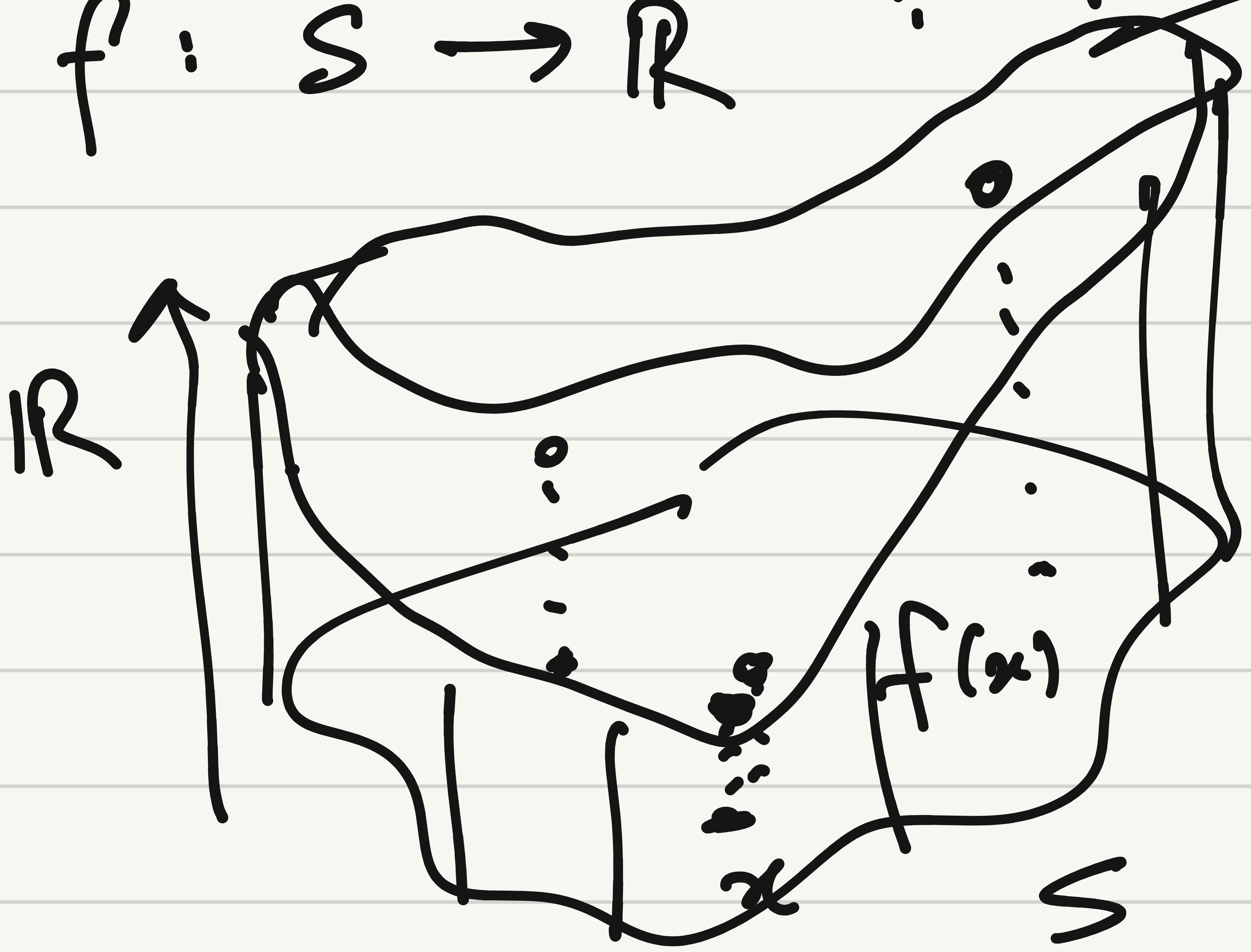
Optimization

$f : S \to \mathbb{R}$

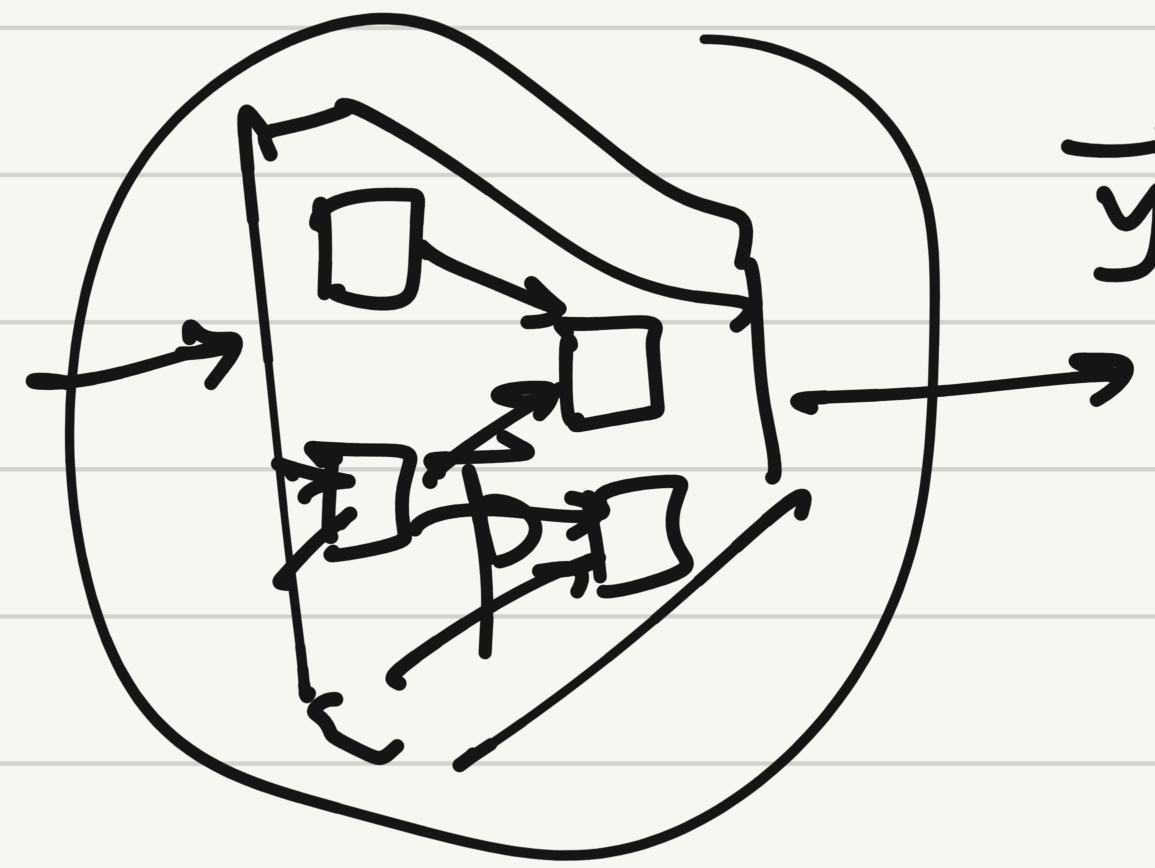
objective fn



find $x^* \in S$ s.t. $f(x^*) \le f(y)\ \forall y \in S$

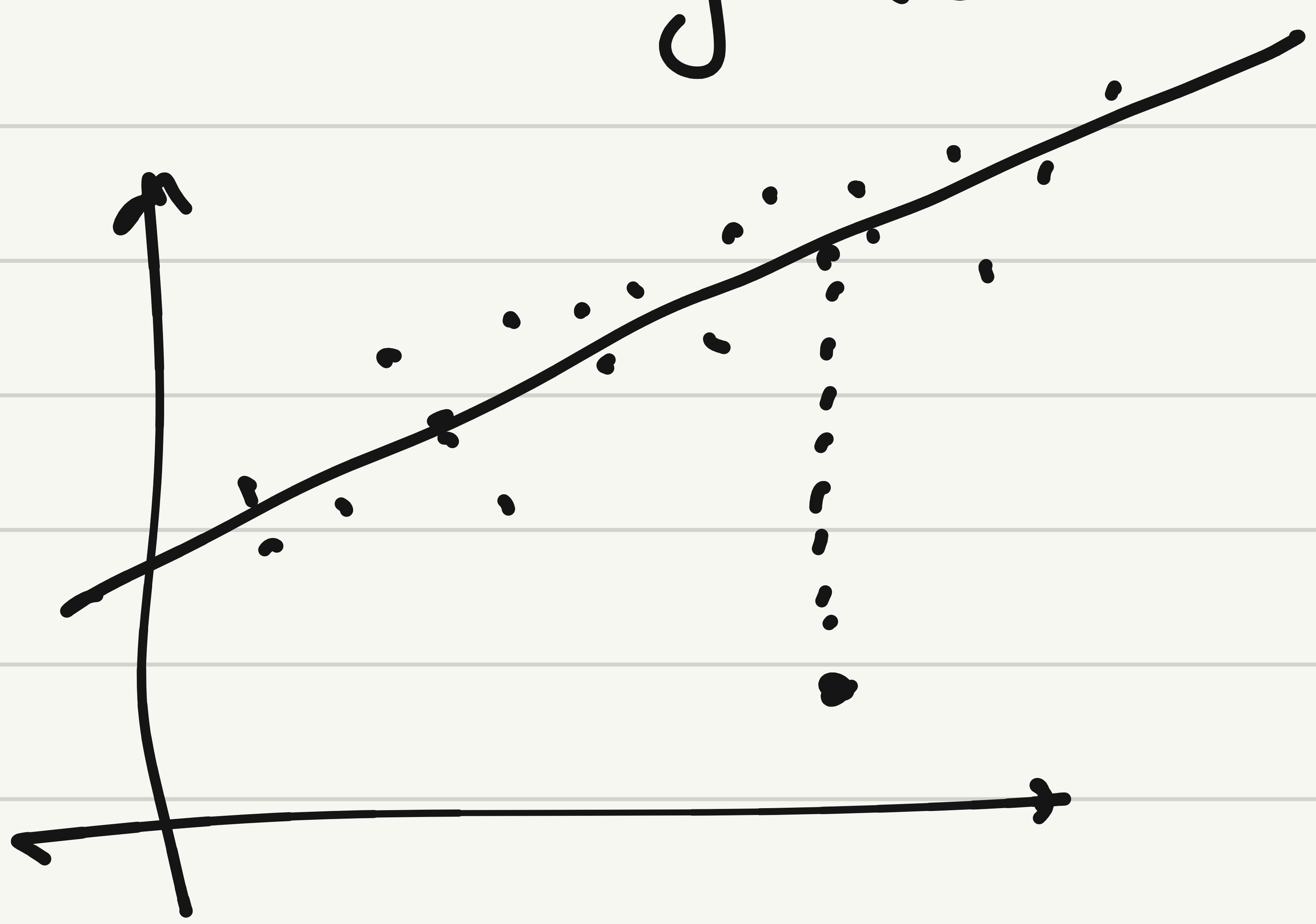
least-squares: $\min \|A\vec{x} - \vec{b}\|_2$

$$\underbrace{\|f(\vec{x}) - \vec{b}\|_p}$$



$$\min \underbrace{\underline{L}(\vec{p})}$$

Robust regression



$y = mx + c$

$$\min \sum \underbrace{(y_i - (mx_i + c))^2} = \|\vec{r}\|_2^2$$

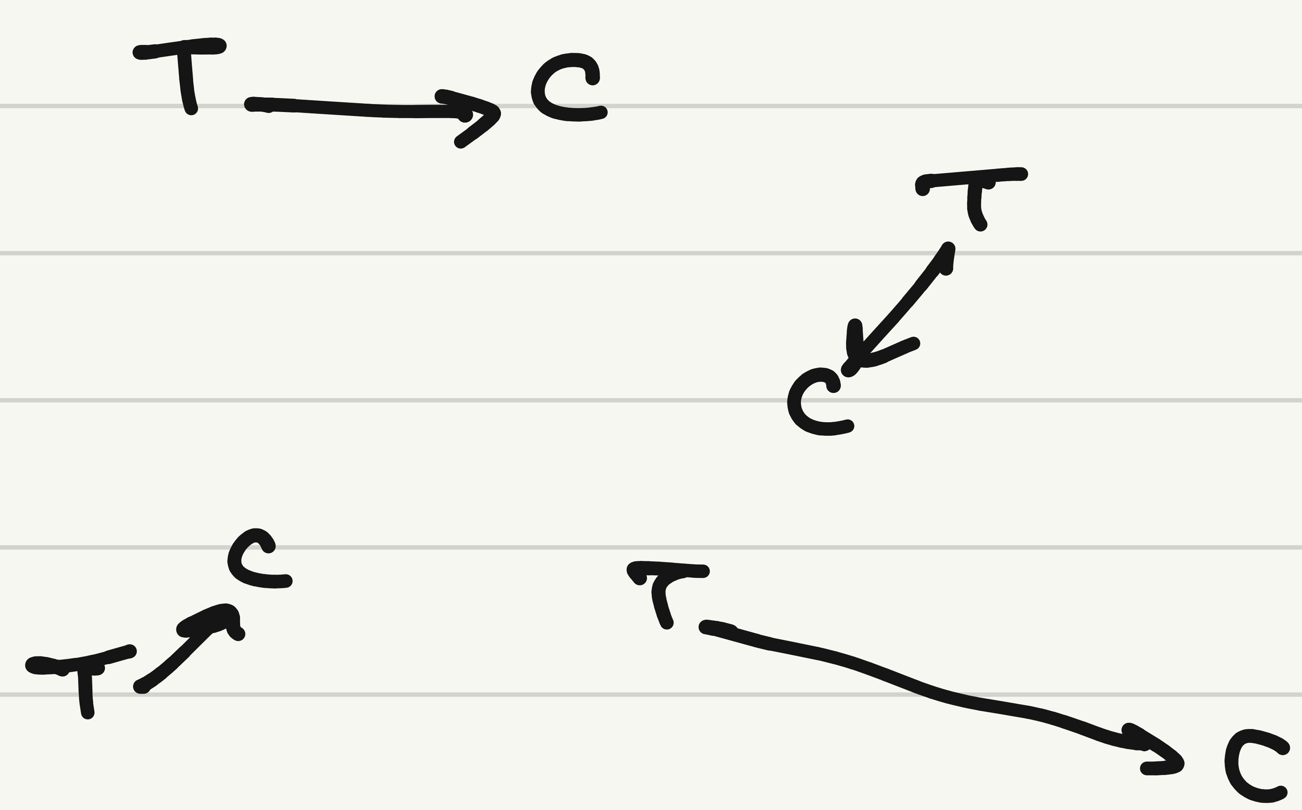
$$\sum |y_i - (mx_i + c)| = \|\vec{r}\|_1$$

$$\begin{bmatrix} y_i - mx_i + c \end{bmatrix}$$

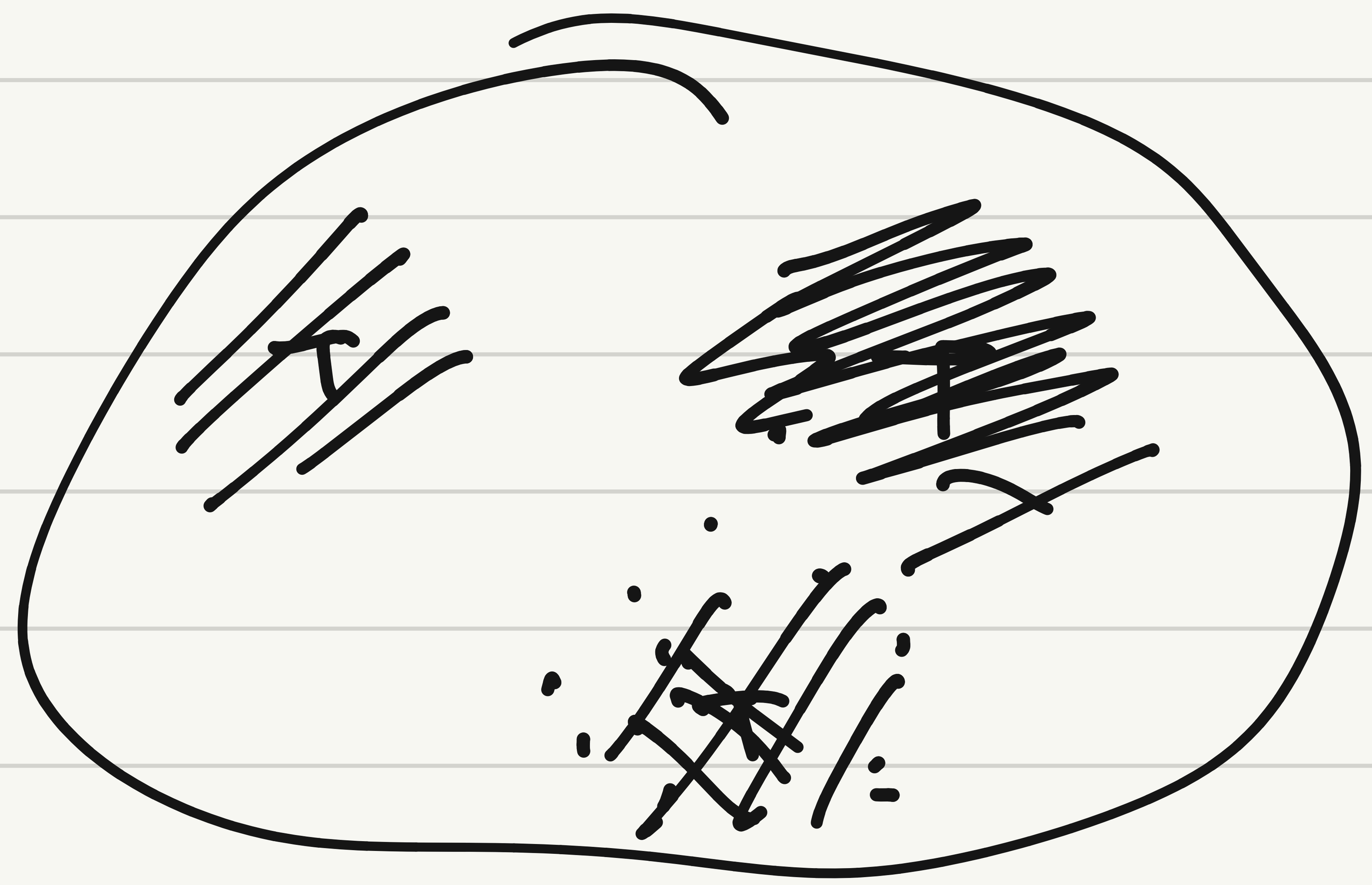
min $f(x)$ over $x \in S$

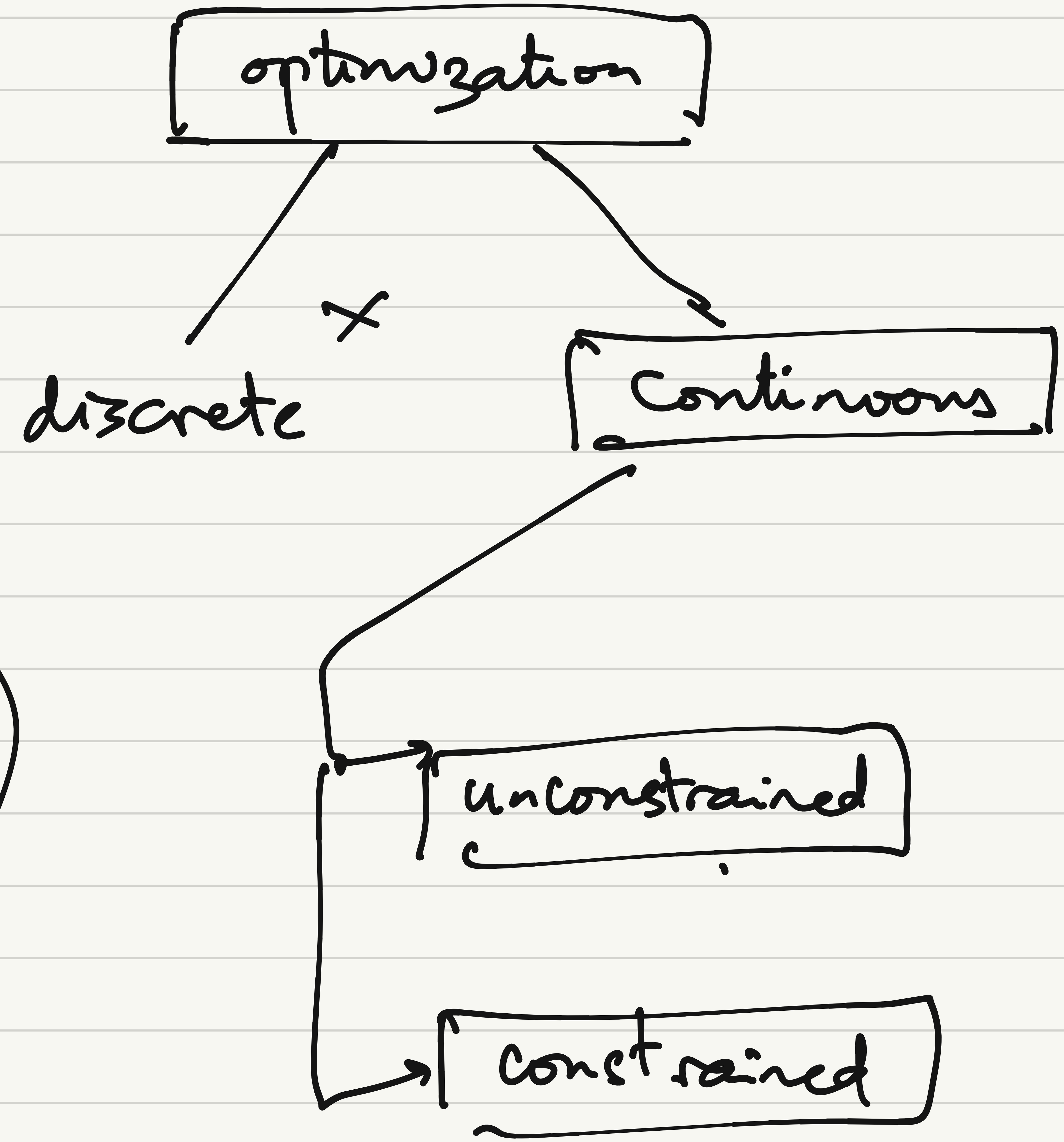
Discrete : S is finite

Continuous: $S \subseteq \mathbb{R}^n$



assignment problem





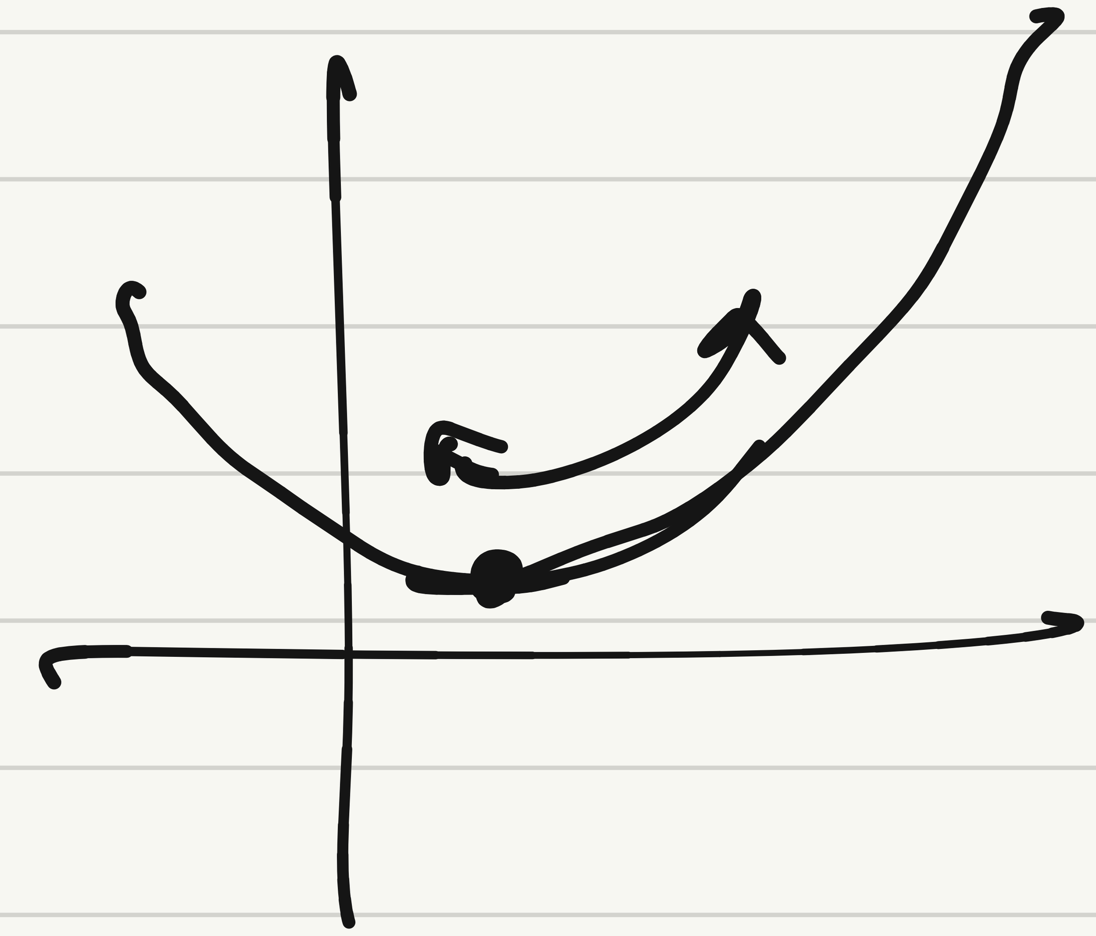
Unconstrained : $S = \mathbb{R}^n$

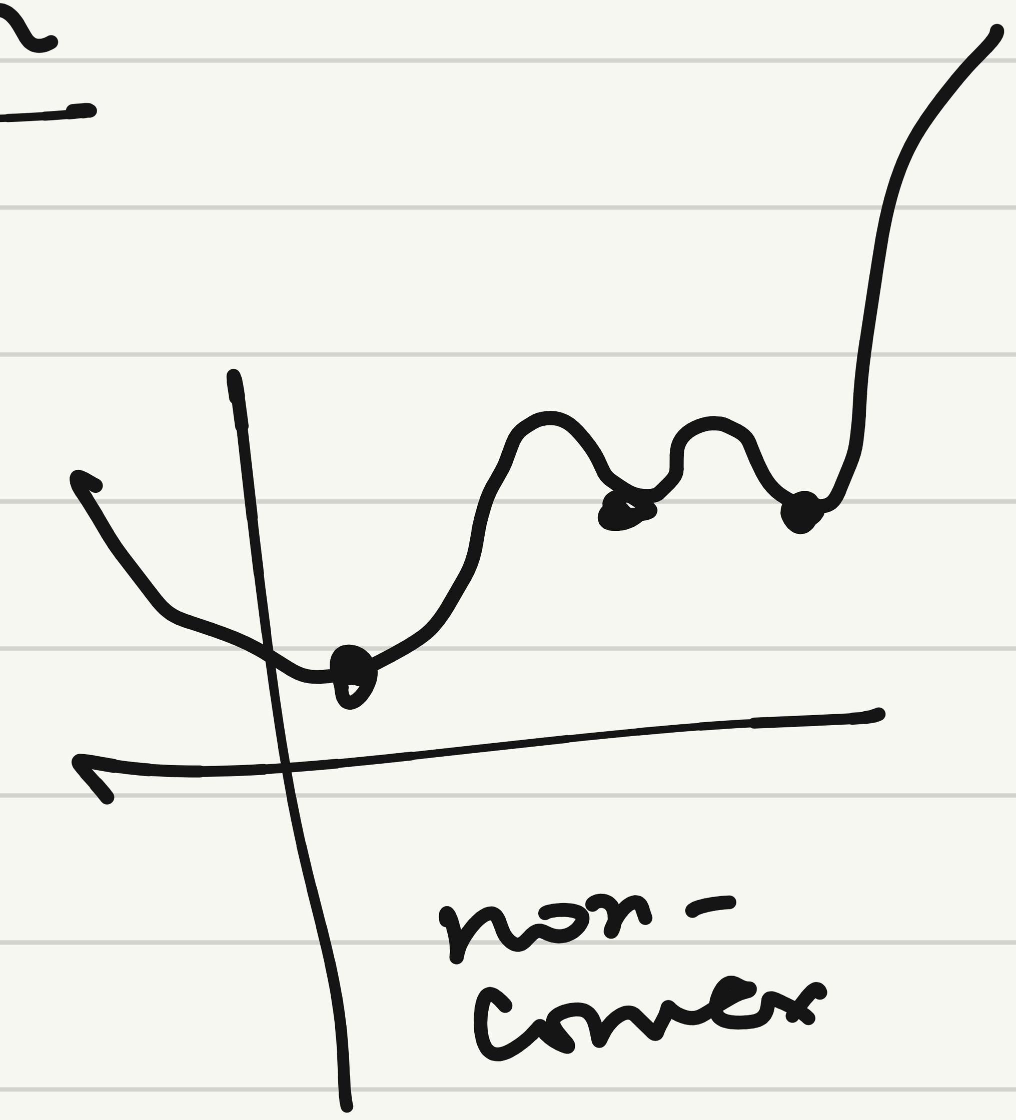
Constrained $S \subset \mathbb{R}^n$: eg. $x_i \geq 0$, $\sum x_i \leq$ budget

Continuous opt.

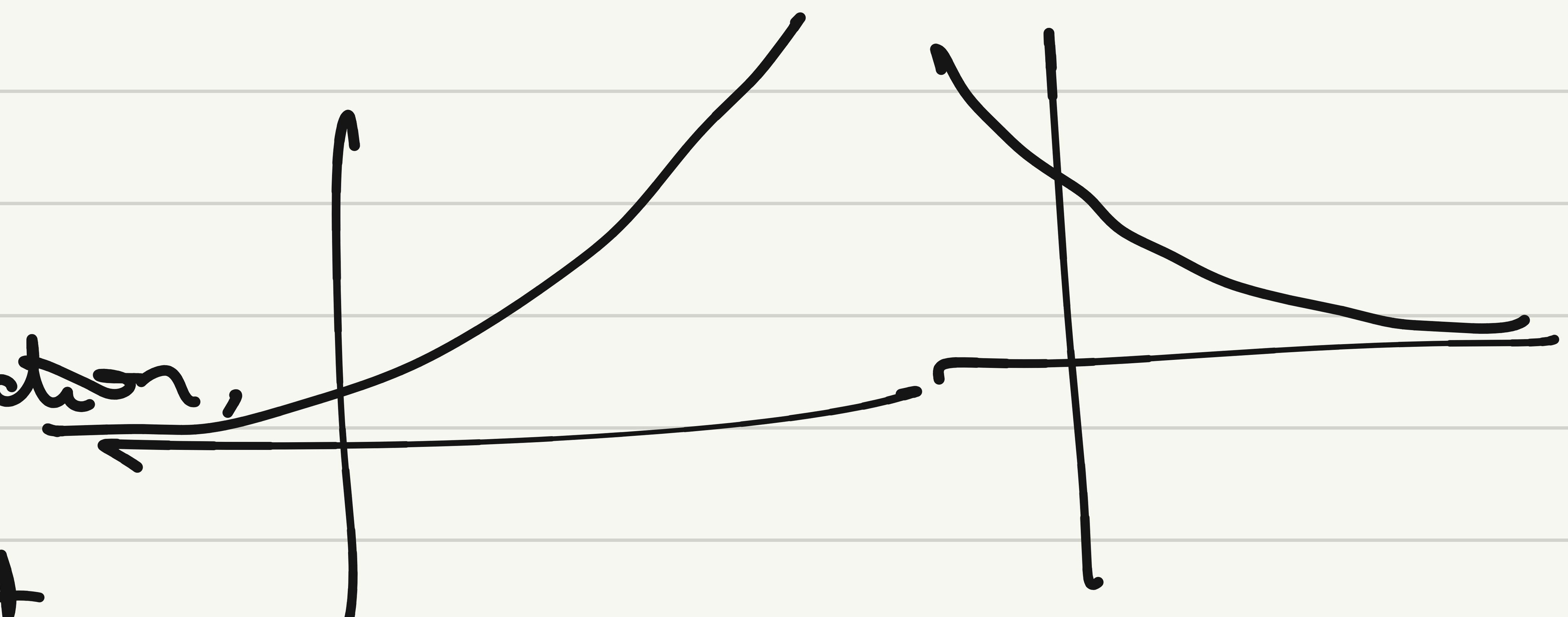
	convex	non convex
uncons.	✓	✓
cons.	.	

Convex function





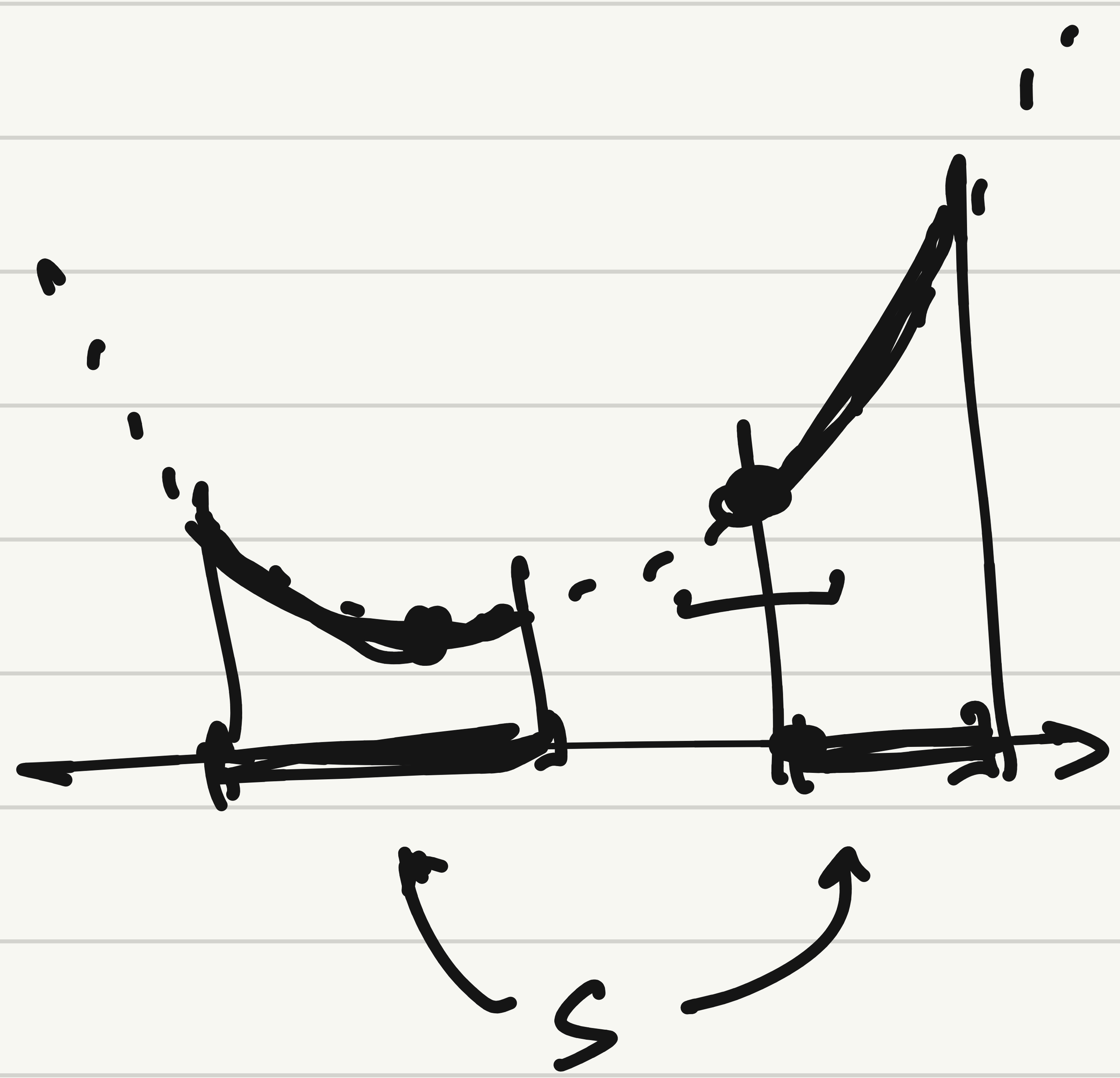
$x^2, \; e^x, \; e^{-x}$



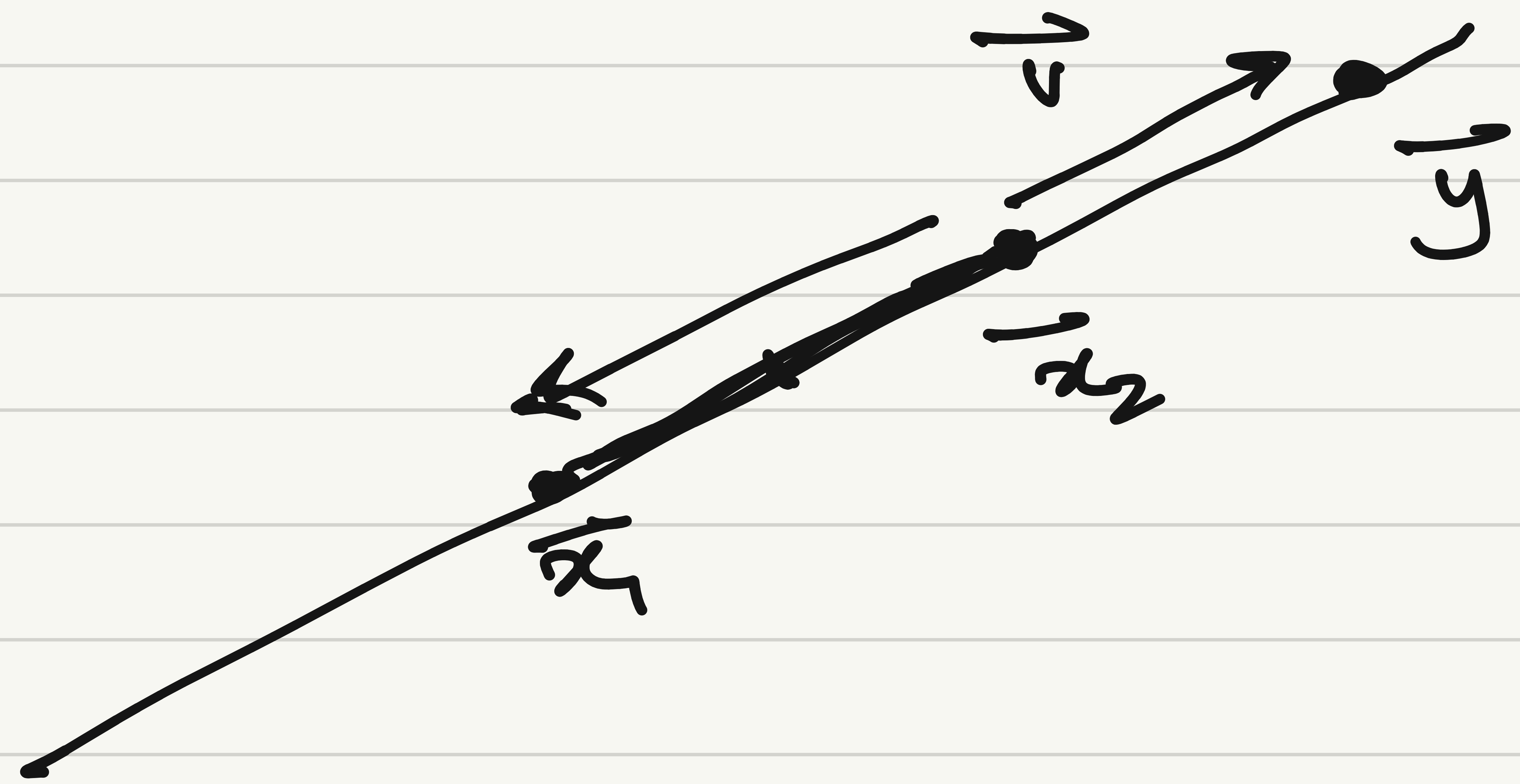
Convex opt. :

objective f is convex function,

domain S is convex set (or $\boxed{\text{feasible set}}$)



Convex sets



$\vec{x}_1 , \vec{x}_2 \in \mathbb{R}^n$

$\vec{y} = \vec{x}_2 + \vec{v}$

$= \vec{x}_2 + \theta(\vec{x}_1 - \vec{x}_2)$: $\boxed{\text{affine combination}}$ of $\vec{x}_1 , \vec{x}_2$

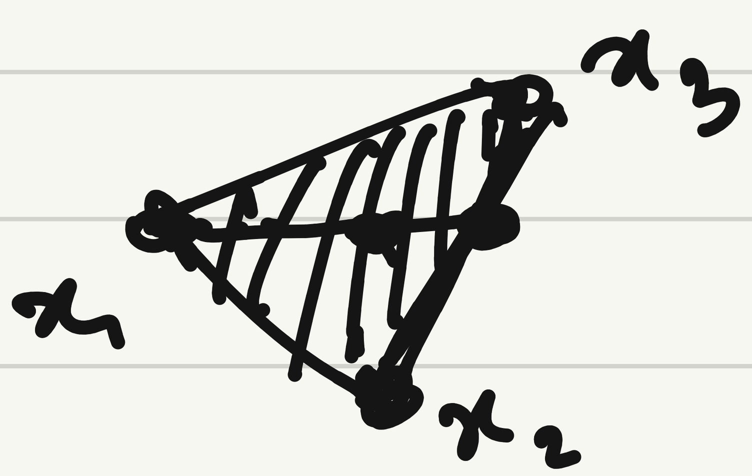
$= \underline{\theta}\,\vec{x}_1 + \underline{(1-\theta)}\,\vec{x}_2$ $\qquad \theta + (1-\theta) = 1$

If $\vec{y}$ is on line segment, $\vec{y} = \theta\vec{x}_1 + (1-\theta)\vec{x}_2$ with $0 \le \theta \le 1$

$\boxed{\text{Convex combination}}$

n points $\vec{x}_1 , \vec{x}_2 , \dots , \vec{x}_n$

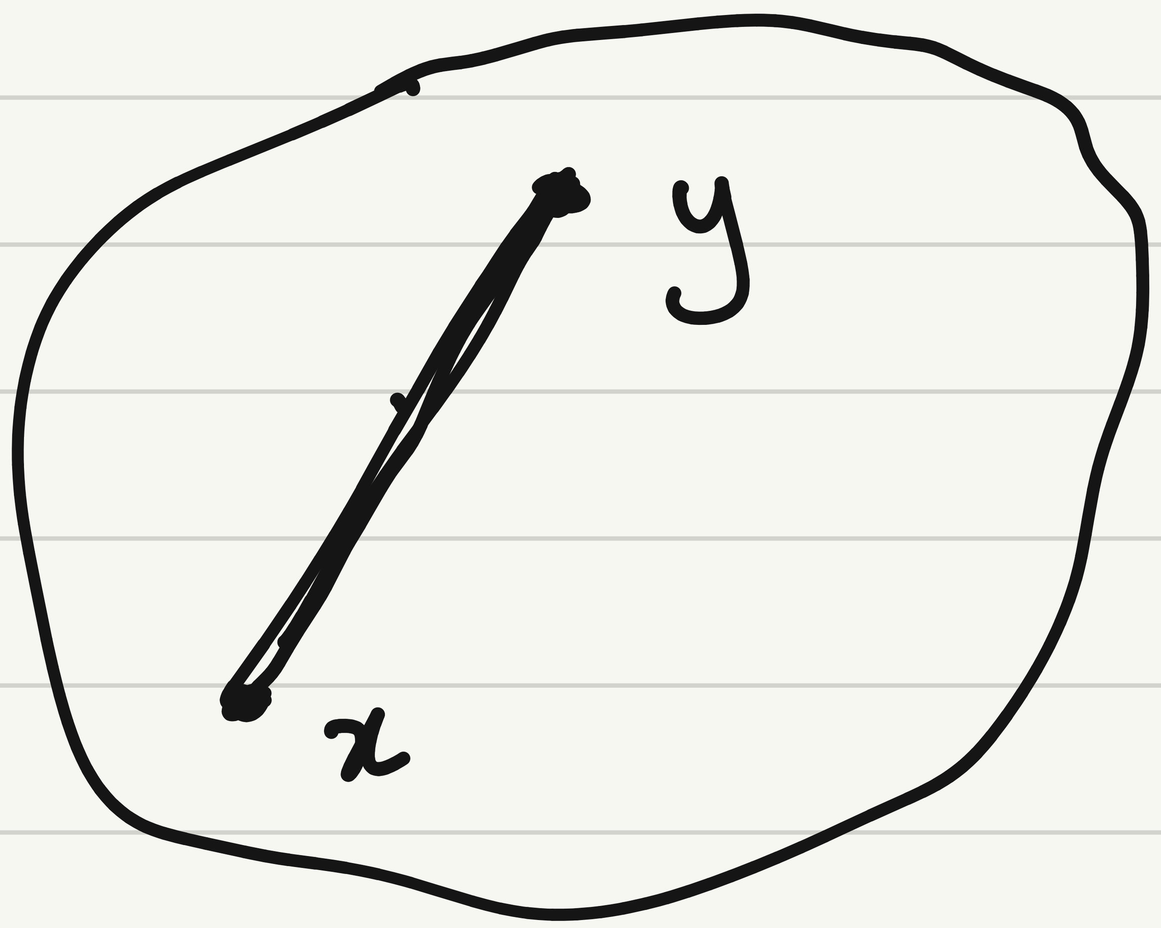
convex comb. $\underbrace{\theta_1\vec{x}_1 + \underbrace{\theta_2\vec{x}_2 + \dots + \theta_n\vec{x}_n}}$: $\underbrace{\theta_i \ge 0}$, $\underbrace{\theta_1 + \theta_2 + \dots + \theta_n = 1}$



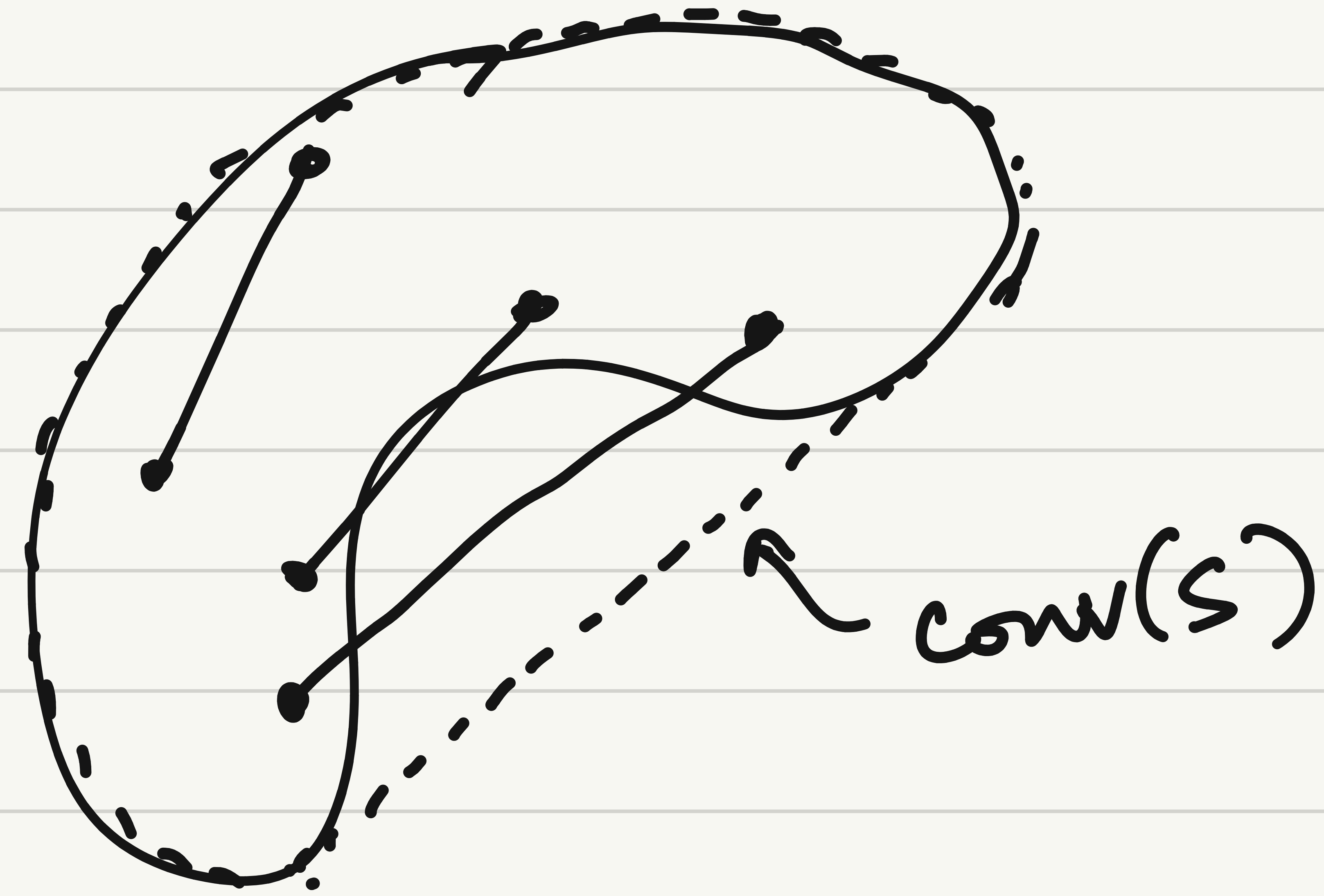
$\theta_1\vec{x}_1 + (1-\theta_1)\underbrace{\vec{x}_2'}$ $\qquad \Rightarrow \theta_i \le 1$

$\boxed{\text{Convex set}}$: S is convex set if $\forall\ x, y \in S$, $\theta x + (1-\theta) y \in S$

$\forall\ \theta \in [0,1]$



convex



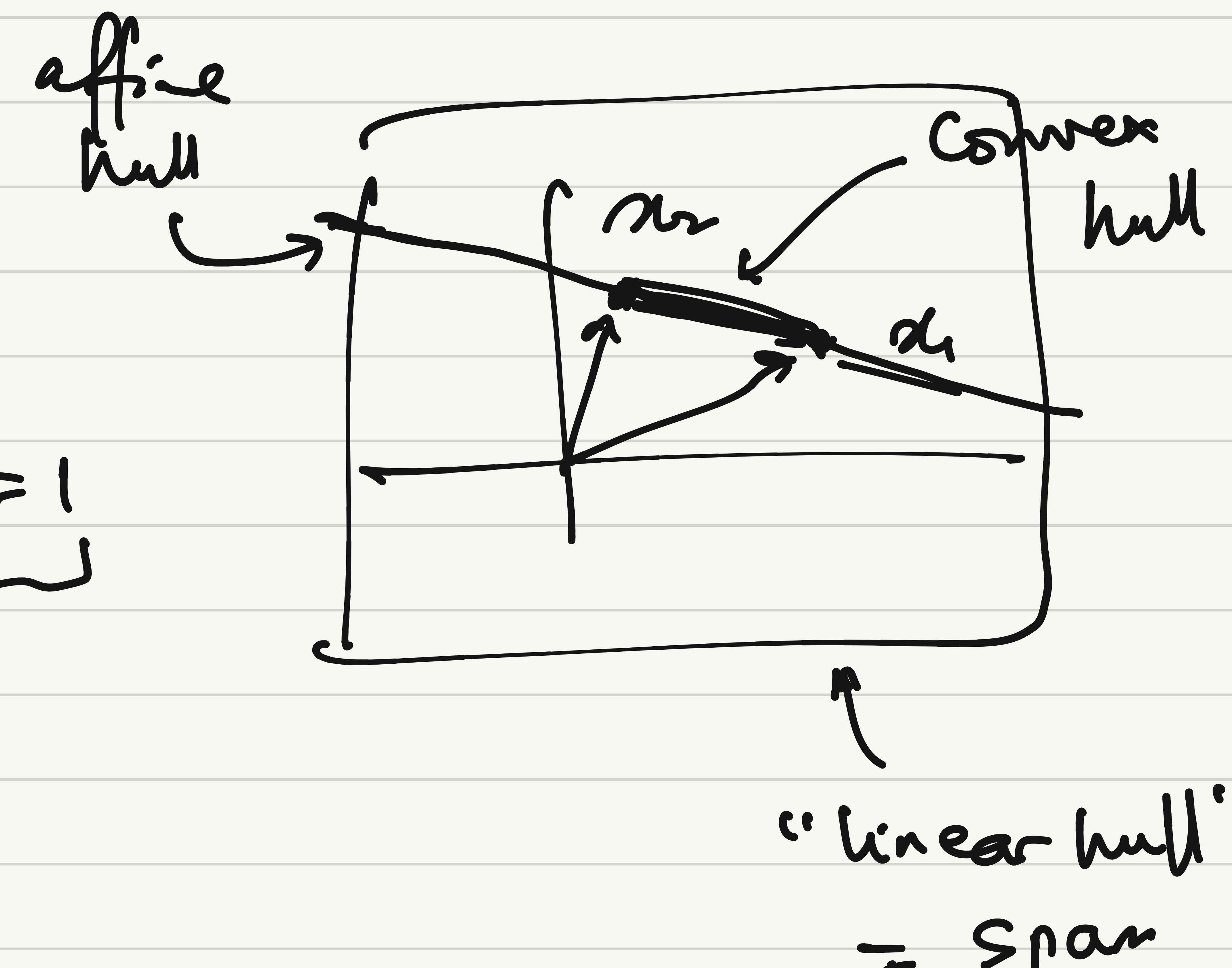
not convex

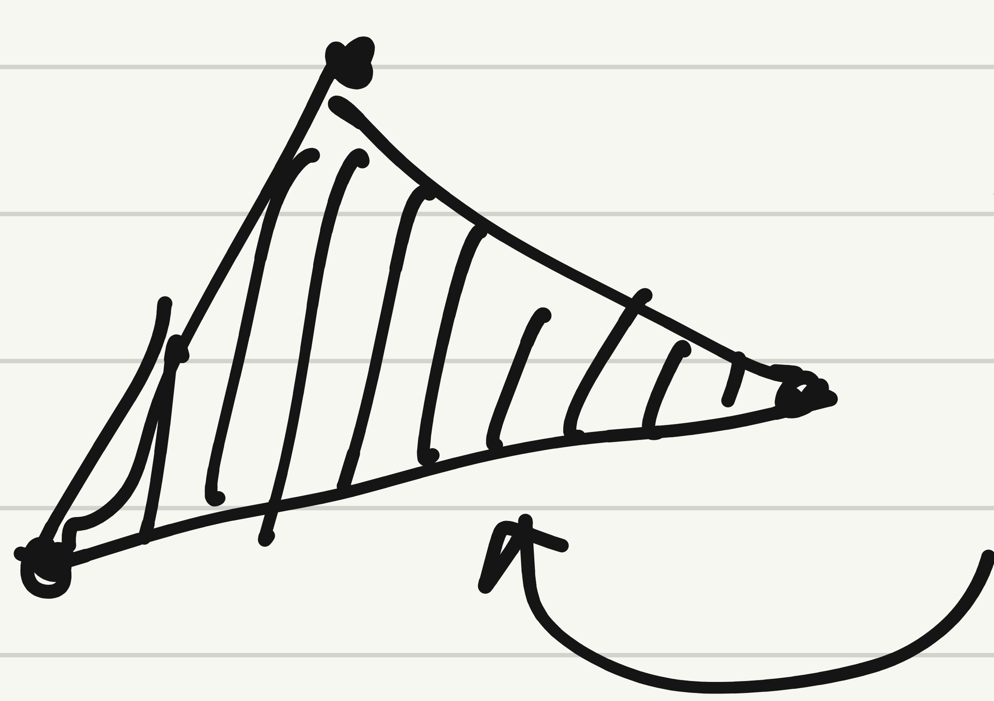
$\boxed{\text{Convex hull}}$ = set of all convex combinations of points in S

affine sets, affine hull

$\cancel{0 \le \theta \le 1}$

$\underbrace{\theta_1 + \cdots + \theta_n = 1}$





$S = \{x_1, x_2, x_3\}$

conv(S)

lin. comb. = $a_1x_1 + \cdots + a_nx_n$

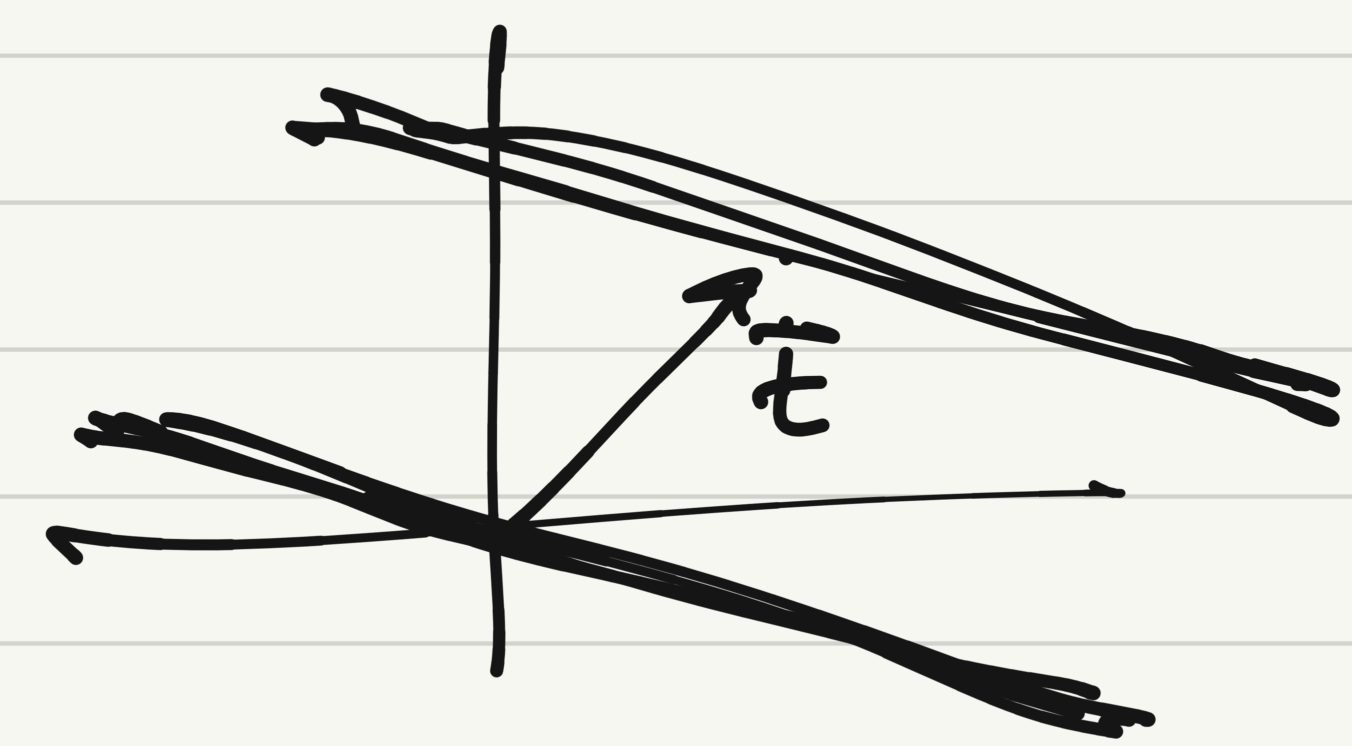
aff. comb. = " , $a_1 + \cdots + a_n = 1$

conv. comb = " , " , $a_1 \geq 0, \ldots, a_n \geq 0$

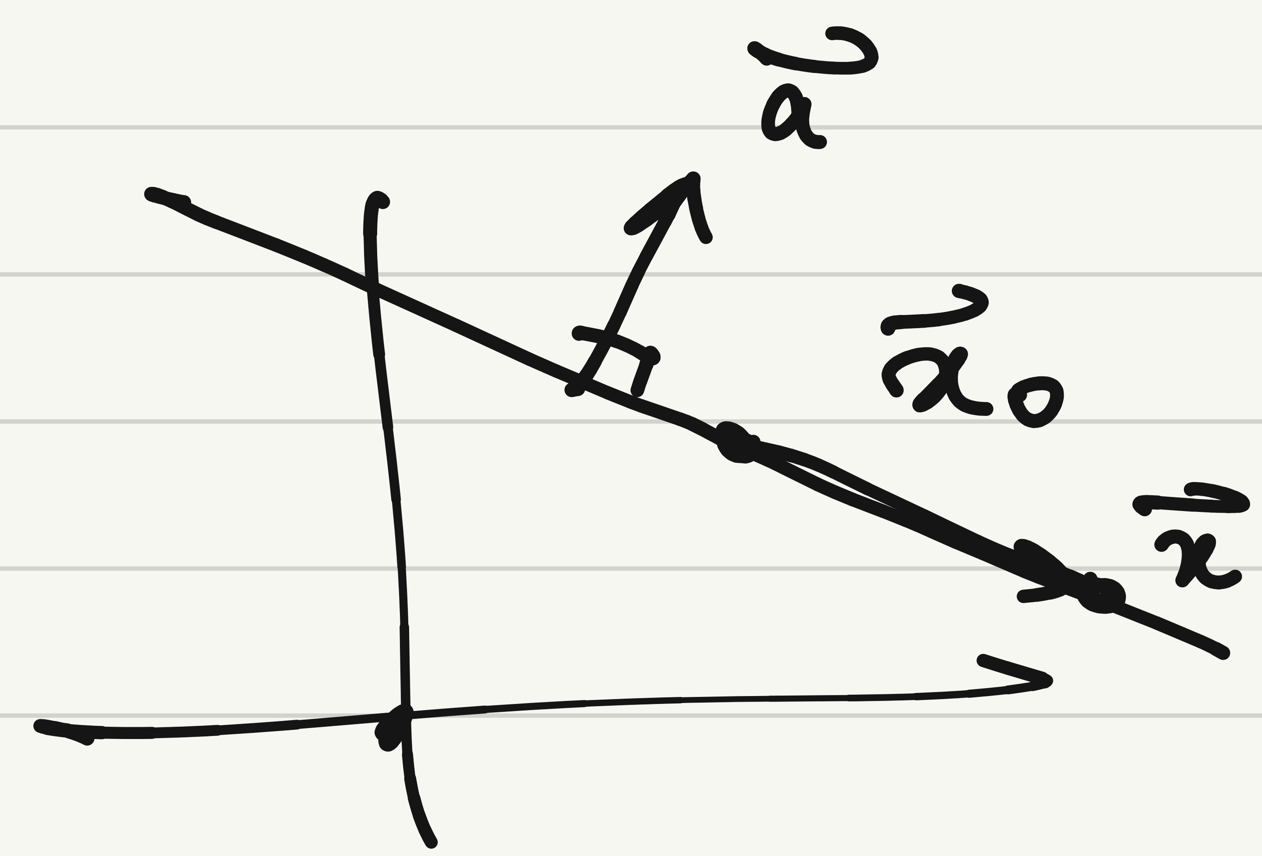
Examples

S = translation of some subspace V : $\vec{x} \in S \iff \vec{x} = \vec{v} + \vec{t}$ for some $\vec{v} \in V$

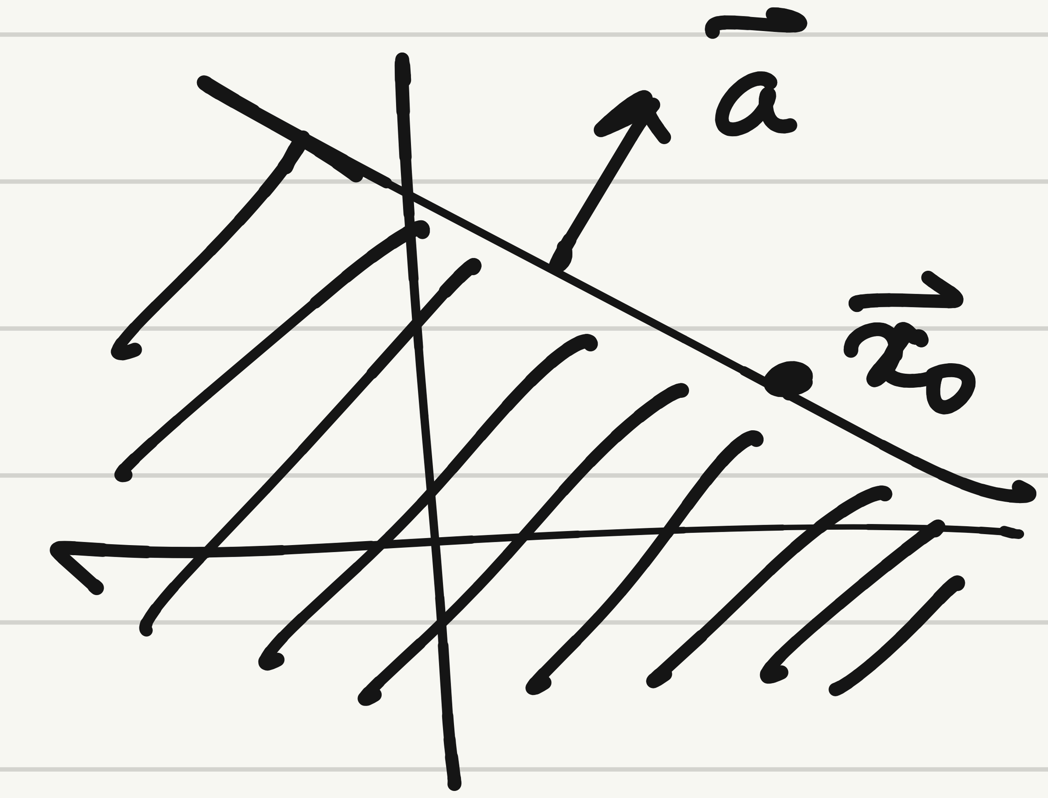
- If S is affine $\implies$ S is convex
- $\boxed{\text{Hyperplane}}$ $= \{\vec{x} : \vec{a}^T\vec{x} = b\}$
  $= \{\vec{x} : \vec{a}^T(\vec{x} - \vec{x}_0) = 0\}$



$\vec{a}$: $\boxed{\text{normal}}$ of hyperplane



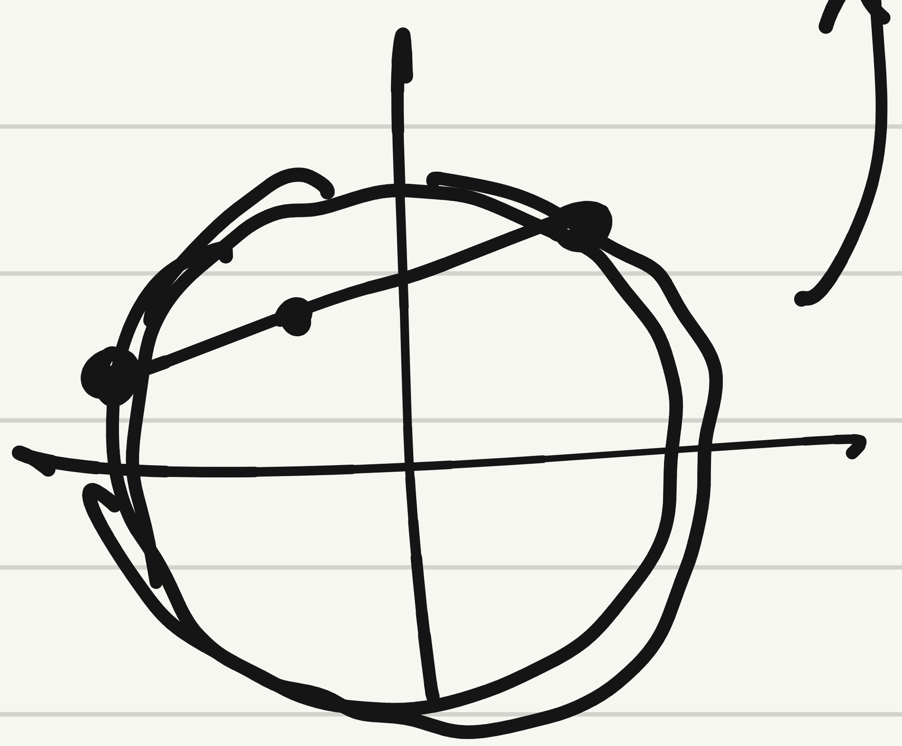
- $\boxed{\text{Half-space}}$: $\{\vec{x} : \vec{a}^T\vec{x} \leq b\}$ or $\{\vec{x} : \vec{a}^T(\vec{x}-\vec{x}_0) \leq 0\}$



- $\boxed{\text{Norm ball}}$: $\{\vec{x} : \|\vec{x}\| \leq 1\}$

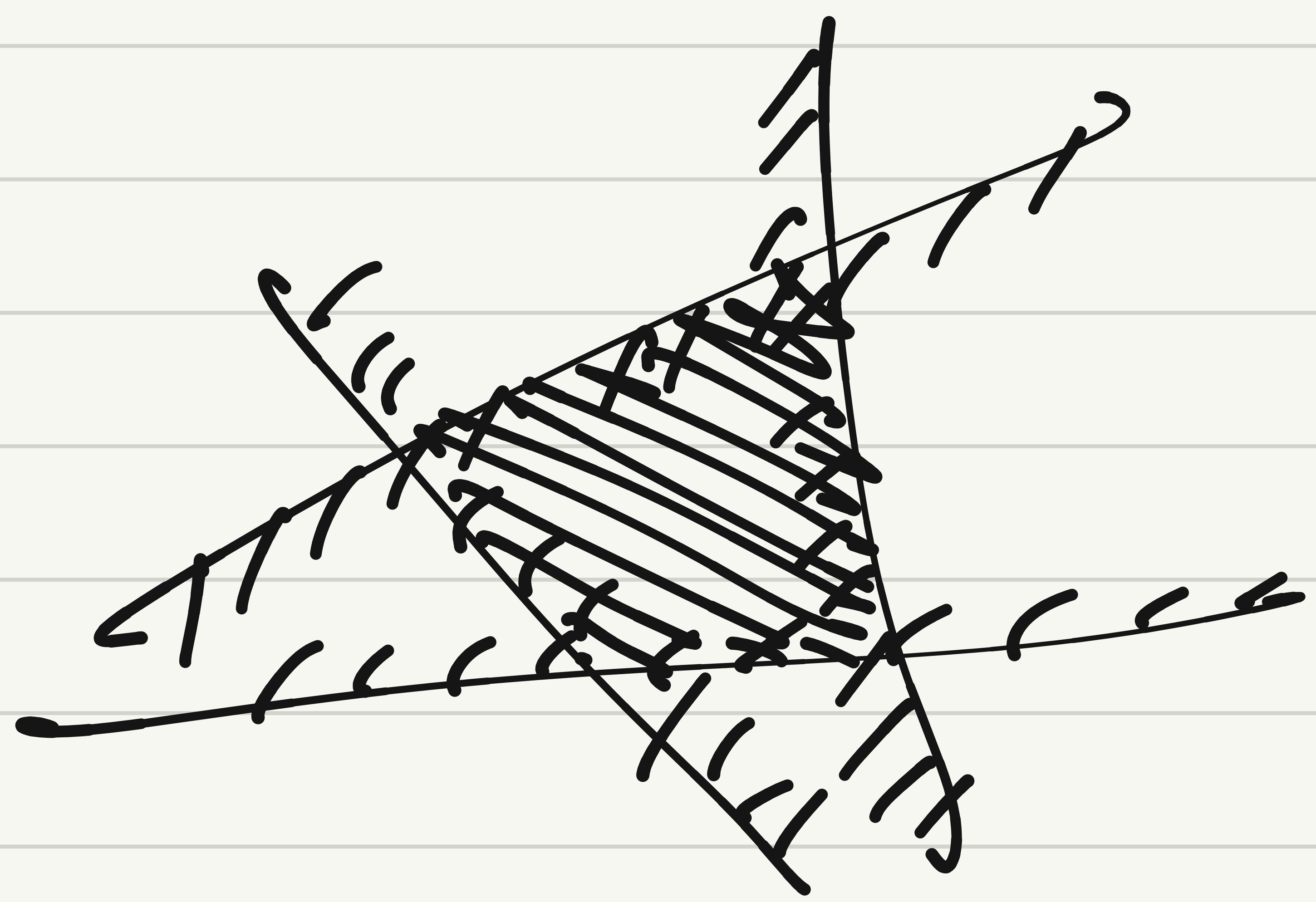
$\{\vec{x} : \|\vec{x}-\vec{x}_c\| \leq r\}$

$\{x : \|x\| = 1\}$



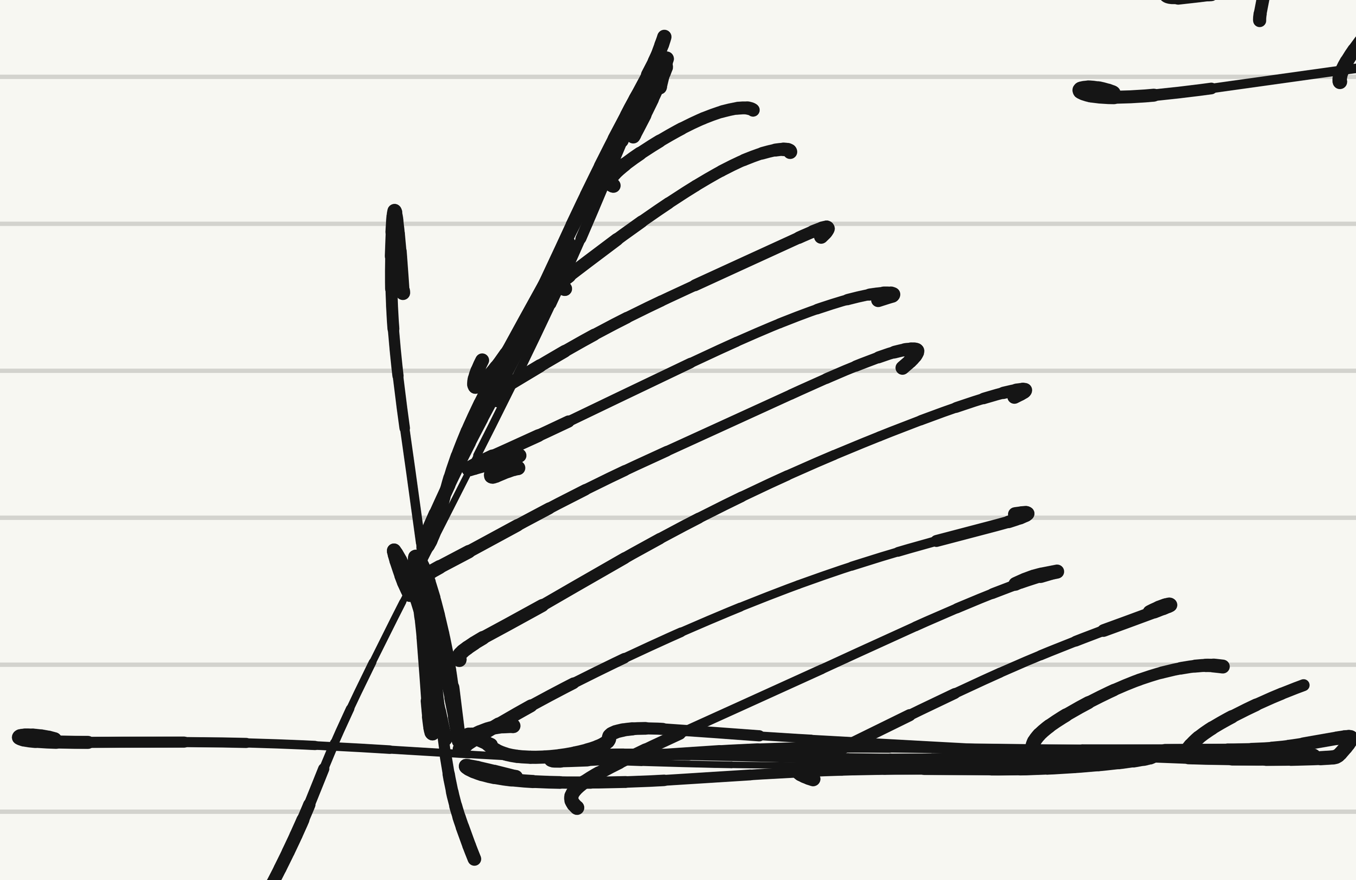
not convex

- $\boxed{\text{Polyhedron}}$ = intersection of finite # of half spaces and hyperplanes



$$P = \{x : a_j^T x \leq b_j : j = 1..m$$
$$c_j^T x = d_j : j = 1...p\}$$
$$= \{\vec{x} : A\vec{x} \preceq \vec{b},\ C\vec{x} = \vec{d}\}$$

$\underbrace{\vec{u} \preceq \vec{v}}$ iff $u_i \leq v_i$ for all i



Example : S_+^n : $\{ X \in \mathbb{R}^{n \times n} : X \text{ is SPSD} \}$

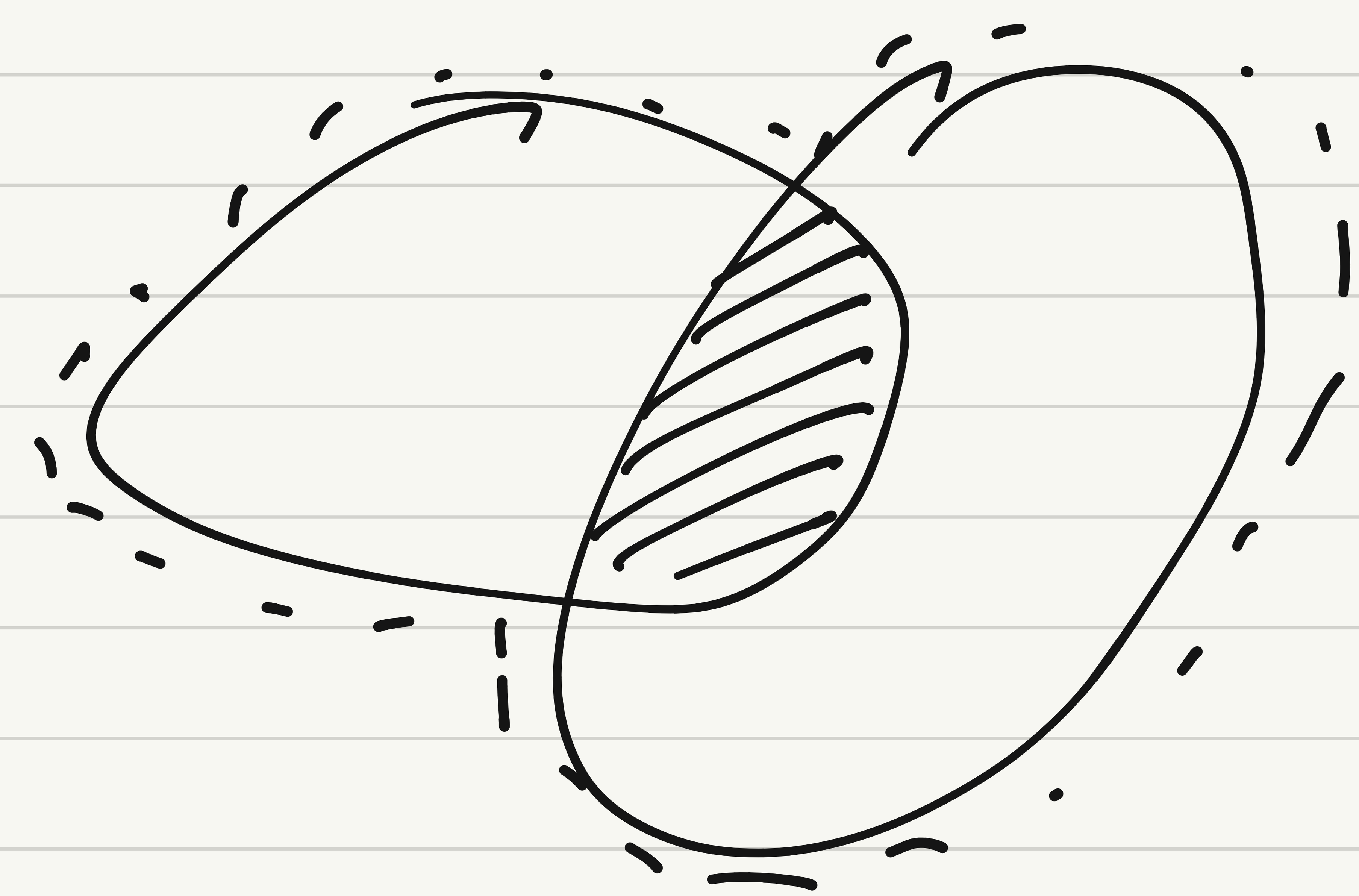
$\begin{bmatrix} a & b \\ b & c \end{bmatrix}$

Is this Convex ?

$X, Y \in S_+^n$, $Z = \underbrace{\theta X}_{\text{SPSD}} + \underbrace{(1-\theta) Y}_{\text{SPSD}}$, $0 \le \theta \le 1$

$\Rightarrow Z \in S_+^n$

Operations that preserve convexity



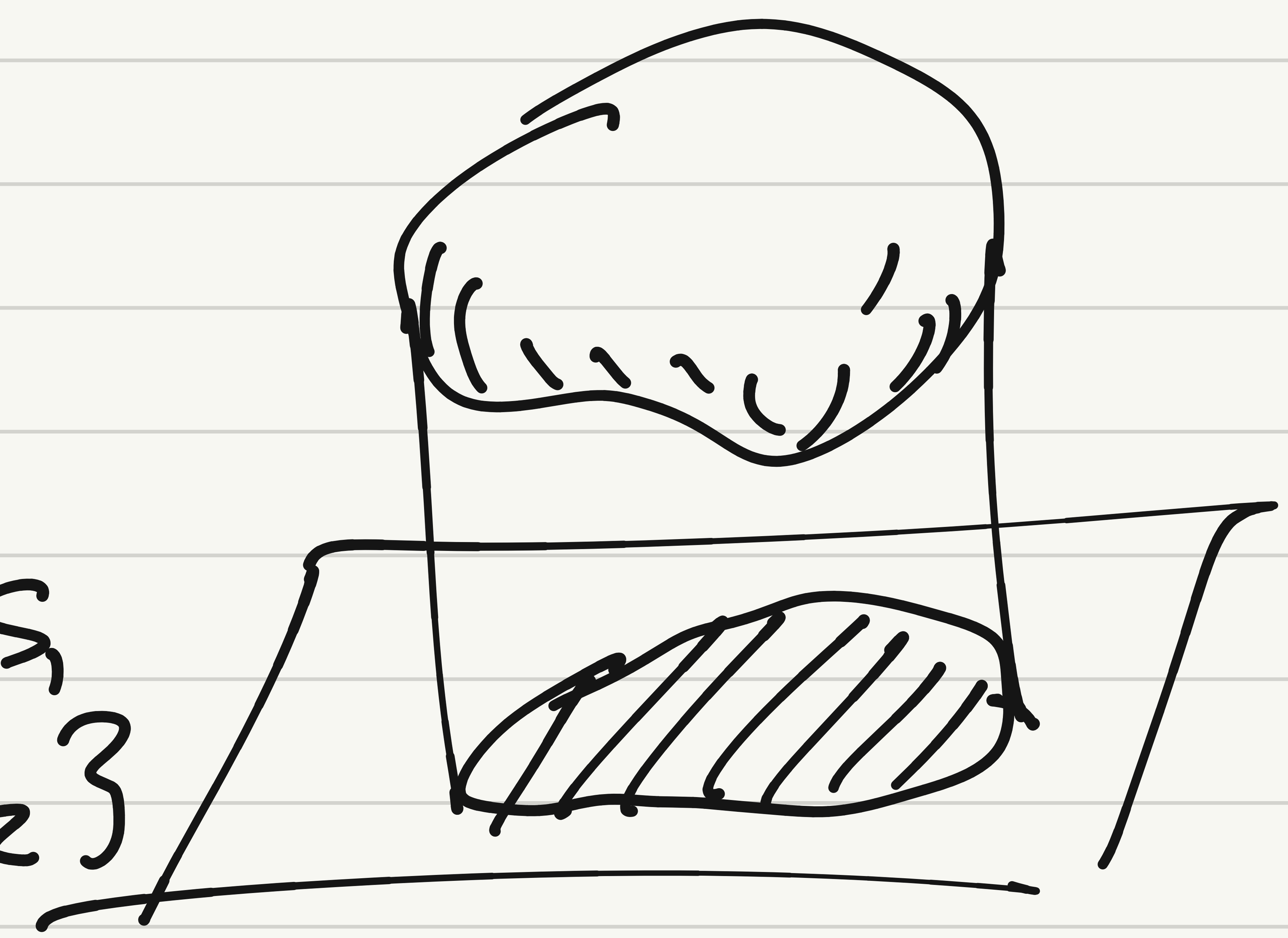
S_1, S_2 Convex

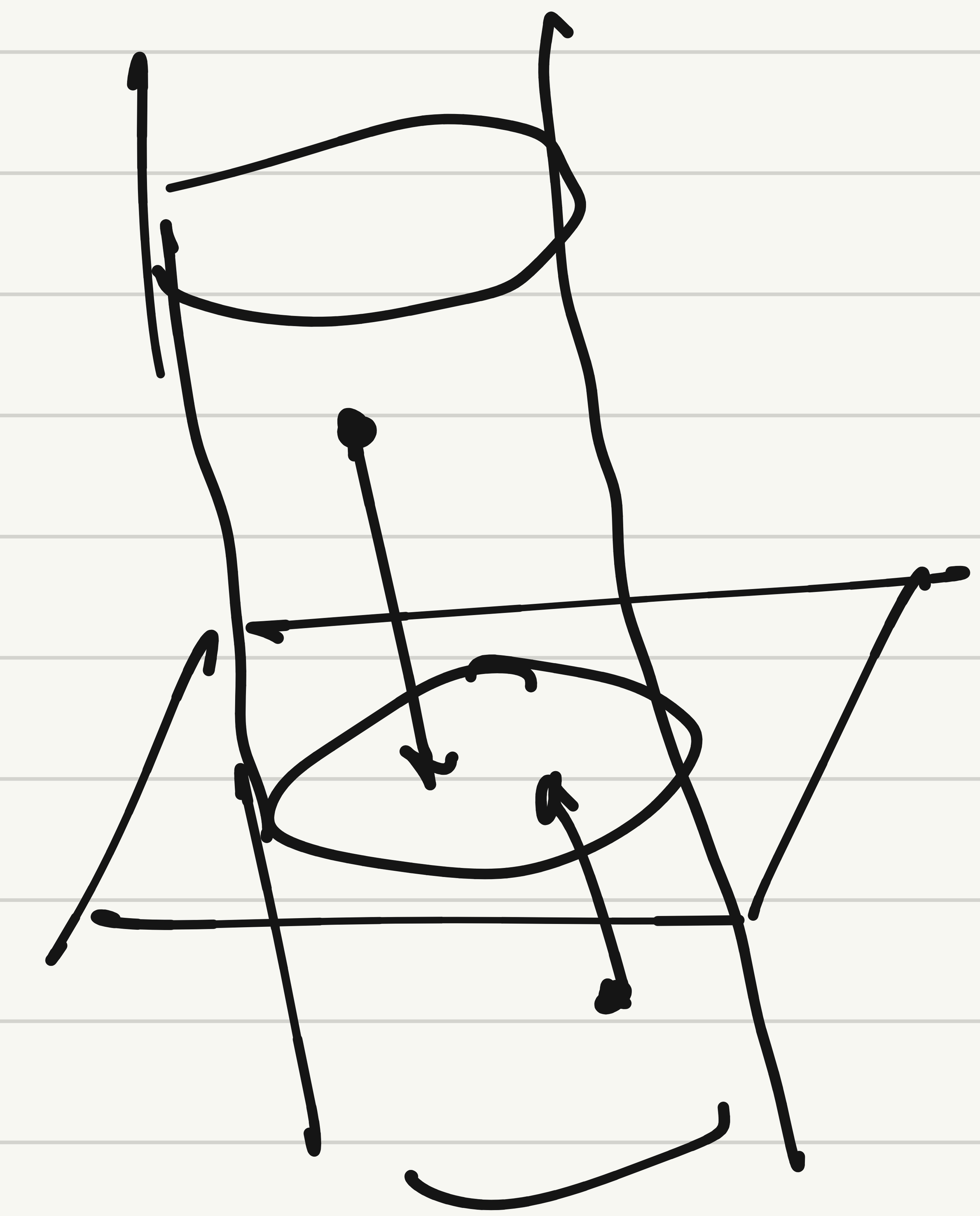
- **Intersection**: $S_1 \cap S_2$ Convex
- $S(x)$: $\bigcap_x S(x)$
- $S \subseteq \mathbb{R}^n$, $\boxed{\text{affine function}}$ $f : \mathbb{R}^n \to \mathbb{R}^m$, $f(\vec{x}) = A\vec{x} + \vec{b}$

$f(S) = \{ f(\vec{x}) : \vec{x} \in S \}$ Convex

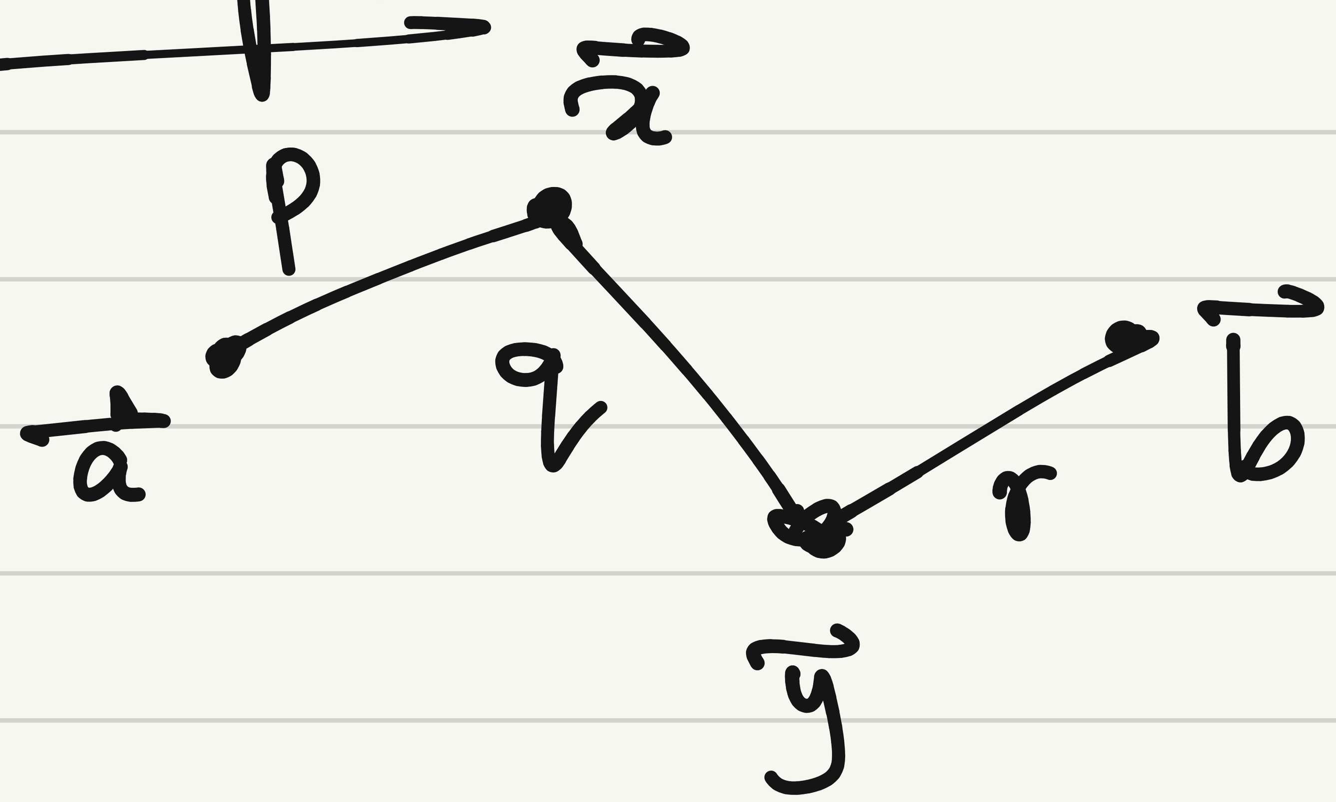
$f : \mathbb{R}^m \to \mathbb{R}^n$, affine, $f^{-1}(S) = \{\vec{y} : f(\vec{y}) \in S\}$ convex

- scaling $\alpha S : \{\alpha \vec{x} : \vec{x} \in S\}$
- translation $S + \vec{t}$
- projection
- sum $S_1 + S_2 = \{x + y : x \in S_1, y \in S_2\}$





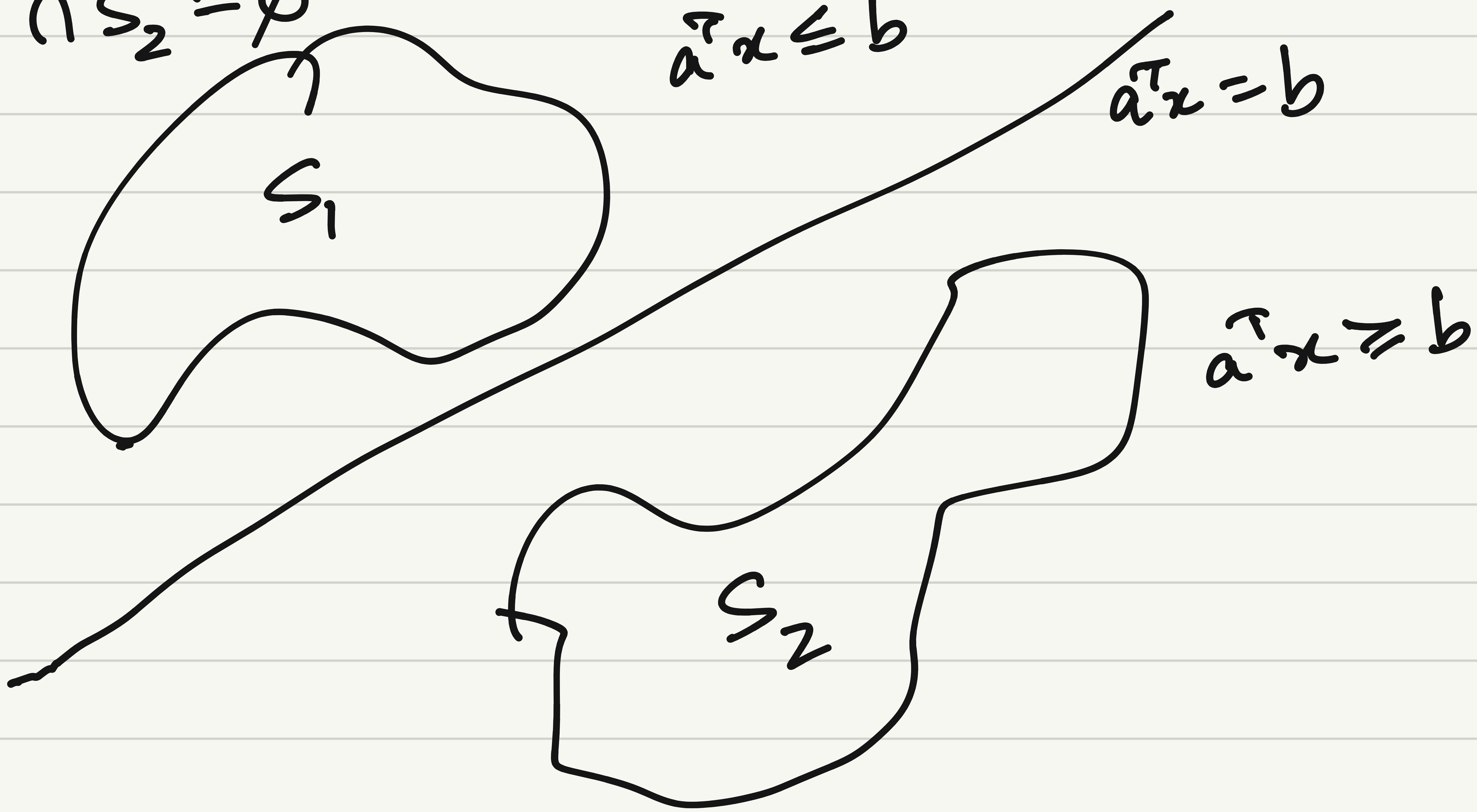
Example:



$\vec{a}, \vec{x}, \vec{y}, \vec{b} \in \mathbb{R}^n$

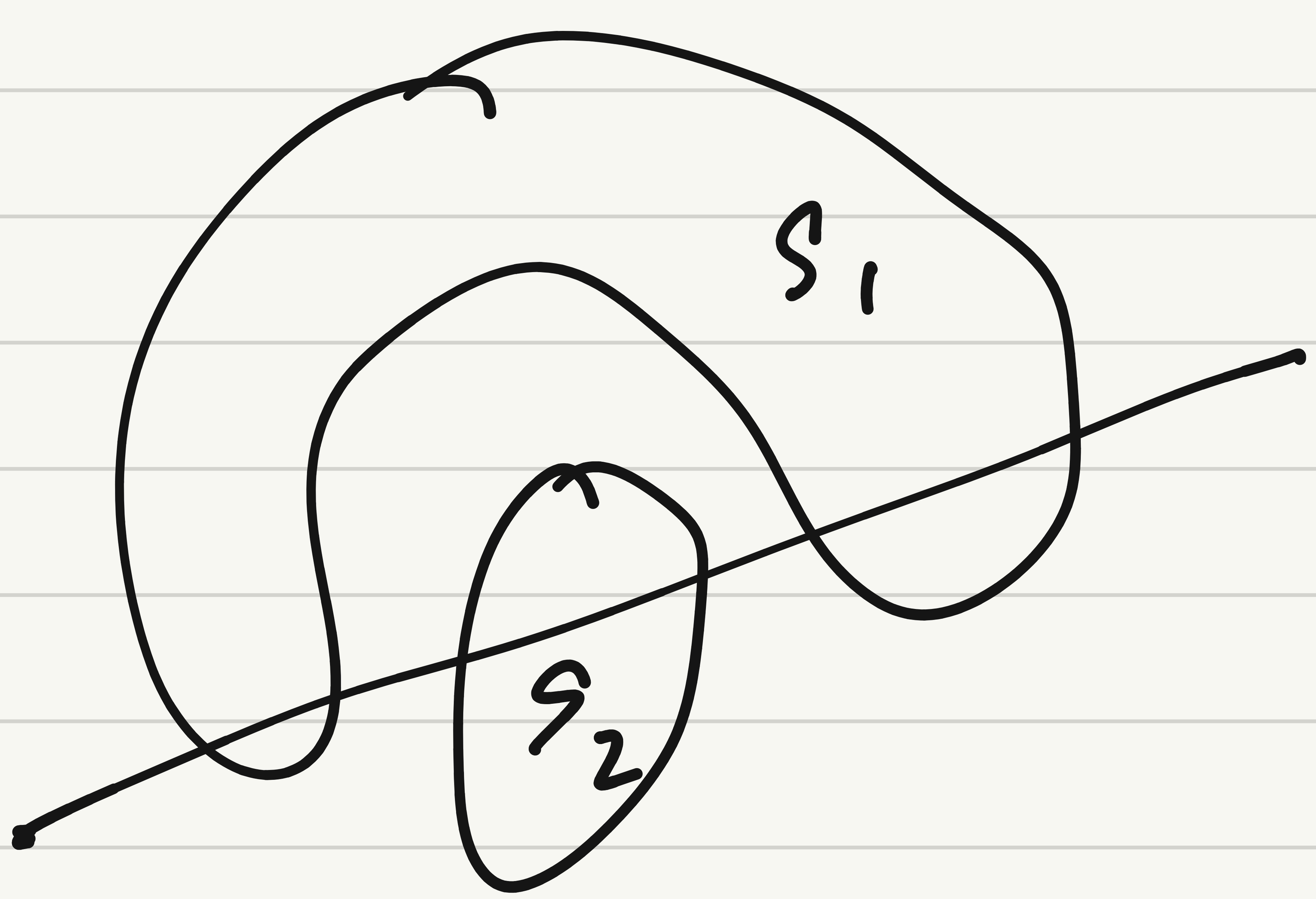
$$S = \left\{ \underbrace{\begin{bmatrix} \vec{x} \\ \vec{y} \end{bmatrix}}_{\in \mathbb{R}^{2n}} : \|\vec{x} - \vec{a}\| \le p, \; \|\vec{x} - \vec{y}\| \le q, \; \|\vec{y} - \vec{b}\| \le r \right\}$$

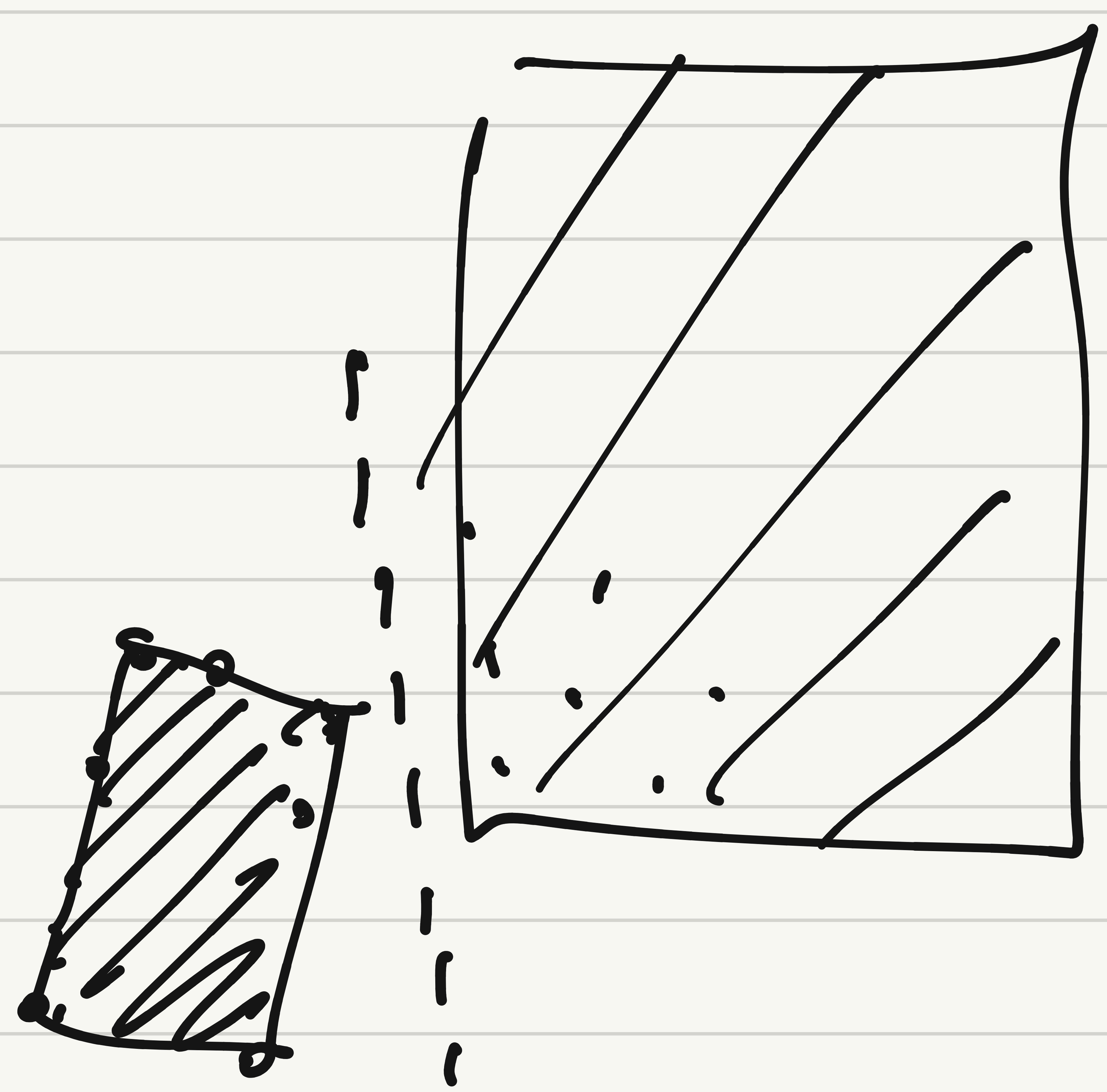
$S_1 \cap S_2 = \emptyset$

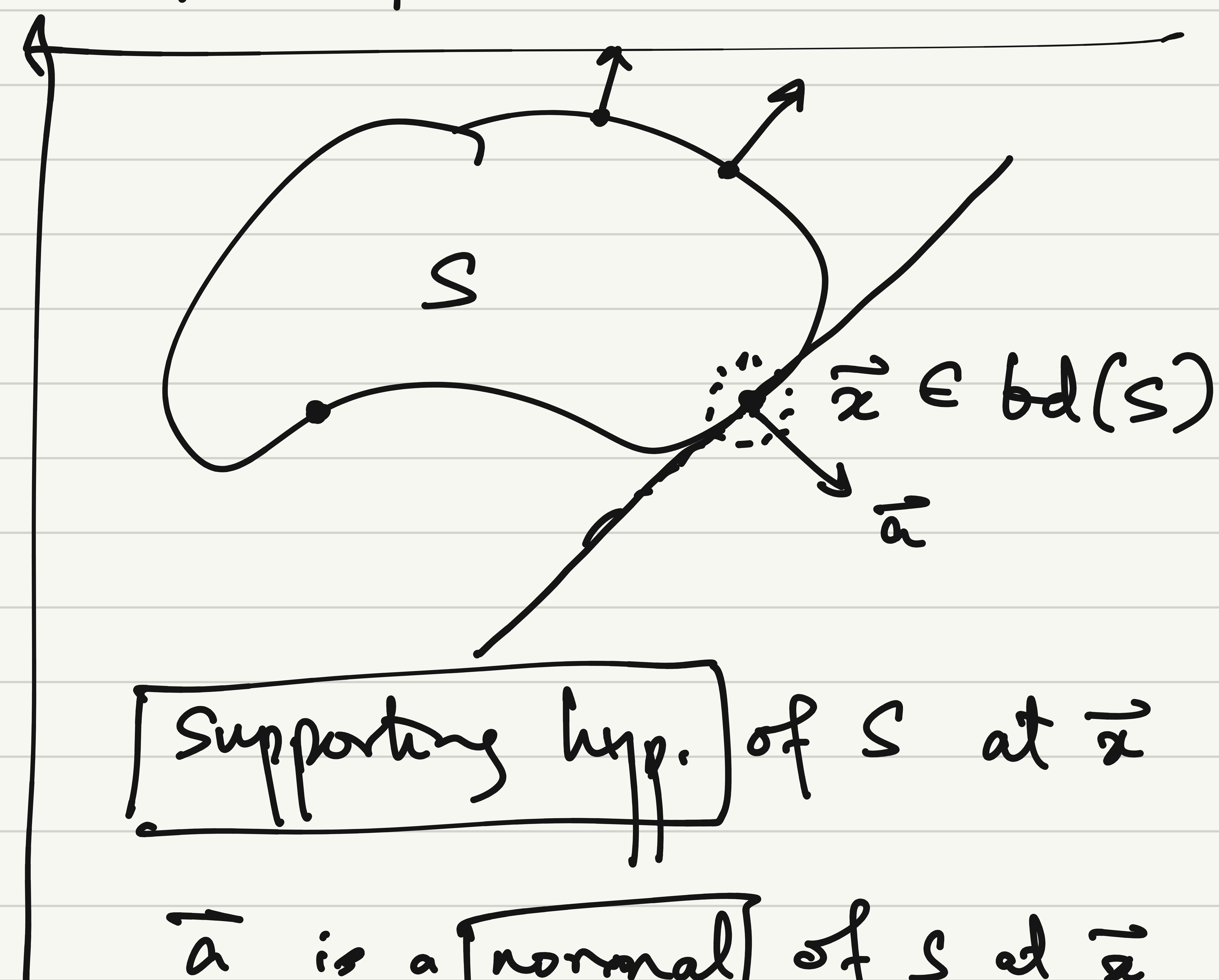


$\{x : a^T x = b\}$ is a $\boxed{\text{separating hyperplane}}$

If S_1, S_2 convex, sep. hyp. always exists







$\boxed{\text{Supporting hyp.}}$ of S at $\vec{x}$

$\vec{a}$ is a $\boxed{\text{normal}}$ of S at $\vec{x}$