Constrained optimization

$$\min f_0(x) \quad \text{s.t. } f_i(x) \le 0, \; h_i(x) = 0$$

primal problem

p^*

$$\longrightarrow \quad \max g(\lambda, \nu) \quad \text{s.t. } \lambda \ge 0$$

dual problem

d^*

$g(\lambda, \nu) \le f_0(x)$ (always)

$d^* \le p^*$: $\boxed{\text{weak duality}}$

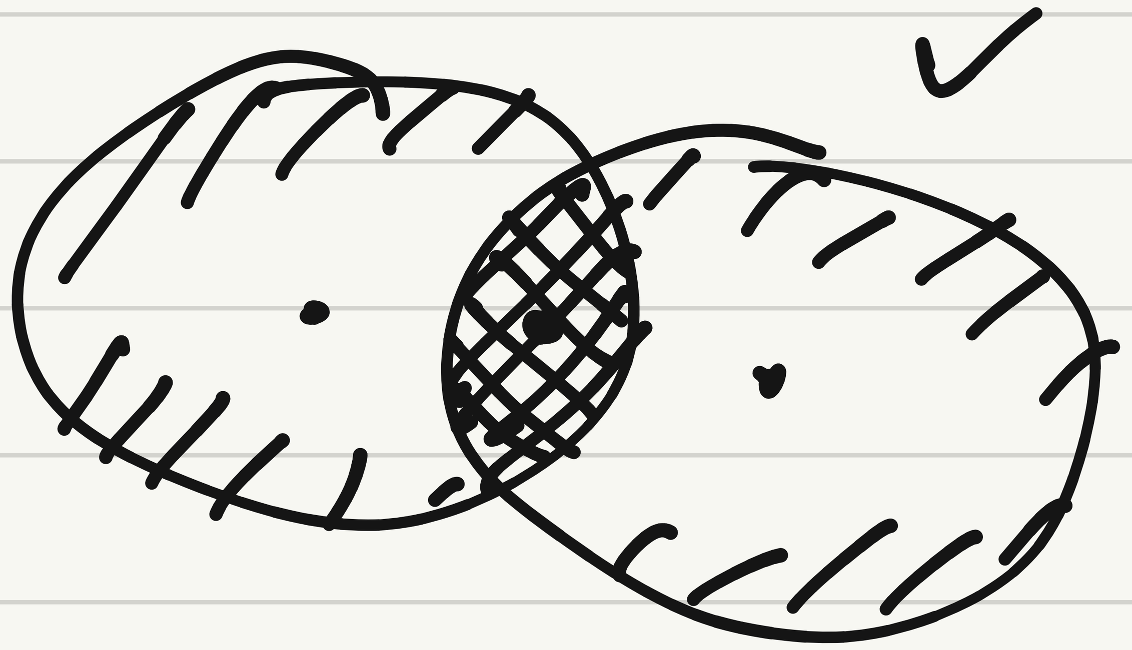
$d^* = p^*$: $\boxed{\text{strong duality}}$

when do we have strong duality?

$\boxed{\text{constraint qualifications}}$ e.g. $\boxed{\text{Slater's condition}}$: there exists at least one strictly feasible pt.

$f_i(x) < 0$, $Ax = b$

for all non-affine ineq. constraints



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for any LP, convex QP, feasibility $\Rightarrow$ Slater's condition

Slater's condition $\Rightarrow$ strong duality

<u>and</u> there exist $(\overset{\geq 0}{\lambda^*}, \nu^*)$ s.t. $g(\lambda^*, \nu^*) = d^*$

(as long as $d^* \geq -\infty$)

Suppose $p^* = d^*$ and $f_0(x^*) = p^* = d^* = g(\lambda^*, \nu^*)$

$$f_0(x^*) = g(\lambda^*, \nu^*)$$
$$= \inf_x L(x, \lambda^*, \nu^*)$$

$$L(x, \lambda, \nu) = f_0(x) + \overbrace{\sum \lambda_i f_i(x)}^{\leq 0} + \underbrace{\sum \nu_i h_i(x)}_{= 0}$$

$$\left\{\begin{array}{l} \leq L(x^*, \lambda^*, \nu^*) \\ \leq f_0(x^*) \end{array}\right.$$

must be = !

$$\leq f_0(x)$$

<u>Lagrangian $\leq f_0$ for all feasible points.</u>

$$\inf_x L(x, \lambda^*, \nu^*) = L(x^*, \lambda^*, \nu^*) \Rightarrow \underline{x^* \text{ minimizes } L(x, \lambda^*, \nu^*)}$$

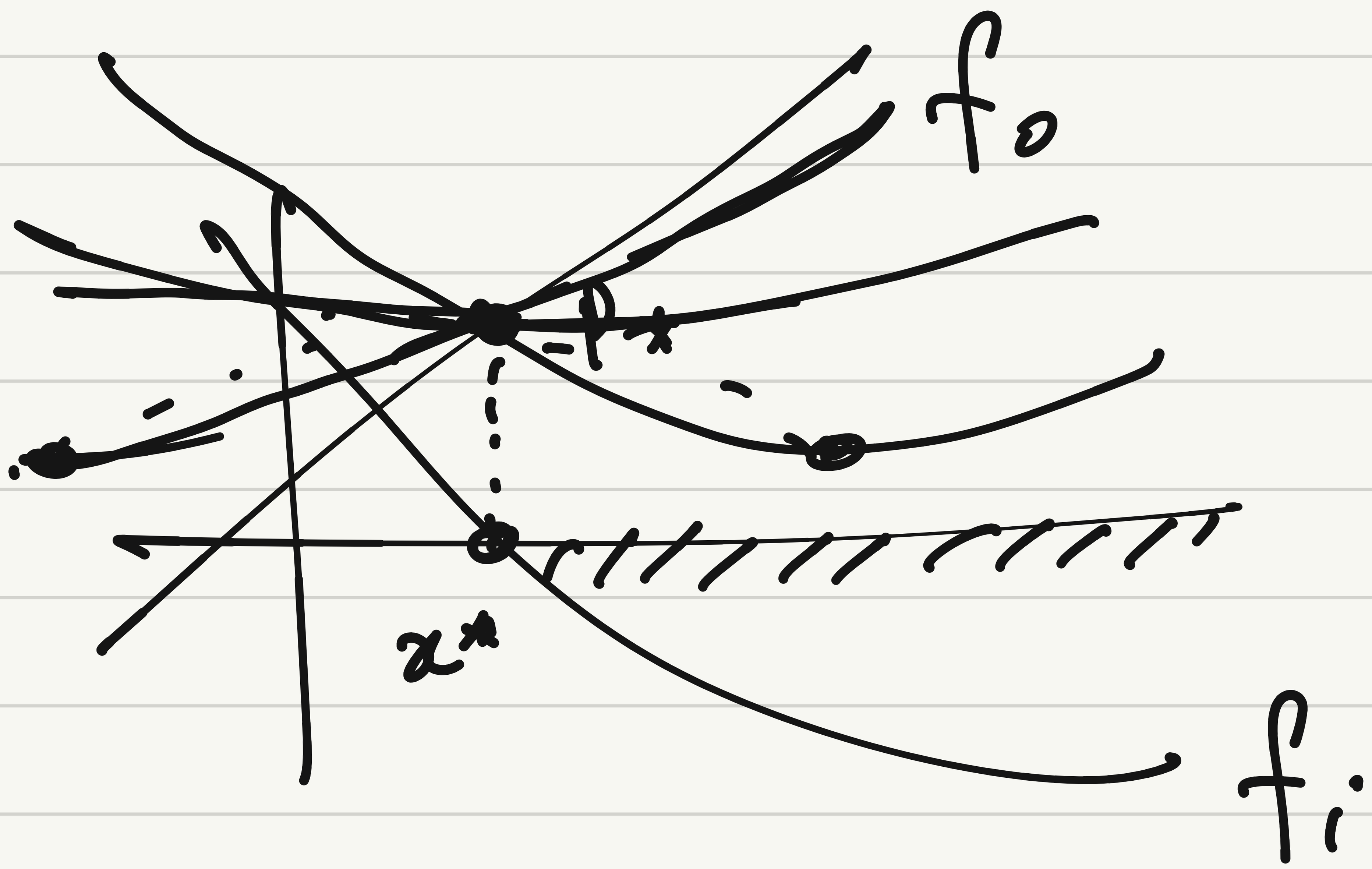
$$\underbrace{L(x^*, \lambda^*, \nu^*)}_{} = \underline{f_0(x^*)} \Rightarrow \underline{\lambda_i^* f_i(x^*) = 0} : \boxed{\text{complementary slackness}}$$

or $\boxed{\text{complementarity}}$

$$\underline{f_0} + \underbrace{\sum \lambda_i^* f_i}_{\leq 0} + \underbrace{\sum \nu_i^* h_i}_{=0}$$

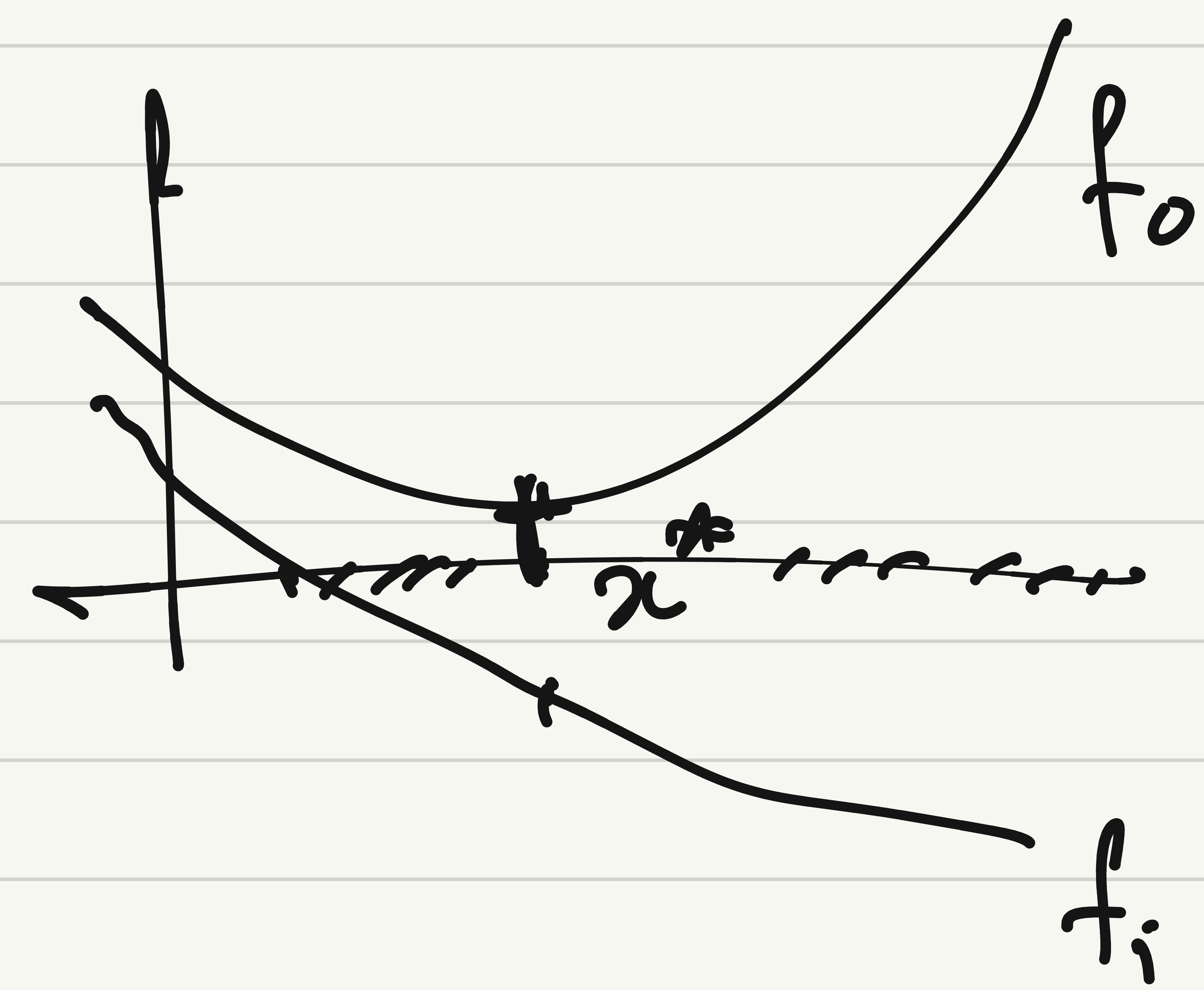
$$\lambda_i^* > 0 \Rightarrow f_i(x^*) = 0$$

$$f_i(x^*) < 0 \Rightarrow \lambda_i^* = 0$$



$$L = f_0 + \lambda_i f_i$$

$$g(\lambda) = \inf_x L(x, \lambda)$$



If f_i, h_i differentiable $\Rightarrow$ first-order cond. for optimality $(\nabla f(x^*) = 0)$

feasibility: $f_i(x^*) \leq 0$

$h_i(x^*) = 0$

x^* minimizes $L(x, \lambda^*, \nu^*)$

$\Rightarrow \nabla_x L(x, \lambda^*, \nu^*) = 0$

Dual feasibility: $\lambda_i^* \geq 0$

Complementarity: $\lambda_i^* f_i(x^*) = 0$

Stationarity: $\nabla f_0(x^*) + \sum \lambda_i^* \nabla f_i(x^*) + \sum \nu_i \nabla h_i(x^*) = 0$

$\boxed{\text{Karush-Kuhn-Tucker (KKT) conditions}}$

Strong duality $\Rightarrow$ KKT holds at optimum $x^*, (\lambda^*, \nu^*)$

If convex, $\begin{cases} \text{Slater's cond} \Rightarrow \text{strong duality} \Rightarrow \text{KKT at } x^*, \lambda^*, \nu^* \\ \text{KKT holds } x, \lambda, \nu \Rightarrow x, \lambda, \nu \text{ are optimal, strong duality holds!} \end{cases}$

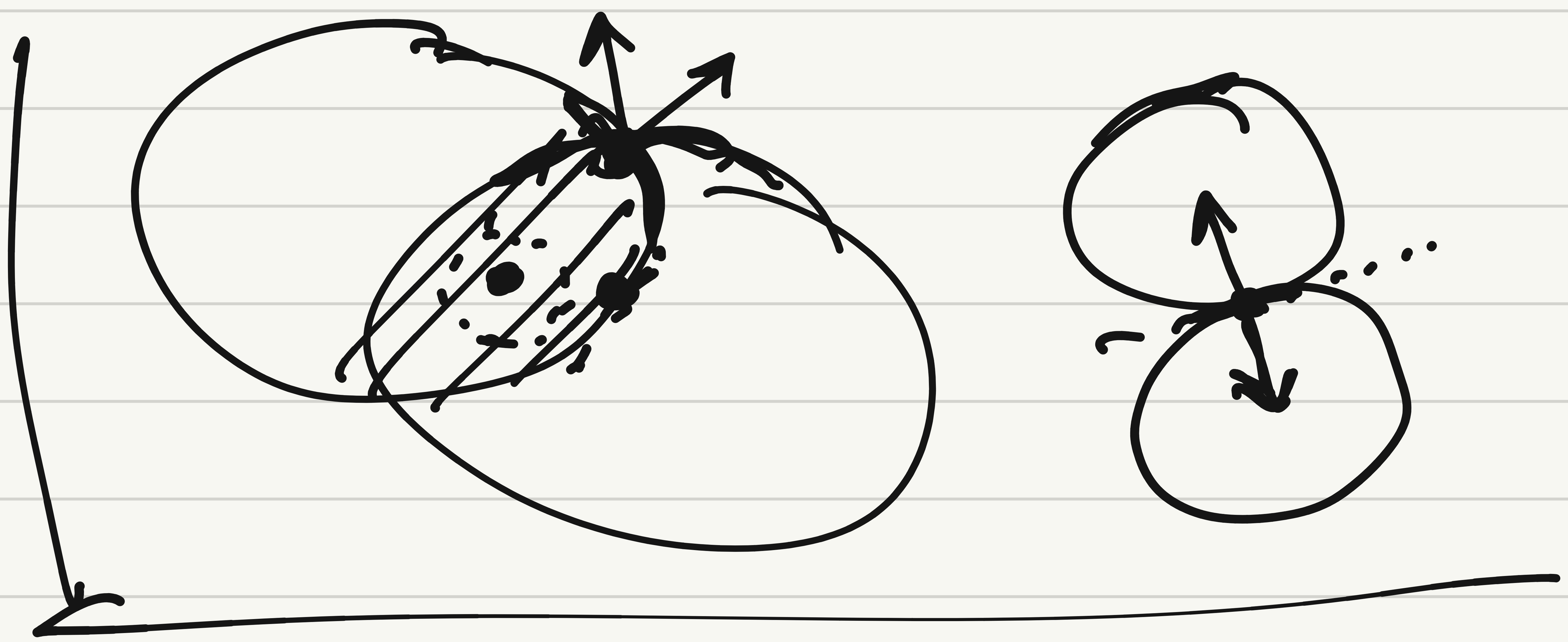
$f_0(x) = p^* = d^* = g(\lambda, \nu)$

for nonconvex problems,

Constraint qualification $\Rightarrow$ KKT holds at local optimum!

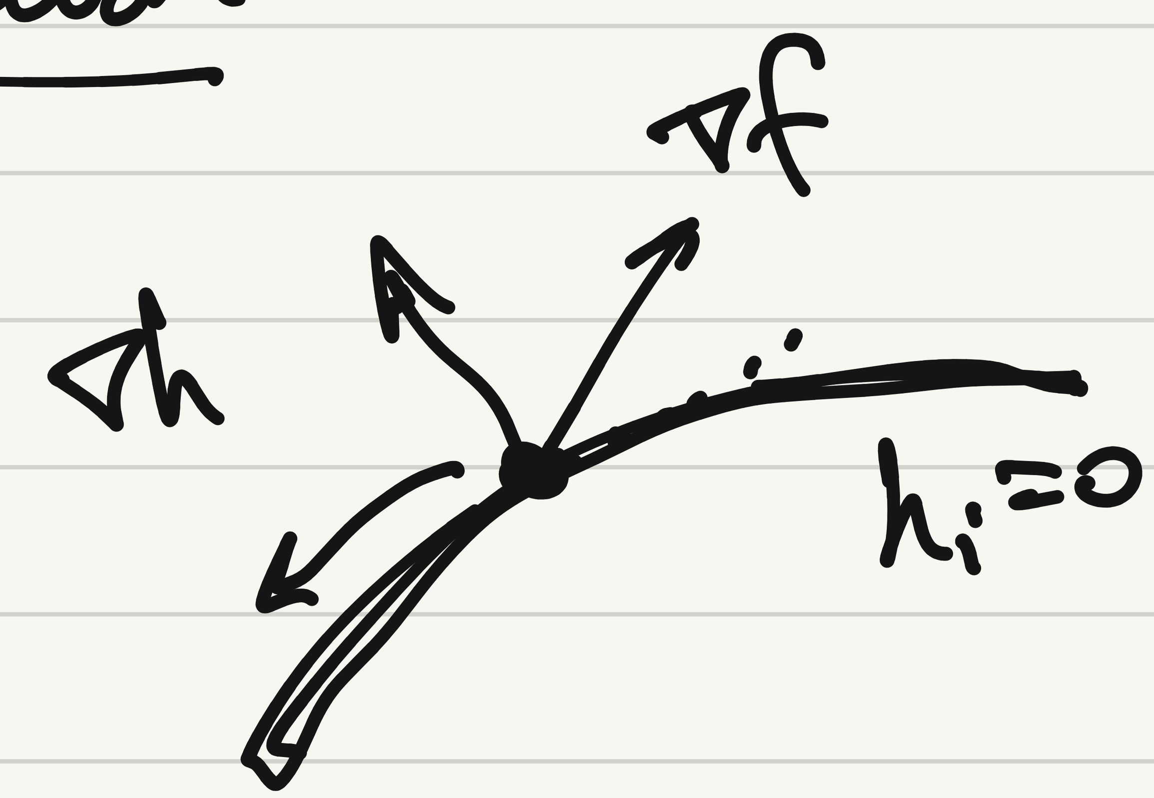
$\hookrightarrow$ Linear Independence con. qual. (LICQ)

$$\{\nabla f_i(x^*) : f_i(x^*) = 0\} \cup \{\nabla h_i(x^*)\} \text{ is linearly indep.}$$



Equality constrained optimization

$$\left.\begin{aligned} &\min f(x) \\ &\text{s.t. } h_i(x) = 0 \end{aligned}\right\}$$



KKT: $\quad h_i(x) = 0$

$$\nabla f(x) + \sum \nu_i \nabla h_i(x) = 0 \iff \nabla f(x) \in \text{span}\{\nabla h_i(x)\}$$

Special case: eq. con. convex QP

$$\min \quad \frac{1}{2} x^T P x + q^T x, \qquad P \succeq 0$$
$$\text{s.t.} \quad Ax = b \qquad \longleftrightarrow \qquad h_i(x) = a_i^T x + b \qquad A = \begin{bmatrix} - & a_1^T & - \\ - & a_2^T & - \\ & \vdots & \end{bmatrix}$$

$A \in \mathbb{R}^{p \times n}$, $p \leq n$, $\text{rank}(A) = p$

KKT:
$$\begin{cases} Ax = b \\ \nabla f(x) + \sum \nu_i \nabla h_i(x) = 0 \\ Px + q + \sum \nu_i a_i \\ \qquad + A^T \nu \end{cases} \Rightarrow \underline{Px + A^T \nu = -q} \qquad \nu = \begin{bmatrix} \nu_1 \\ \nu_2 \\ \vdots \end{bmatrix}$$

$$\begin{bmatrix} P & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} x \\ \nu \end{bmatrix} = \begin{bmatrix} -q \\ b \end{bmatrix}$$

$\boxed{\text{KKT system}}$,

KKT matrix : $\mathbb{R}^{(n+p) \times (n+p)}$

$\begin{bmatrix} P & A^T \\ A & 0 \end{bmatrix}$ not SPSD, n nonneg. eigenvalues

p negative eigenvalues !

If nonsingular $\Rightarrow$ unique sol $\begin{bmatrix} x^* \\ \nu^* \end{bmatrix}$ is optimal primal-dual pair

If singular but has sol $\Rightarrow$ all sols are optimal

" " no sol $\Rightarrow$ problem is unbounded below or infeasible

KKT matrix is nonsingular $\Leftrightarrow$ P is SPD on null space of A

$\Leftrightarrow$ $\forall\, v \neq 0$ s.t. $Av = 0$, $v^T P v > 0$

$\Leftrightarrow$ If $\text{null}(A) = \text{range}(F)$, $F^T P F \succ 0$

General case: $f(x)$ convex fn.

1. Eliminate eq. constraints
2. Solve dual problem
3. Newton's method

① Elim. Cons:

$$\min f(x) \quad \text{s.t. } x_1 + x_2 + \cdots + x_n = 1 \iff \min f(x_1, x_2, \ldots, x_{n-1}, 1 - x_1 - x_2 - \cdots x_{n-1})$$

$F \in \mathbb{R}^{n \times (n-p)}$

$Ax = b \iff x = \hat{x} + Fz$ where $\text{range}(F) = \text{null}(A)$

$\min f(\hat{x} + Fz)$ over z

② Dual problem

$$g(\nu) = \inf_x \left(f(x) + (Ax-b)^T \nu \right) = -b^T\nu + \underbrace{\inf_x \left(f(x) + \nu^T A x \right)}$$

If you have closed form for inf term.

(1) $\min g(\nu) \longrightarrow \nu^*$

(2) $\min L(x, \nu^*) \longrightarrow x^*$

<u>Example</u> : $\min \frac{1}{2} x^T P x + q^T x \quad s.t. \ Ax = b$

$$g(\nu) = -b^T\nu + \inf_x \left(\underbrace{\tfrac{1}{2} x^T P x + q^T x + \nu^T A x} \right)$$

$x^*(\nu) = -P^{+}(q + A^T\nu)$

$$= -b^T\nu + \ldots$$

③ Newton's method

Already have $Ax = b$

$\hat{f}(x+\Delta x) = f(x) + \nabla f(x)^T \Delta x + \frac{1}{2} \Delta x^T \nabla^2 f(x) \Delta x$

$x + \Delta x$ is feasible $\Leftrightarrow$ $A \underline{\Delta x} = 0$: Δx is a $\boxed{\text{feasible direction}}$

$\min \underbrace{\hat{f}(x+\Delta x)}$ s.t. $A \Delta x = 0$

$$\rightarrow \begin{bmatrix} \nabla^2 f(x) & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} \Delta x \\ \nu \end{bmatrix} = \begin{bmatrix} -\nabla f(x) \\ 0 \end{bmatrix}$$

solve to get Δx (and ν), then do line search $\underline{x} + t \Delta x$

$A \Delta x = 0$

feasible

≤ 1

Another interp.:

Solve KKT cond. using Newton's method

$$\begin{cases} \underbrace{\nabla f(x) + A^T \nu}_{r_{dual}(x,\nu)} = 0 \\ \underbrace{Ax - b}_{r_{primal}(x,\nu)} = 0 \end{cases}$$

$r_{dual}(x, \nu)$ measures stationarity

$r_{primal}(x, \nu)$ measures feasibility

$$r(x,\nu) = \begin{bmatrix} r_{dual}(x,\nu) \\ r_{primal}(x,\nu) \end{bmatrix}$$

Solve $r(x,\nu) = 0$ using Newton's method

$$\begin{bmatrix} \nabla^2 f(x) & A^T \\ A & 0 \end{bmatrix} \begin{pmatrix} \Delta x \\ \Delta \nu \end{pmatrix} = \begin{bmatrix} -r_{dual} \\ -r_{primal} \end{bmatrix}$$

min $f(x)$ s.t. $Ax = b$ with eq. con. Newton

$\Updownarrow$

Interpretation 3: $Ax = b \iff x = \hat{x} + Fz$. min $f(\hat{x} + Fz)$ with uncon Newton

Infeasible start Newton's method

$Ax^{(0)} \neq b$