is $\boxed{\text{linearly independent}}$ if no nontrivial l.c. is zero. $s_1 v_1 + \cdots + s_n v_n = \vec{0} \Leftrightarrow s_1 \ldots s_n = 0$



$\{v_1, \dots, v_n\}$ are not lin. indep. $\iff \exists\, i$ s.t. $v_i = s_1 v_1 + \dots + s_n v_n$ (not including i)

$\boxed{\text{basis}}$ of V is a linearly independent set that spans V.

$\langle b_1, \dots, b_n \rangle = V$

$\beta = \{\vec{b}_1, \dots, \vec{b}_n\}$

for any $\vec{x} \in V$, $\exists\, s_1, \dots, s_n$ s.t. $\vec{x} = s_1 \vec{b}_1 + \dots + s_n \vec{b}_n$

and $s_1, \dots s_n$ are unique

$\Rightarrow$ any vector $\vec{x} \in V$ can be represented uniquely as $\begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix} \in \mathbb{K}^n$

$\boxed{\text{coordinate vector}}$ for $\vec{x}$ $\longrightarrow$ $[\vec{x}]_\beta$

$\mathbb{R}^n$

$\mathbb{C}^n$

standard basis of $\mathbb{K}^n$ $= \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{bmatrix}, \dots \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \right\}$

$\vec{e}_1 \quad \vec{e}_2 \quad \dots \quad \vec{e}_n$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \cdots = x_1\vec{e}_1 + x_2\vec{e}_2 + \cdots + x_n\vec{e}_n$$

$$[\vec{x}]_e = \vec{x}$$

$$\begin{tikzcd} T : U \arrow[r] \arrow[d] & V \arrow[d] \\ \mathbb{K}^n & \mathbb{K}^m \end{tikzcd}$$

T is a $\boxed{\text{linear transformation}}$ if

$$T(u_1 + u_2) = T(u_1) + T(u_2) \quad \text{and} \quad T(su) = sT(u)$$

$$\Leftrightarrow \; T(s_1 u_2 + \cdots + s_n u_n) = s_1 T(u_1) + \cdots + s_n T(u_n)$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \longrightarrow T(\vec{x}) = y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} \qquad \begin{bmatrix} | \\ \vec{a}_1 \\ | \end{bmatrix} \cdots \begin{bmatrix} | \\ \vec{a}_n \\ | \end{bmatrix} \in \mathbb{K}^m \longrightarrow A = \underset{m \times n}{\begin{bmatrix} | & & | \\ a_1 & \cdots & a_n \\ | & & | \end{bmatrix}}$$

$$T(\vec{x}) = T(x_1\vec{e}_1 + \cdots + x_n\vec{e}_n) = x_1 \underbrace{T(\vec{e}_1)} + \cdots + x_n \underbrace{T(\vec{e}_n)} = x_1\vec{a}_1 + \cdots + x_n\vec{a}_n$$

$y = T(\vec{x}) = x_1\vec{a}_1 + \cdots + x_n\vec{a}_n$ where $\vec{a}_j = T(\vec{e}_j)$ $\quad A = \begin{bmatrix} | & & | \\ a_1 & \cdots & a_n \\ | & & | \end{bmatrix}$

$= \sum x_j \vec{a}_j$

$y_1 = ?$

$y_2 = ?$

$\vdots$

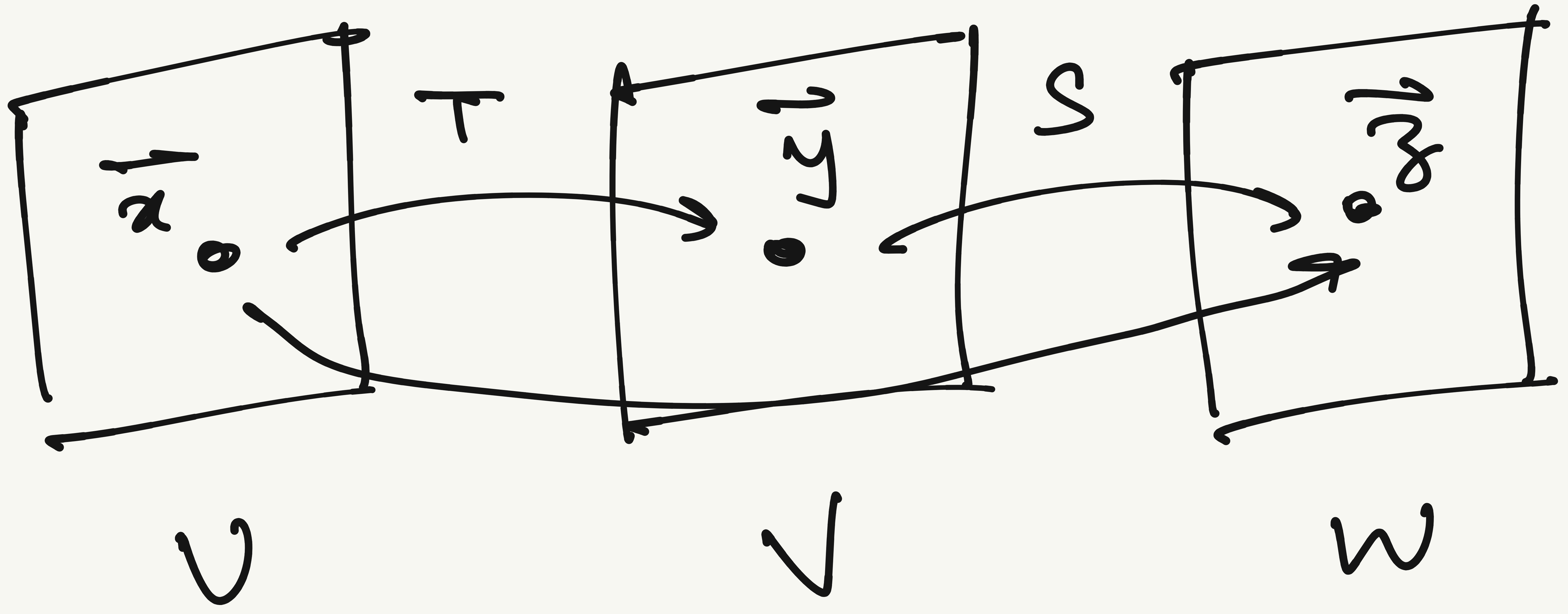
$\underline{y_i = \sum x_j a_{ij}} \longrightarrow$ matrix-vector multiplication

$$\underbrace{\begin{bmatrix} | & & | \\ a_1 & & a_n \\ | & & | \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}}_{\vec{x}} = \sum x_j \vec{a}_j \quad \text{--- } 1$$

2 j^{th} col. of $A = A\vec{e}_j = \underline{T(\vec{e}_j)}$

$$\underline{T : \mathbb{K}^n \to \mathbb{K}^m} \qquad \underline{A} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \in \mathbb{K}^{m \times n}$$

$T(\vec{x}) = A\vec{x}$



$T(\vec{x}) = B\vec{x}$

$S(\vec{y}) = A\vec{y}$

$S \circ T : U \to W$

$(S \circ T)(\vec{x}) = A(B\vec{x})$

$C\vec{x} = (S \circ T)(\vec{x}) = A(B\vec{x})$

$C\vec{e}_j = A(B\vec{e}_j)$

$\downarrow$

$\vec{c}_j = A\vec{b}_j$

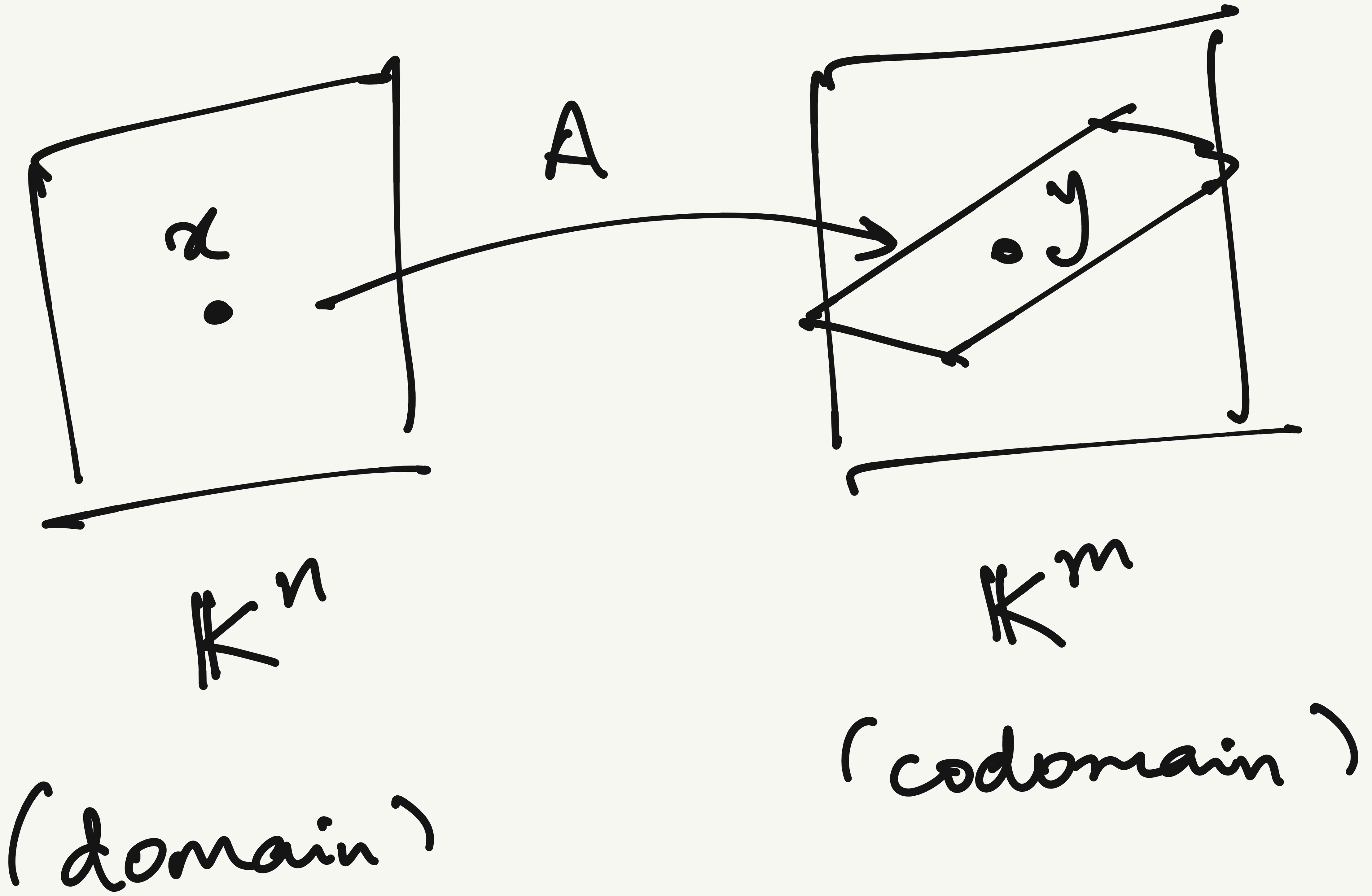
$$\underbrace{\begin{bmatrix} \vec{c}_1 & \cdots & \vec{c}_n \end{bmatrix}}_{C} = \underline{\begin{bmatrix} A\vec{b}_1 & \cdots & A\vec{b}_n \end{bmatrix}}$$

$C = AB$

$$\begin{bmatrix} \vec{c}_1^* \\ \vdots \\ \vec{c}_m^* \end{bmatrix} = \begin{bmatrix} \vec{a}_1^* B \\ \vdots \\ \vec{a}_m^* B \end{bmatrix}$$

$\underline{c_{ik} = \sum a_{ij} b_{jk}}$

$A \in \mathbb{K}^{m \times n}$

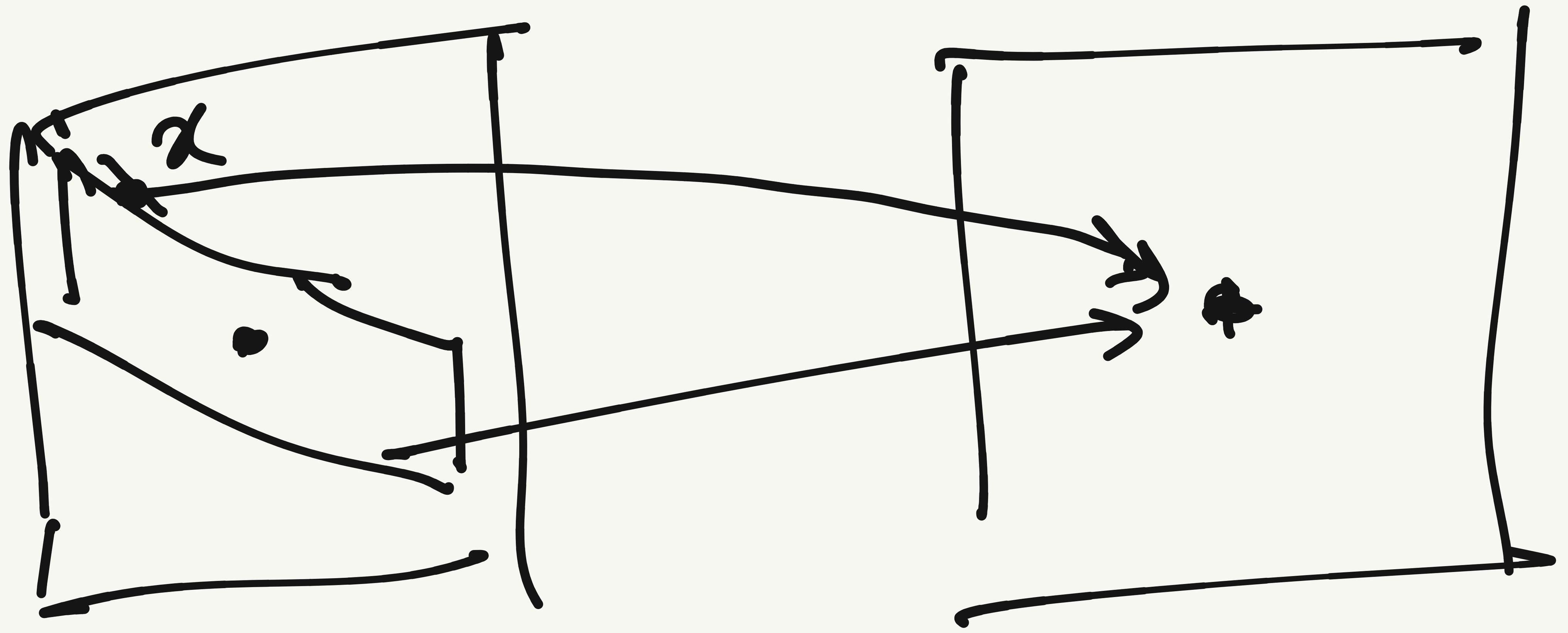


$\boxed{\text{range}}(A) = \{ \underbrace{A\vec{x}}_{\vec{a}_1 x_1 + \dots + \vec{a}_n x_n} : \vec{x} \in \mathbb{K}^n \}$

$A = \left[\vec{a}_1 \middle| \cdots \middle| \vec{a}_n \right]$

$= \langle \vec{a}_1, \dots, \vec{a}_n \rangle$

$= \text{Col}(A) \subseteq \text{codomain}$



$\boxed{\text{null}}(A) = \{ \vec{x} : A\vec{x} = \vec{0} \} \subset \text{domain}$

Solve $A\vec{x} = \vec{b}$

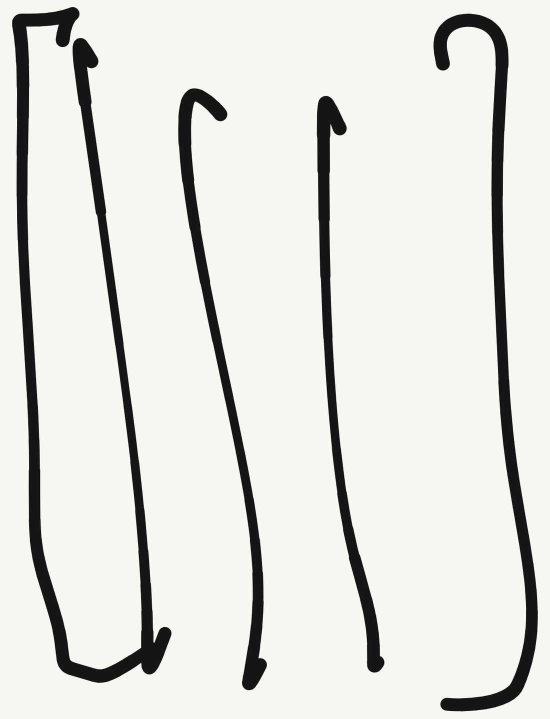
1. Does sol. exist? Iff $\vec{b} \in \text{range}(A)$
2. If $\vec{x}$ is a sol., what are all other sol.?

$A\vec{x} = b,\ A\vec{y} = b \Rightarrow A(\vec{x} - \vec{y}) = 0$

$A\vec{x} = \vec{b}$ has sol. $\Leftrightarrow$ $\vec{b} \in \text{range}(A)$

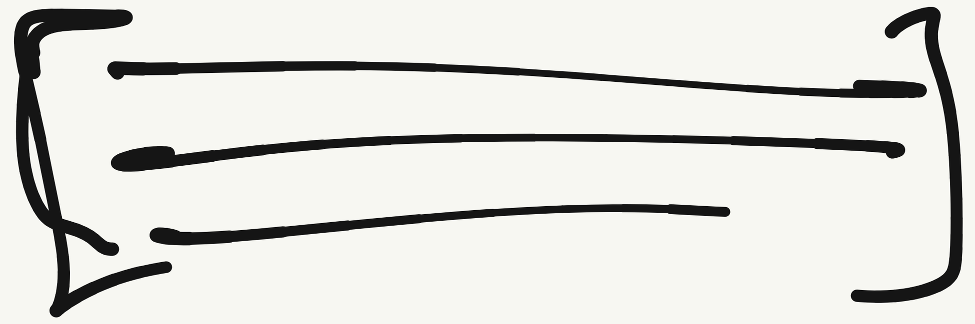
If $\vec{x}, \vec{y}$ are sol. $\Rightarrow$ $\vec{x} - \vec{y} \in \text{null}(A)$

$\vec{x}$ sol. and $\vec{z} \in \text{null}(A) \Leftrightarrow \vec{x} + \vec{z}$ is also sol.



col $\boxed{\text{rank}}$ $=$ # lin. indep. cols. of A

$= \dim(\text{range}(A))$



row rank $=$ # lin indep. rows $=$ col $\boxed{\text{rank}}$

$\text{rank}(A) \leq \min(m, n)$

If $\text{rank}(A) = \min(m, n)$:

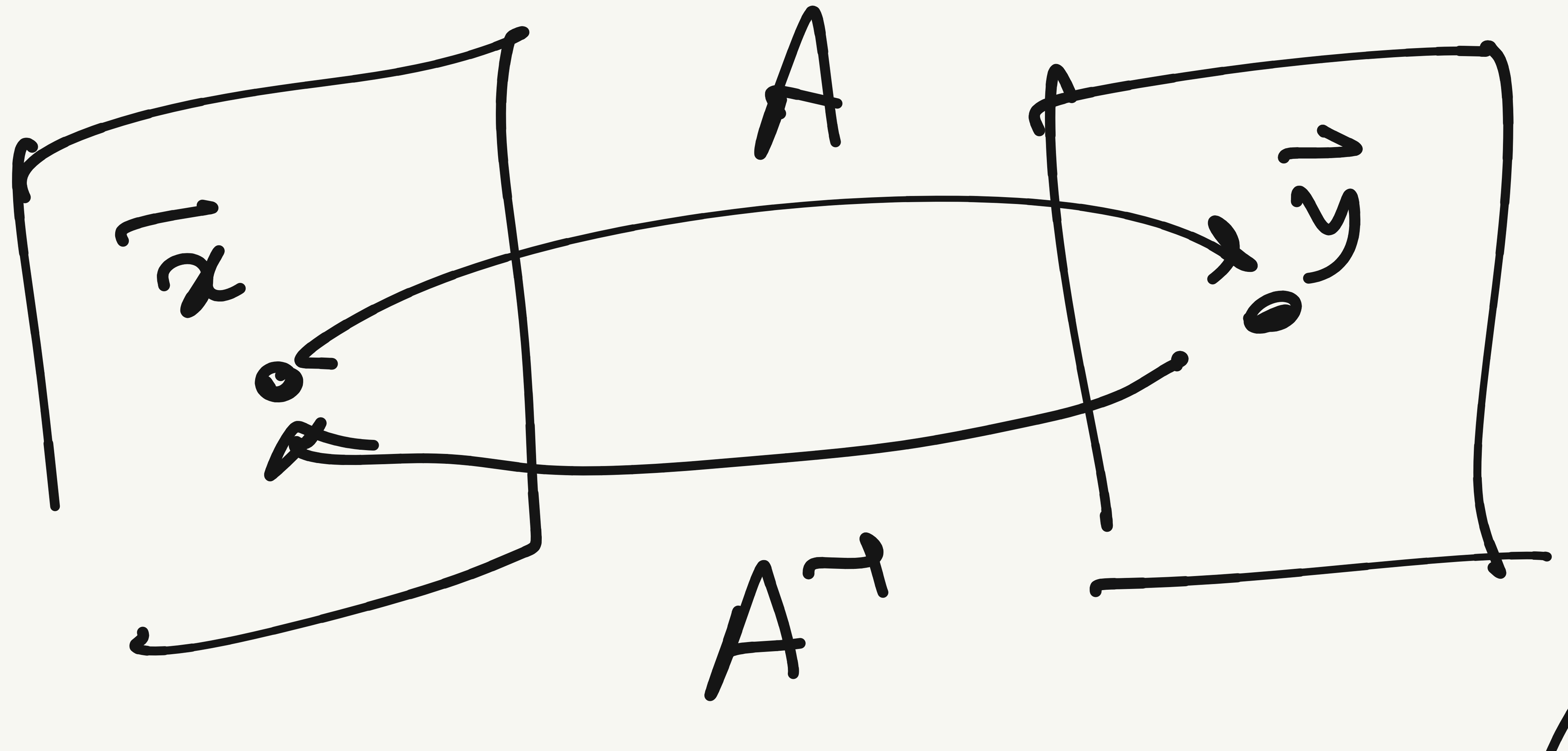
A has $\boxed{\text{full rank}}$

$\boxed{\text{Thm}}$: A "tall" matrix $A \in \mathbb{K}^{m \times n}$, $m \geq n$ is full rank

iff. $\vec{x} \mapsto A\vec{x}$ is <u>injective</u> ie. $A\vec{x}_1 = A\vec{x}_2 \Rightarrow \vec{x}_1 = \vec{x}_2$.

Inverse of matrix A : $A^{-1}(A\vec{x}) = \vec{x}$ exists only if $m=n$

$A(A^{-1}\vec{y}) = \vec{y}$



If $A \in \mathbb{K}^{m\times m}$,

$$\left.\begin{array}{l} A^{-1} \text{ exists} \\ \iff \operatorname{rank}(A) = m \\ \iff \operatorname{range}(A) = \mathbb{K}^{m} \\ \iff \operatorname{null}(A) = \{\vec{0}\} \\ \iff 0 \text{ is not eigenvalue of } A \\ \iff 0 \text{ is not singular value of } A \\ \iff \det(A) \neq 0 \end{array}\right.$$

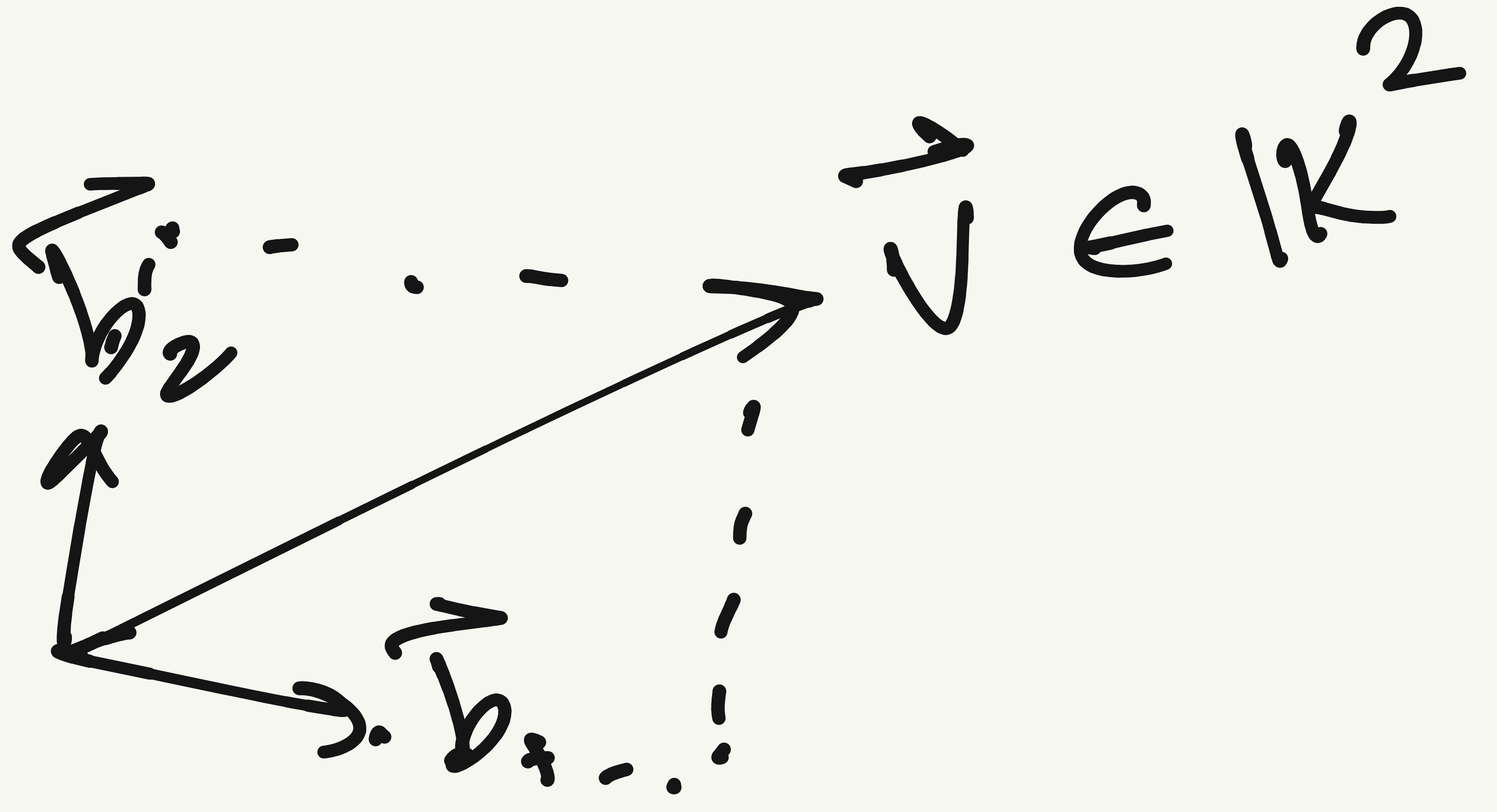
if A is not invertible,

A is $\boxed{\text{singular}}$

$T(\vec{x}) = \vec{y} = A\vec{x}$

$T^{-1}(\vec{y}) = \vec{x} = A^{-1}\vec{y}$

$$\beta = \{\vec b_1, \dots, \vec b_m\} \Longrightarrow B = \begin{bmatrix} \vec b_1 & \cdots & \vec b_m \end{bmatrix}$$



$$\vec v = \underline{x_1\vec b_1 + x_2\vec b_2 + \dots + x_m\vec b_m}$$

$$= B\vec x$$

$$\vec x = \underline{B^{-1}\vec v} = [\vec v]_\beta$$

$B \to B^{-1} \to B^{-1}\vec v$

unstable!!

Solve for $\vec x$ s.t. $B\vec x = \vec v$

$$\vec x = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

solve $A\vec x = \vec b$

$$A = \begin{bmatrix} 1.00 & 2.01 \\ 1.01 & 2.03 \end{bmatrix} \qquad b = \begin{bmatrix} 1.01 \\ 1.02 \end{bmatrix}$$

① by A^{-1}

② by elimination