Constrained optimization

$\boxed{\text{Gauss-Newton alg.}}$

$$\begin{cases} \min & f_0(x) \\ \text{s.t.} & f_i(x) \le 0 \quad i = 1, 2, \dots, m \\ & h_i(x) = 0 \quad i = 1, \dots, p \end{cases}$$

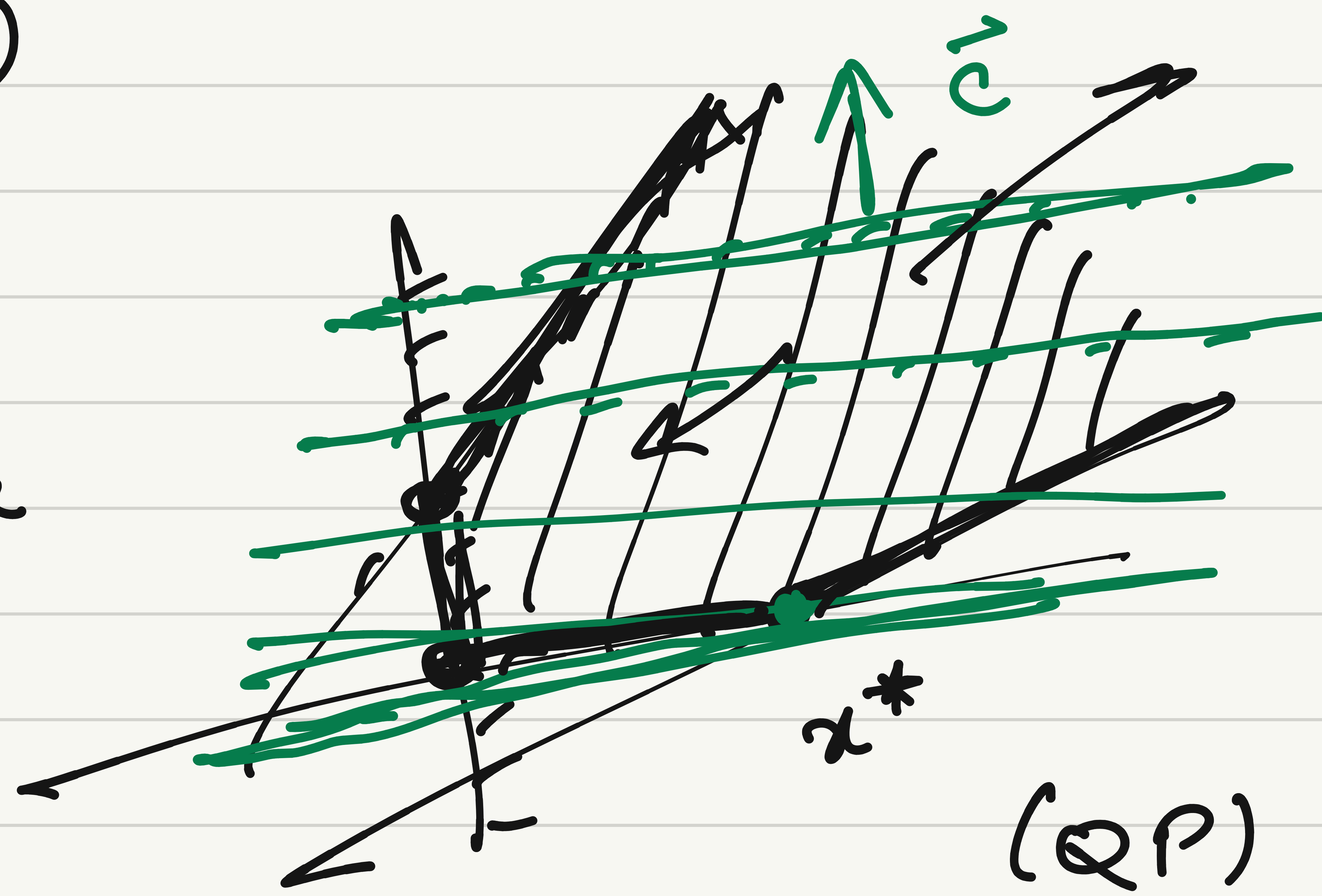
convex opt. problems if f_0, f_i convex, h_i affine $\to Ax = b$

if all f_0, f_i, h_i affine : $\boxed{\text{linear programming}}$ (LP)

$$\min \; c^T x + d$$
$$\text{s.t.} \; \begin{cases} Gx \preceq h \\ Ax = b \end{cases}$$

$g_i^T x \le h_i$: halfspace

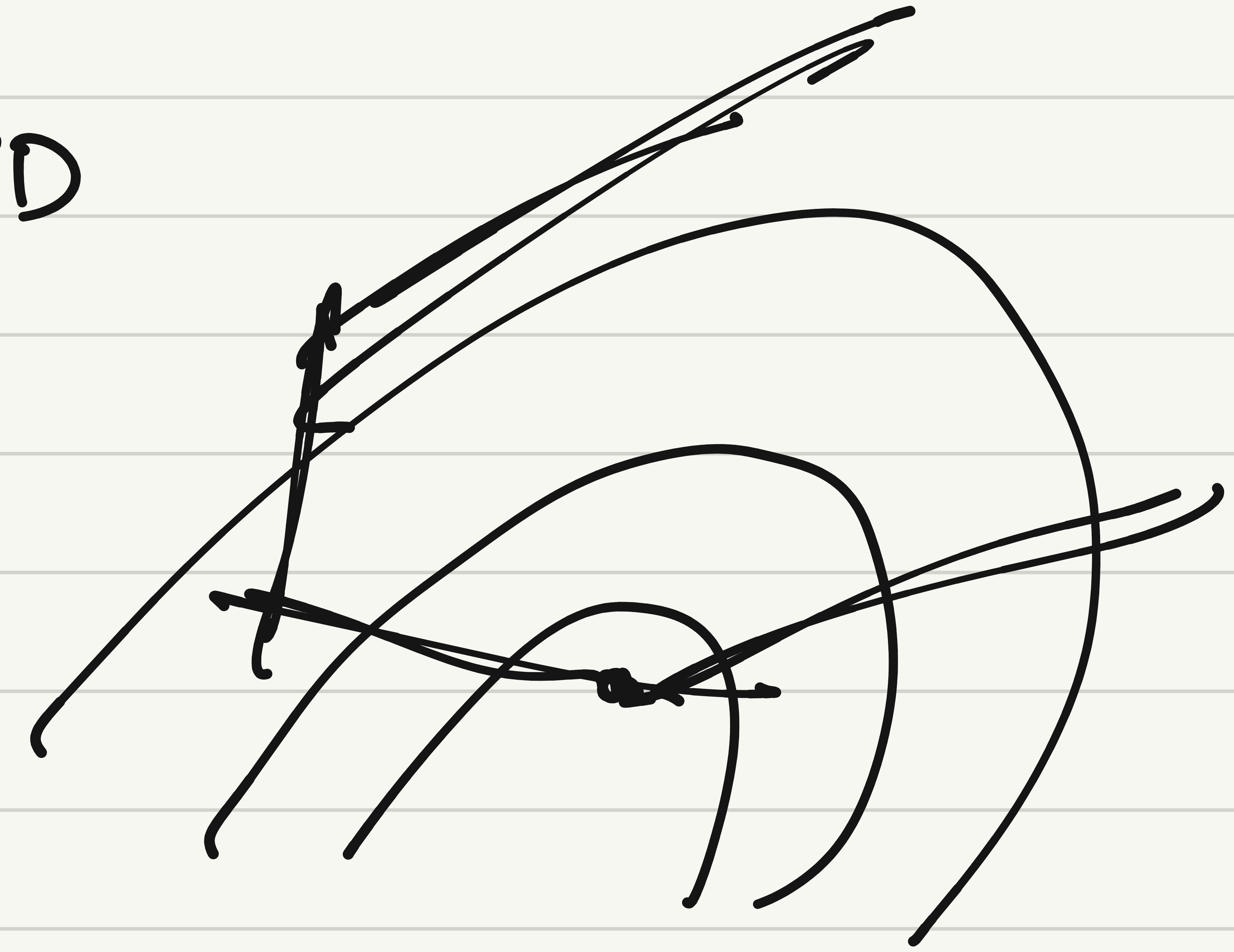
$a_i^T x = b_i$: hyperplane



Constraint fns $f_1, \dots f_m$, h_i affine,

obj f_0 is quadratic $f_0(x) = \frac{1}{2} x^T P x + q^T x + r$: $\boxed{\text{quadratic prog.}}$ (QP)

$f_0(x) = \frac{1}{2} x^T P x + q^T x + r$ convex $\iff$ P is SPSD



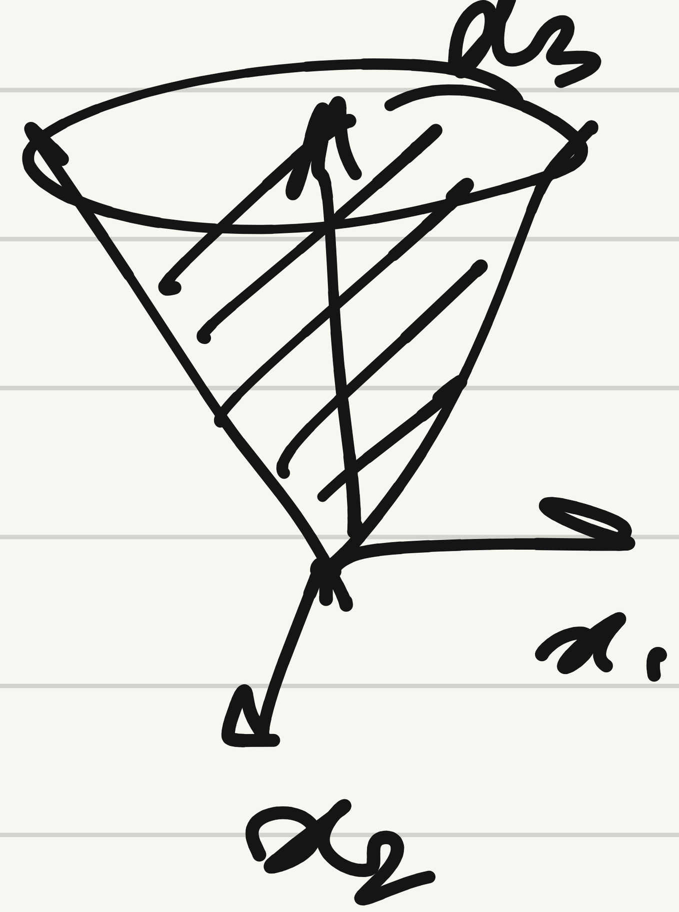
If P is nonconvex: NP-hard!

- Quadratically constrained QP (QCQP)
- Second-order cone programming (SOCP): $\|Ax + b\| \le c^T x + d$

$$\left\| \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\| \le x_3$$

- Semidefinite programming (SDP):

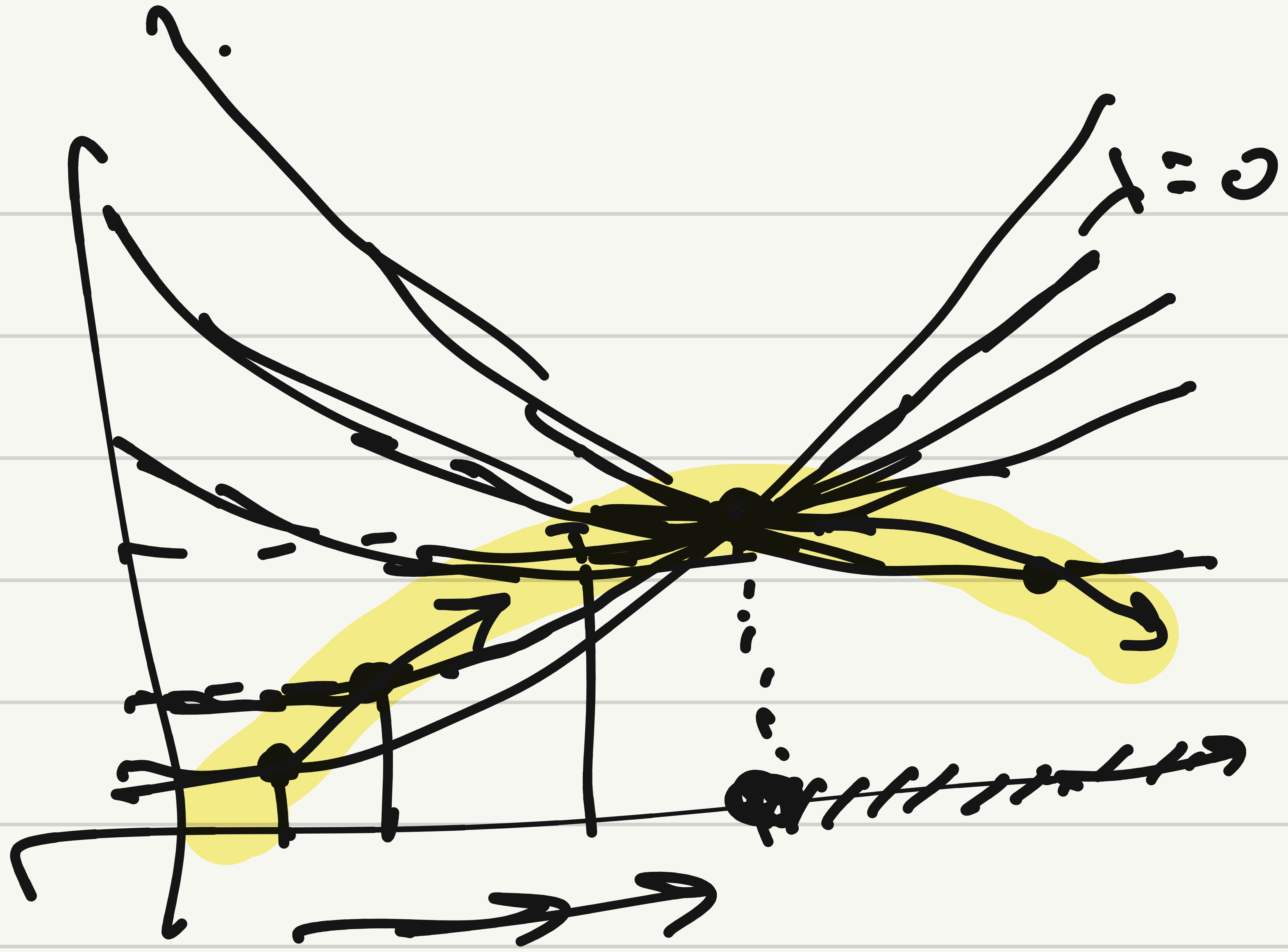
$$\sum x_i F_i + G \preceq 0$$



$LP \subseteq QP \subseteq QCQP \subseteq SOCP \subseteq SDP$

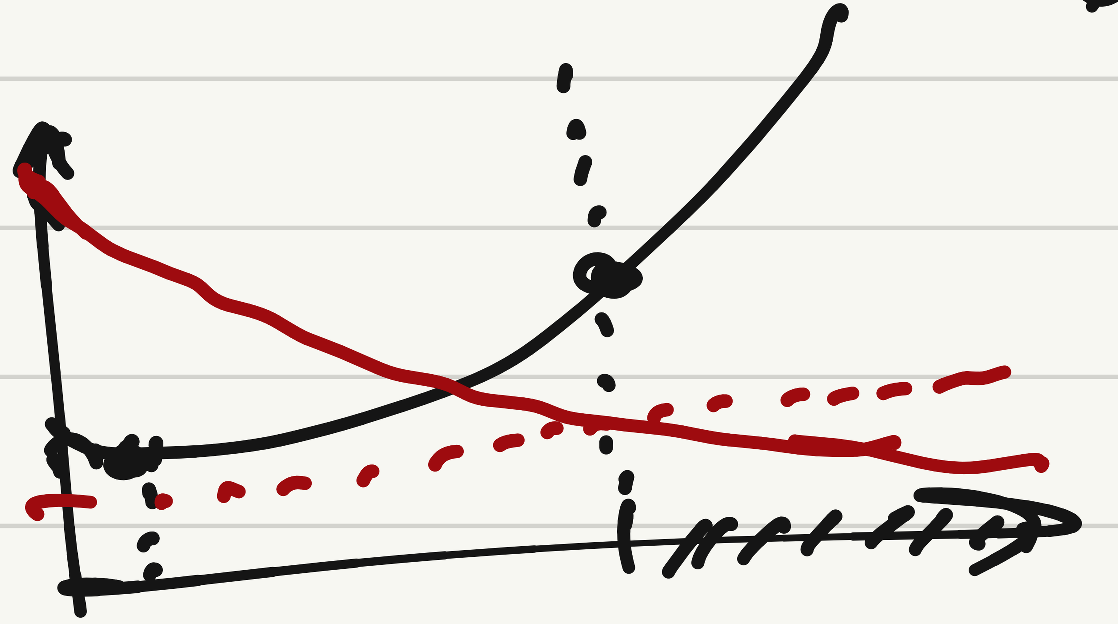
Lagrange duality

$$\left.\begin{array}{ll} \min & f_0(x) \\ \text{s.t.} & f_i(x) \le 0 \\ & h_i(x) = 0 \end{array}\right\} \longrightarrow \min f_0(x) + \cdots$$



$f_0(x)$: operating cost

$f_1(x)$: amount of excess pollution



$$\boxed{\min f_0(x) \quad \text{s.t.} \quad f_1(x) \le 0}$$

$$\text{fine} = \lambda \; f_1(x) \longrightarrow \min f_0(x) + \lambda f_1(x)$$

$$\boxed{\text{Lagrangian}} : L(x,\lambda) = f_0(x) + \lambda f_1(x)$$

min $f_0(x)$
s.t. $f_i(x) \le 0$
$h_i(x) = 0$

$\longrightarrow$ Lagrangian $L(\vec{x}, \vec{\lambda}, \vec{\nu}) = f_0(x) + \sum \lambda_i f_i(x) + \sum \nu_i h_i(x)$

lambda

nu

Lag. mult. vec. or $\boxed{\text{dual variables}}$

$\boxed{\text{primal variable}}$

λ_i, ν_i : $\boxed{\text{Lagrange multipliers}}$

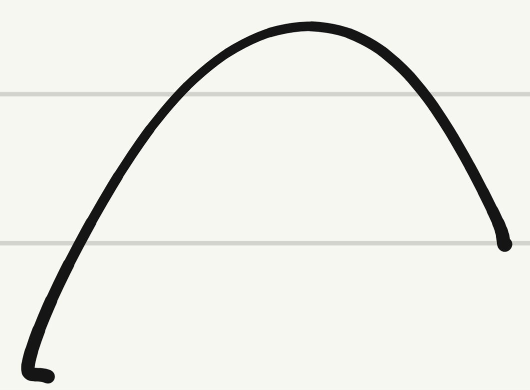
$\lambda_i \ge 0$ for ineq. cons.

$\nu_i \in \mathbb{R}$ for eq. cons

for any value of $\lambda \ge 0$, ν,
there is some $x^*(\lambda, \nu)$, some opt value of L

$g(\lambda, \nu) = \inf_x L(x, \lambda, \nu)$: $\boxed{\text{Lagrange dual function}}$

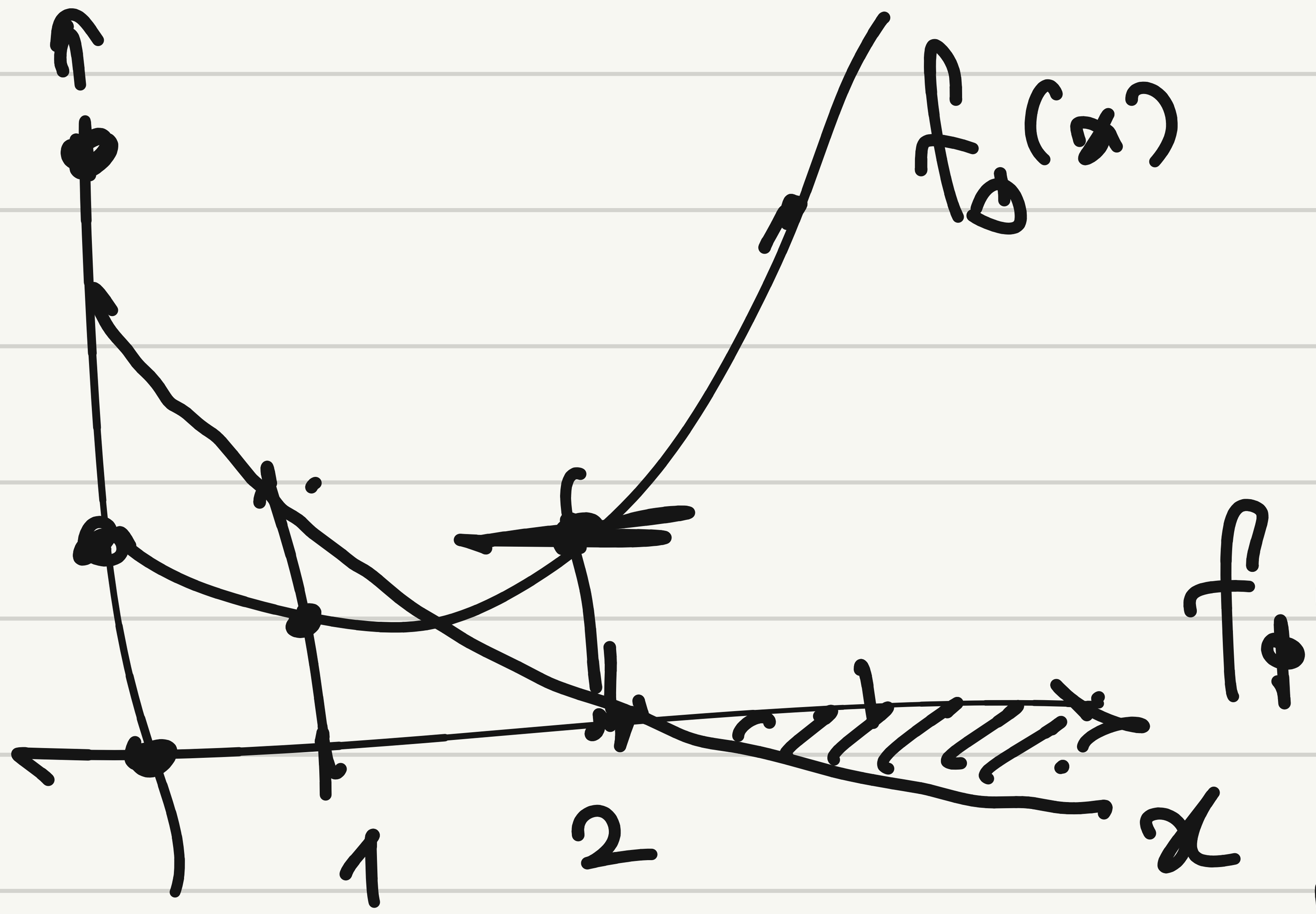
$g(\lambda,\nu)$ is <u>concave</u> function! for all problems (not just convex!)



$$g(\lambda,\nu) = \inf_x \underbrace{\left(f_0(x) + \sum \lambda_i f_i(x) + \sum \nu_i h_i(x) \right)}$$

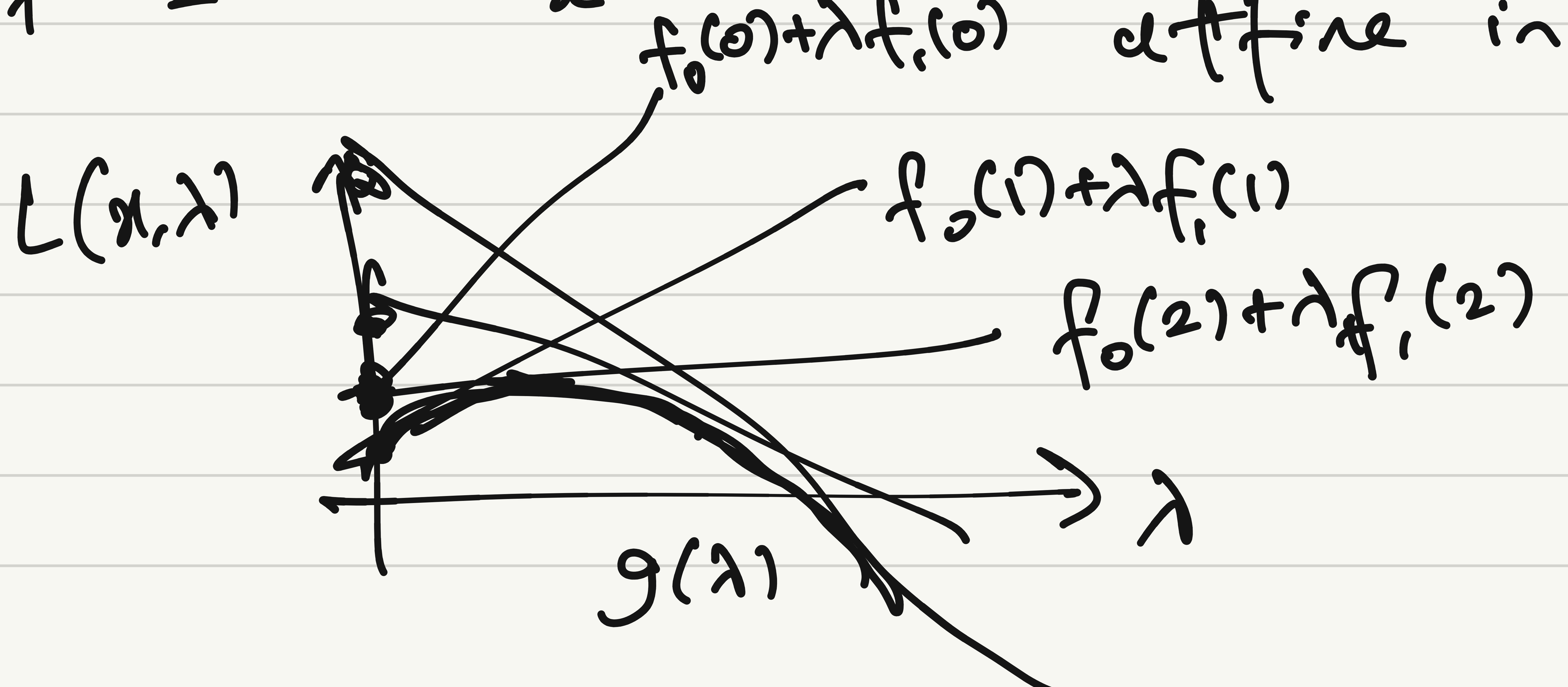
$$= \inf_x L(x,\lambda,\nu)$$

$$= \min \left(L(0,\lambda,\nu),\ L(1,\lambda,\nu),\ L(2,\lambda,\nu),\ \vdots \right)$$



$$\inf_x L(x,\lambda,\nu)$$

for any x, $L(x,\lambda,\nu)$ is affine in λ,ν



$g(\lambda,\nu)$ is inf. of family of <u>affine</u> fns.

$\Rightarrow$ concave

$\min f_0(x)$

s.t. $f_i(x) \leq 0$

$h_i(x) = 0$

$\longrightarrow \quad g(\lambda,\nu) = \inf_x \left(f_0(x) + \cdots \right)$

$=$

Example: dual fn of LP

LP: $\min c^T x$

s.t. $Ax = b$

$x \succeq 0$

$$L(x,\lambda,\nu) = c^T x - \sum \lambda_i x_i + \sum \nu_i \cdot (Ax-b)_i$$

$$= c^T x - \lambda^T x + (Ax-b)^T \nu$$

$$= \underbrace{(c - \lambda + A^T\nu)^T}_{p} x - \underbrace{b^T \nu}_{q}$$

$L = p^T x - q$

$$g(\lambda,\nu) = \inf_x L(x,\lambda,\nu)$$

$$= \begin{cases} -b^T\nu & \text{if } A^T\nu - \lambda + c = 0 \\ -\infty & \text{otherwise} \end{cases}$$

Ex: find g for QCQP:

$\min \frac{1}{2} x^T A x + b^T x$ s.t. $\|x\|^2 \leq 1$

Dual function $g(\lambda, \nu)$ gives lower bound on opt. value p^*

$\underline{\lambda \succeq 0}$

$$g(\lambda,\nu) = \underline{\inf L(x,\lambda,\nu)} \quad : \text{ over all } x$$

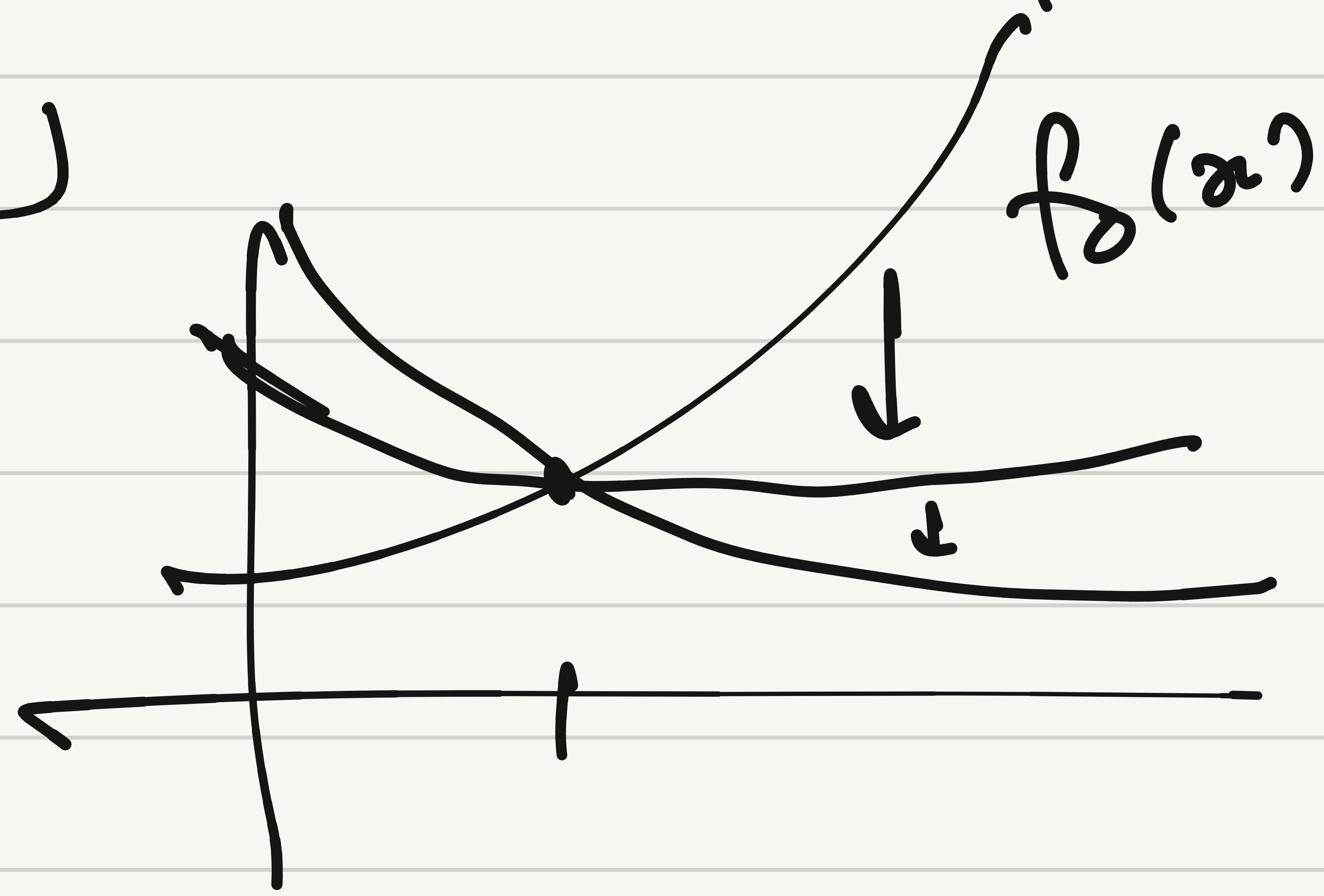
$$\le \underline{\inf L(x,\lambda,\nu)} \quad : \text{ over all feasible } x$$

$$\le \inf_{x \text{ feasible}} f_0(x) = p^*$$

if x feasible

$$L(x,\lambda,\nu) = f_0(x) + \underbrace{\sum \lambda_i \cancelto{\le 0}{f_i(x)}} + \underbrace{\sum \nu_i \cancelto{=0}{h_i(x)}}$$

$$\le \underline{\underline{f_0(x)}}$$



for any $\lambda \succeq 0$, ν,
for any feasible x, $\Big\}$ $g(\lambda, \nu) \leq p^* \leq f_0(x)$

$f_0(x) - p^*$: suboptimality

$\underbrace{f_0(x) - g(\lambda, \nu)}_{\boxed{\text{duality gap}}} \geq$ suboptimality

$\boxed{\text{Lagrange dual problem}}$

$$\left.\begin{aligned} \max\ & g(\lambda, \nu) \\ \text{s.t.}\ & \lambda \succeq 0 \end{aligned}\right\}$$

(λ, ν) are $\boxed{\text{dual feasible}}$ if
$\lambda \succeq 0$, and $g(\lambda, \nu) > -\infty$

always convex problem!

Example : LP

$$\boxed{\begin{aligned} \min\ & c^T x \\ \text{s.t. } & Ax = b \\ & x \succeq 0 \end{aligned}} \longrightarrow \quad g(\lambda, \nu) = \begin{cases} -b^T \nu & \text{if } A^T \nu - \lambda + c = 0 \\ -\infty & \text{otherwise} \end{cases}$$

Dual problem: $\max\ g(\lambda, \nu)$ s.t. $\lambda \succeq 0$ $\iff$ $\max\ -b^T \nu$ s.t. $\lambda \succeq 0$, $A^T \nu - \lambda + c = 0$

$$\iff \boxed{\begin{aligned} \max\ & -b^T \nu \\ \text{s.t. } & A^T \nu + c \succeq 0 \end{aligned}}$$

Dual of LP is LP

QP $\leftrightarrow$ QP

primal: $x \in \mathbb{R}^n$, $b \in \mathbb{R}^m$

Dual: $\nu \in \mathbb{R}^m$, $c \in \mathbb{R}^n$

Primal :

min $f_0(x) \longrightarrow p^*$
s.t. ...

Dual problem:

max $g(\lambda, \nu) \longrightarrow d^*$
s.t. ...

$d^* \leq p^*$: $\boxed{\text{weak duality}}$

$p^* - d^* \geq 0$: $\boxed{\text{optimal duality gap}}$

Ideally we want $p^* = d^*$ $\boxed{\text{strong duality}}$

If strong duality holds and p^*, d^* are attained

then solving primal $\iff$ finding (x^*, λ^*, ν^*) s.t. $f_0(x^*) = p^* = d^* = g(\lambda^*, \nu^*)$

$\iff$ " " s.t. $f_0(x^*) = g(\lambda^*, \nu^*)$

If I find $(\underbrace{x^*}_{\text{feasible}}, \underbrace{\lambda^*, \nu^*}_{\text{dual feasible}})$ s.t. $f_0(x^*) = g(\lambda^*, \nu^*)$

then x^* is optimal,

(λ^*, ν^*) is $\boxed{\text{certificate}}$ proving optimality of x^*