COL726: Eigenvalues / Nonlinear equations

$$
\begin{tikzcd}
\underline{Q}^{(0)} \arrow[r] \arrow[d] & \underline{Q}^{(1)} \arrow[r] \arrow[d] & \underline{Q}^{(2)} \arrow[r] \arrow[d] & \cdots \arrow[r, "\lim"] & Q \\
A^{(0)} \arrow[r] & A^{(1)} \arrow[r] & A^{(2)} \arrow[r] & \cdots \arrow[r, "\lim"] & \Lambda
\end{tikzcd}
$$

Simultaneous iteration

QR algorithm

$$A^{(0)} = \underline{Q}^{(0)T} A \underline{Q}^{(0)}$$

Conditioning of eigenvalues

$$A = \underline{V} \Lambda \underline{V}^{-1}$$

μ is eigenvalue of perturbed matrix $A + \delta A$

$$V^{-1}(A + \delta A)V = \underbrace{V^{-1}AV}_{\Lambda} + \underbrace{V^{-1}\,\delta A\, V}_{\delta\Lambda}$$

μ is an eigenvalue of $\Lambda + \delta\Lambda$

$\Lambda \longrightarrow \quad \lambda_1, \lambda_2, \ldots, \lambda_j, \ldots, \lambda_m$

$\Lambda + \delta\Lambda \longrightarrow \quad \mu \qquad \Rightarrow \quad |\mu - \lambda_j| \leq \|\delta\Lambda\|_2$

$\exists \vec{v} : \quad (\Lambda + \delta\Lambda)\vec{v} = \mu \vec{v}$

$\sigma_{min}(\mu I - \Lambda)\cancel{\|\vec{v}\|_2} \leq \|(\mu I - \Lambda)\vec{v}\|_2 = \|\delta\Lambda \vec{v}\|_2 \leq \|\delta\Lambda\|_2 \cdot \cancel{\|\vec{v}\|_2}$

$\underbrace{\min_{j=1..m} |\mu - \lambda_j|}$

$A = V\Lambda V^{-1}, \qquad \delta\Lambda = V^{-1}\delta A\, V$

$|\mu - \lambda_j| \leq \|\delta\Lambda\|_2 \leq \|V^{-1}\|\,\|\delta A\|\,\|V\|$

$= \underbrace{\kappa(V)}\|\delta A\|$

$+0.01 \longrightarrow \lambda = 1.5 \pm 0.86i$

$A = \begin{bmatrix} 101 & -110 \\ 90 & -98 \end{bmatrix} \qquad \lambda = 2, 1$

$\kappa(V) \approx 402$

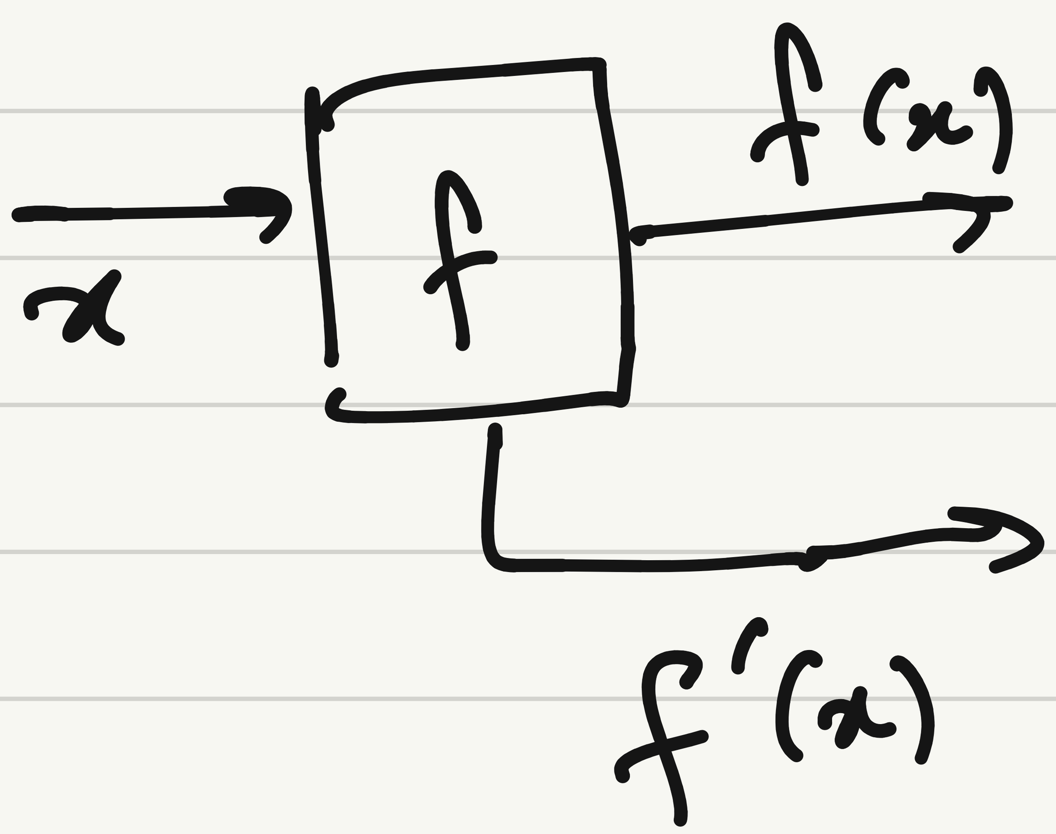
But if $A^TA = AA^T \Rightarrow \kappa(V) = 1$

Nonlinear equations

$f:\mathbb{R}\to\mathbb{R}$ find x : $f(x)=0$: $\boxed{\text{root finding}}$

$f:\mathbb{R}^n\to\mathbb{R}^n$ x is a $\boxed{\text{root}}$ of f

$f(x)=y \iff \underbrace{f(x)-y}=0$



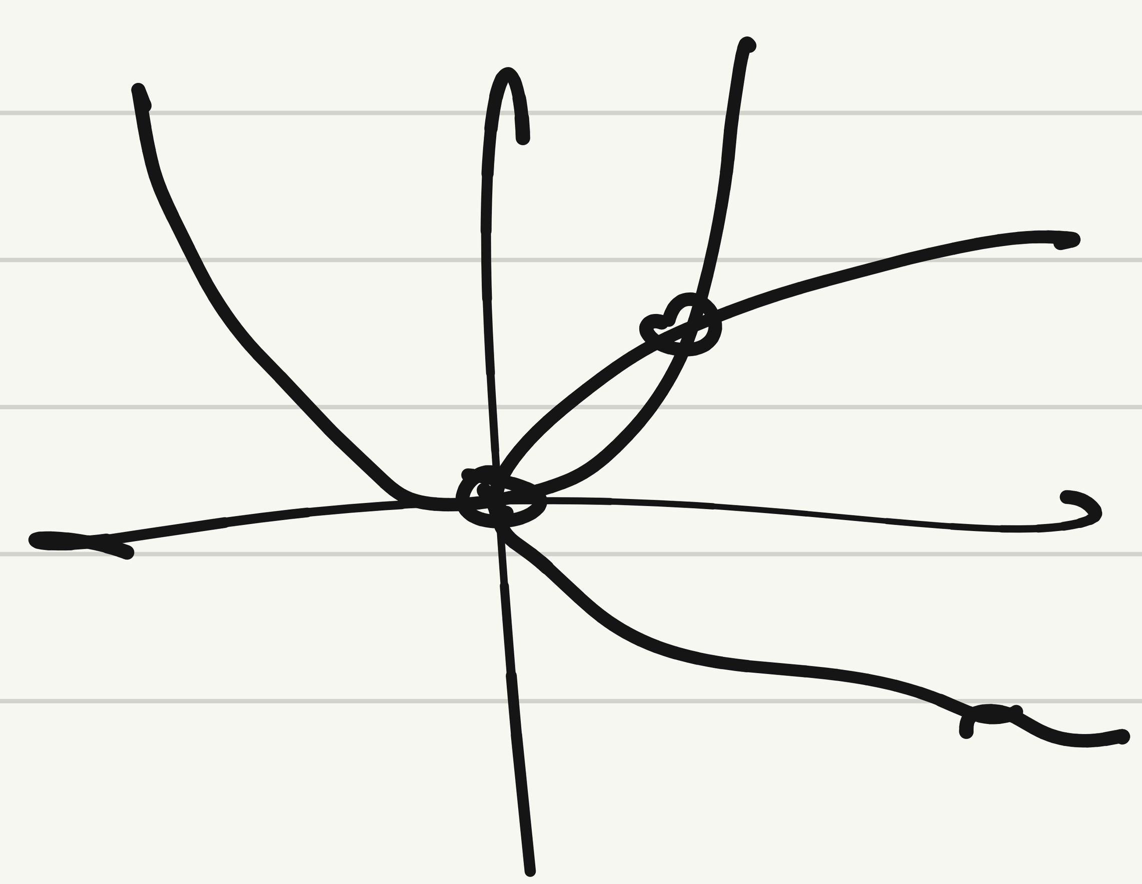
Existence & uniqueness

Only consider equal # eqs. & vars

$\vec{x}=\begin{bmatrix}x_1\\x_2\end{bmatrix}$ $f(\vec{x})=\begin{bmatrix}g_1(x_1,x_2)\\g_2(x_1,x_2)\end{bmatrix}$

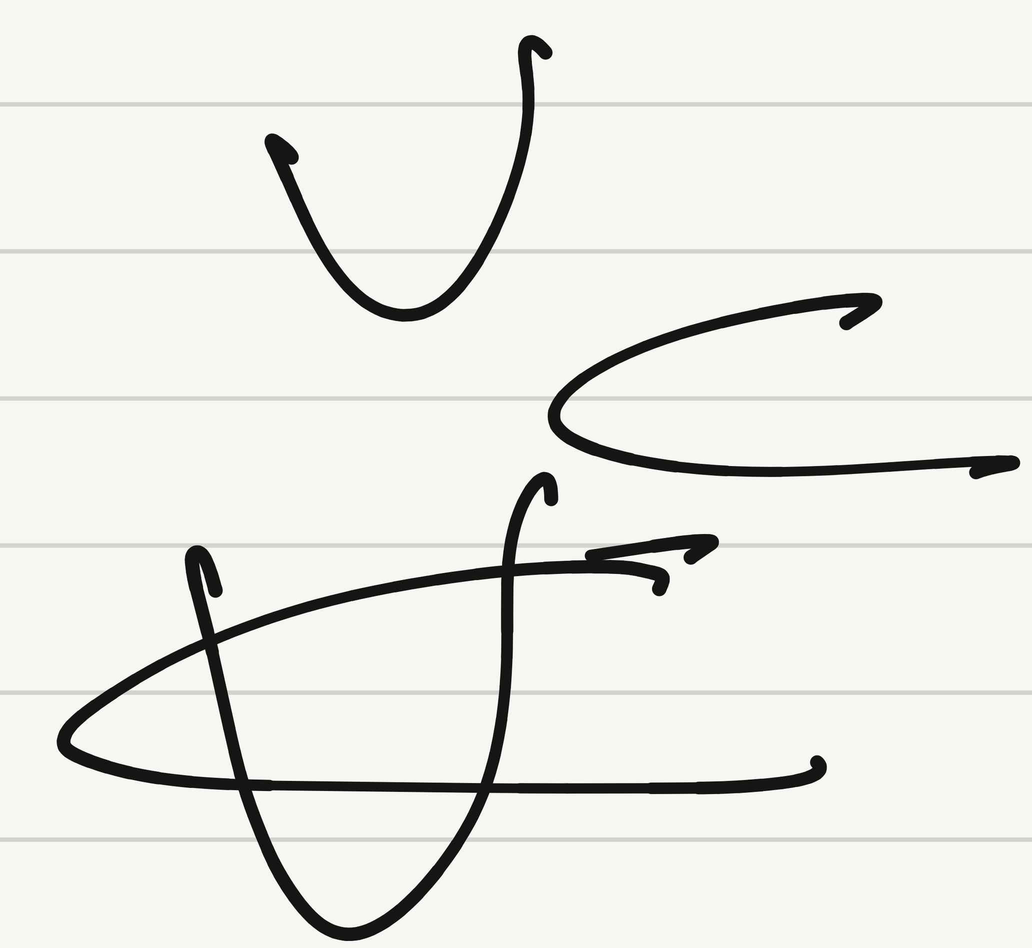
$g_1(x_1,x_2)=0$

$g_2(x_1,x_2)=0$



$c_1+x_1^2=x_2$

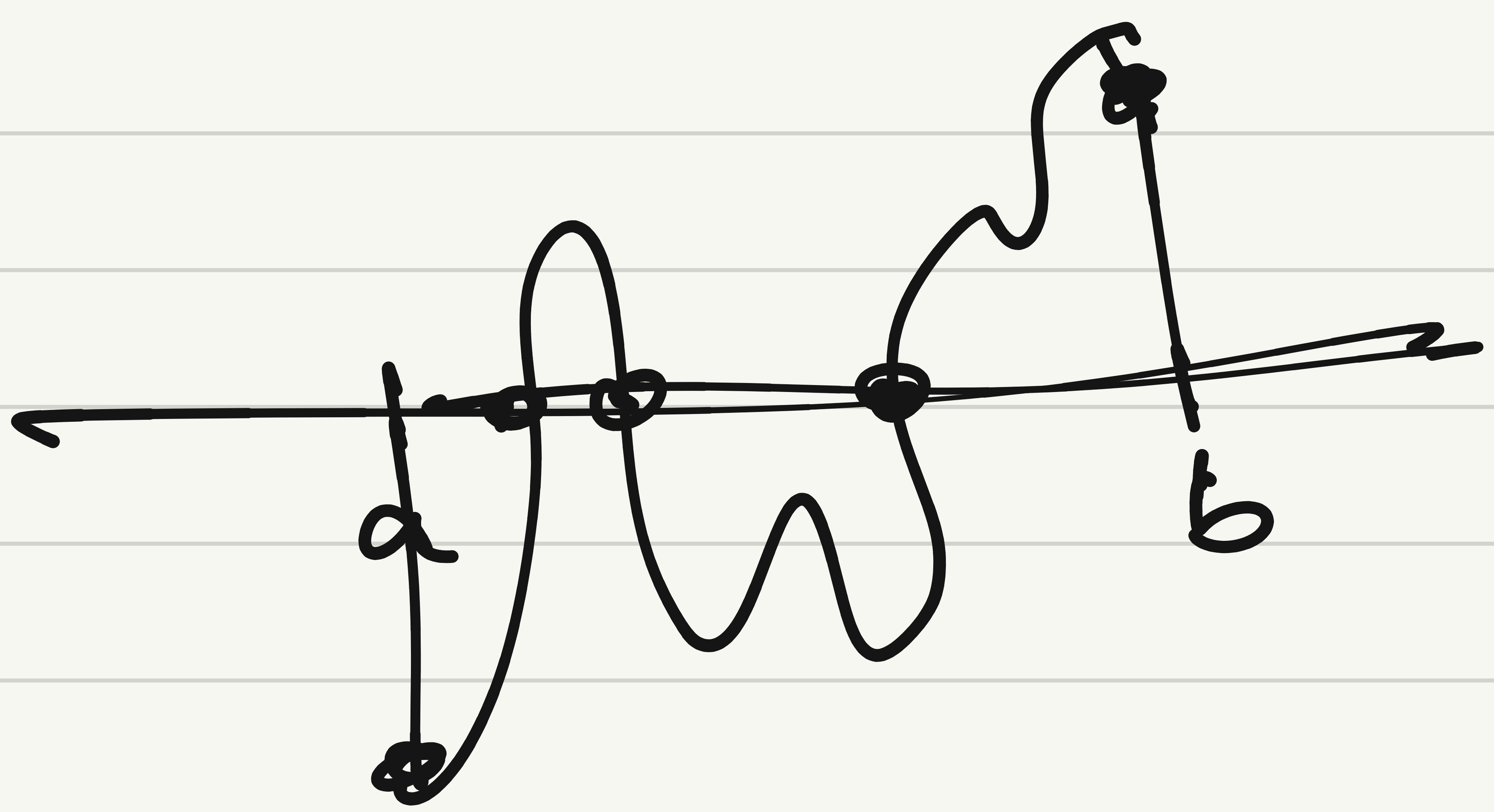
$c_2+x_2^2=x_1$



local guarantees of existence & uniqueness

<u>Intermediate value theorem</u> : $f : \mathbb{R} \to \mathbb{R}$ continuous,

If $f(a) < 0$, $f(b) > 0$ then $\exists\, x \in [a, b]$ s.t. $f(x) = 0$



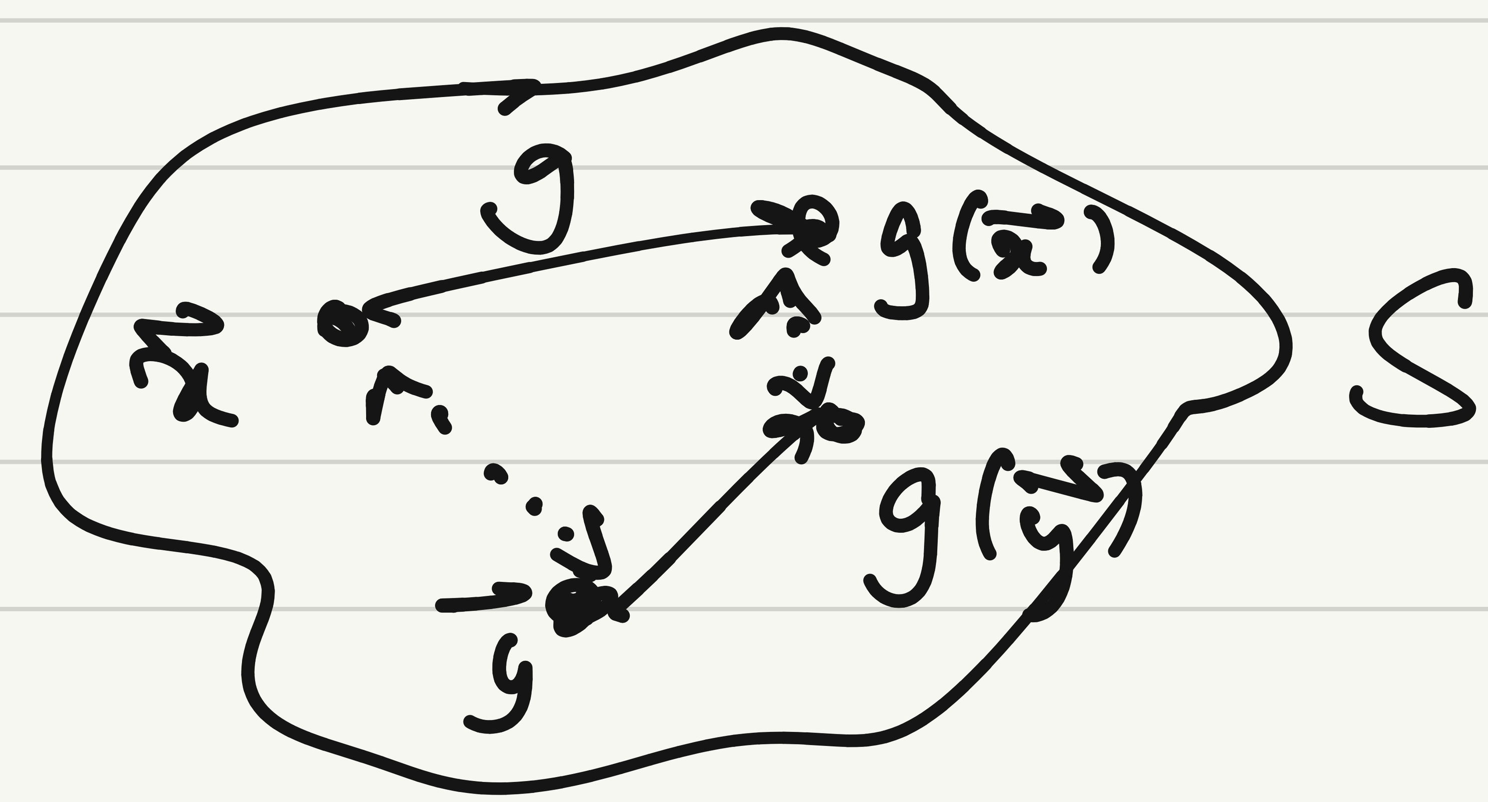
Interval $[a,b]$ with $f(a)$, $f(b)$ having diff. signs : $\boxed{\text{Bracket}}$

<u>Contraction mapping thm.</u> :

If $g : S \to S$ is such that $\|g(\vec{x}) - g(\vec{y})\| < \|\vec{x} - \vec{y}\| \ \forall\, \vec{x}, \vec{y}$

then g is a $\boxed{\text{Contraction}}$

$S \subseteq \mathbb{R}^n$



If g is contraction on compact set S then it has unique point $\vec{x} \in S$
s.t. $g(\vec{x}) = \vec{x}$: $\boxed{\text{fixed point}}$ of g

Let $f(\vec{x}) = x - g(\vec{x})$: f has unique solution!

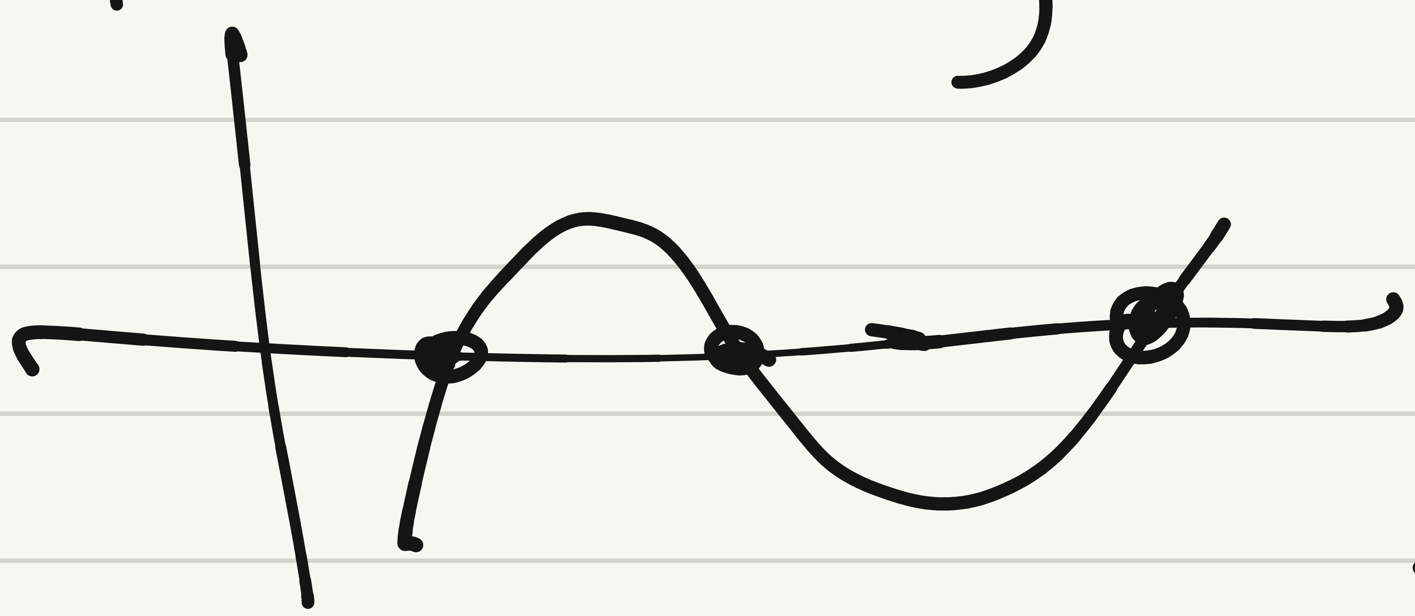
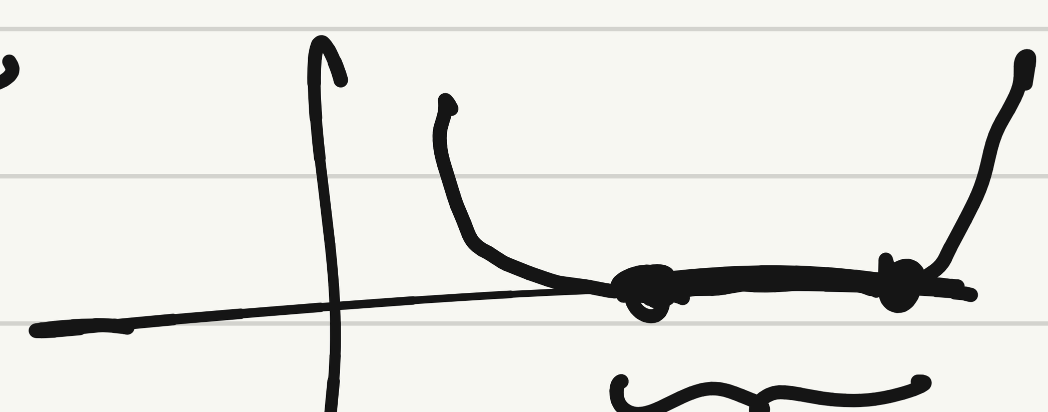
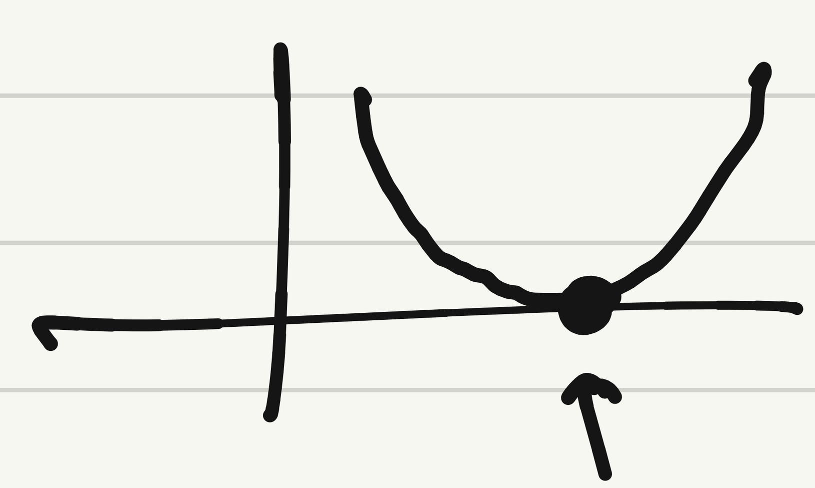
<u>Inverse function Thm.</u>

If Jacobian is nonsingular
then f is locally invertible
$\Rightarrow$ solution $\vec{x}_*$ is locally unique
If not, we say solution is $\boxed{\text{degenerate}}$

$\boxed{\text{Jacobian}}$ of $f: \mathbb{R}^n \to \mathbb{R}^n$

$$J(\vec{x}) = \begin{bmatrix} \partial f_1/\partial x_1 & \partial f_1/\partial x_2 & \cdots & \partial f_1/\partial x_n \\ \partial f_2/\partial x_1 & \partial f_2/\partial x_2 & \cdots & \vdots \\ \vdots & \vdots & & \\ \partial f_n/\partial x_1 & \cdots & & \partial f_n/\partial x_n \end{bmatrix}$$

$$f(\vec{x} + \delta\vec{x}) = f(\vec{x}) + J(\vec{x}) \cdot \delta\vec{x} + O(\|\delta\vec{x}\|^2)$$

$$\left.\begin{array}{l} \text{If } f(x)=0,\ f'(x)=0,\ f''(x)=0,\ \dots,\ f^{(n-1)}(x)=0 \\ f^{(n)}\neq 0 \end{array}\right\} \Rightarrow \text{multiplicity of root is } n.$$

<u>Conditioning:</u>

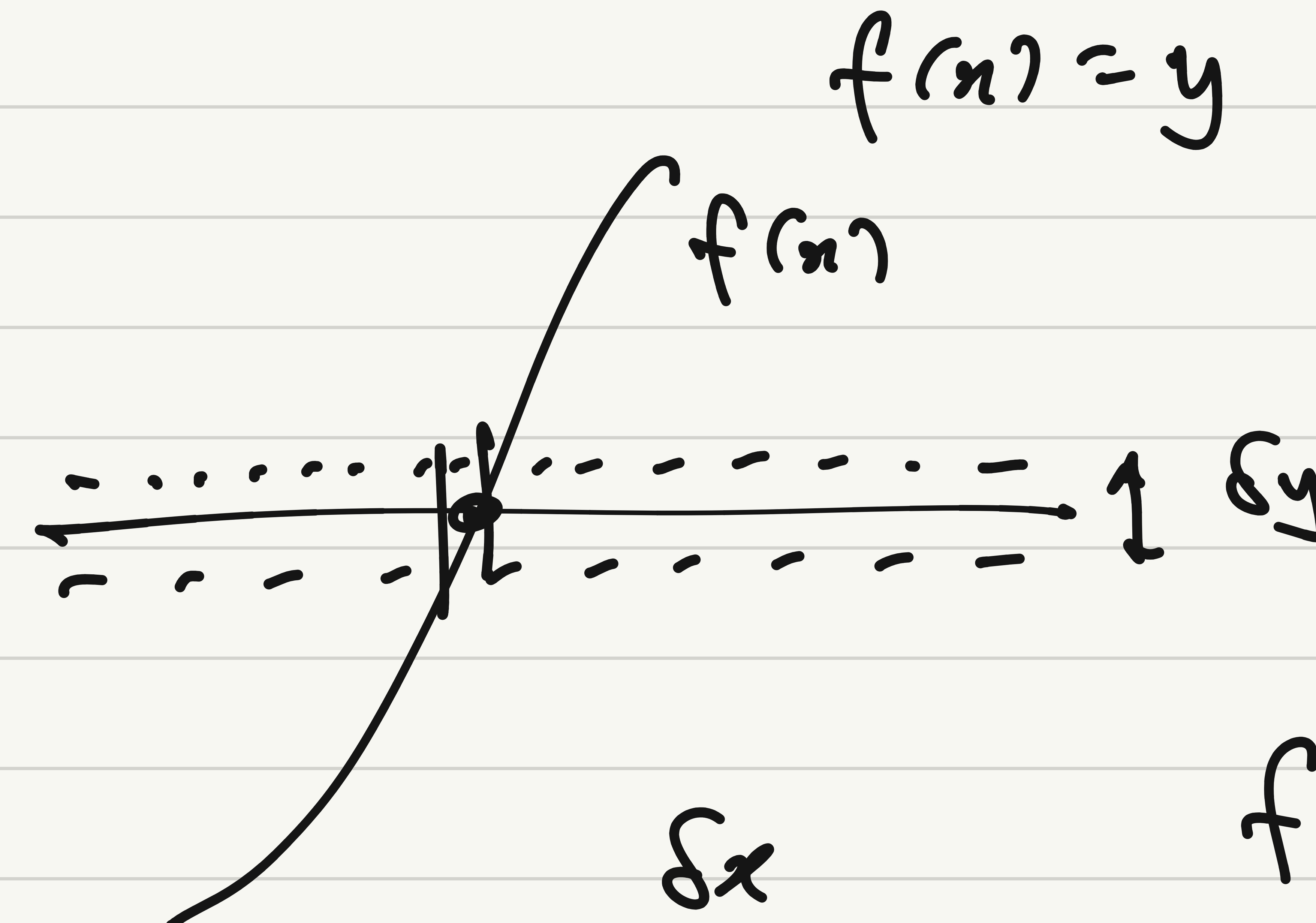
$$f(\vec{x}_*)=0, \longrightarrow f(x)=y$$

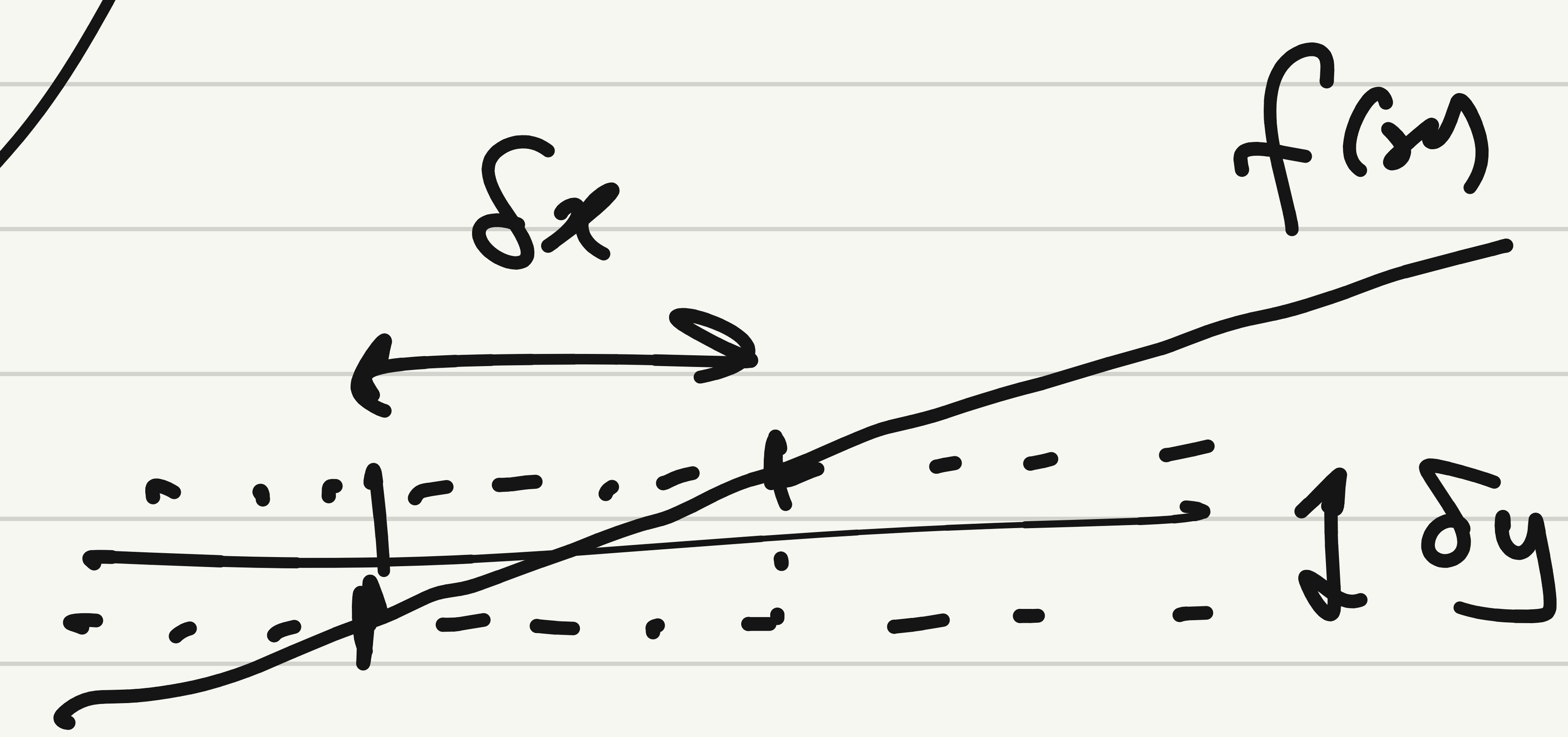
$$f^{-1}=g, \quad g'(y)=1/f'(x)$$

$$f:\mathbb{R}\to\mathbb{R} \Rightarrow x_* = f^{-1}(y)$$

$$\text{Abs. cond. num} = \frac{|\delta x|}{|\delta y|} = \left|\frac{1}{f'(x_*)}\right|$$

$$\text{In } \mathbb{R}^n, \text{ abs. cond. num.} = \|J^{-1}(x_*)\|$$





Convergence

$$x_0 \longrightarrow x_1 \longrightarrow x_2 \longrightarrow \cdots \longrightarrow x_k \longrightarrow \cdots \xrightarrow{\lim} x_*$$

$\underbrace{}_{\#\text{flops}}$

error : $e_k = x_k - x_*$

residual : $f(x_k)$

$\boxed{\text{rate of convergence}}$

$$\lim_{k\to\infty} \frac{\|e_{k+1}\|}{\|e_k\|} = C < 1$$: convergece is <u>linear</u> with rate const. C

gain $\sim -\log_{10} C$ decimal digits per iter

eg. power iter : $C = |\lambda_2| / |\lambda_1|$

$$\lim_{k\to\infty} \frac{\|e_{k+1}\|}{\|e_k\|^r} = C$$ (not neccessarily < 1) : convergece is $\boxed{\text{superlinear}}$ if $r>1$

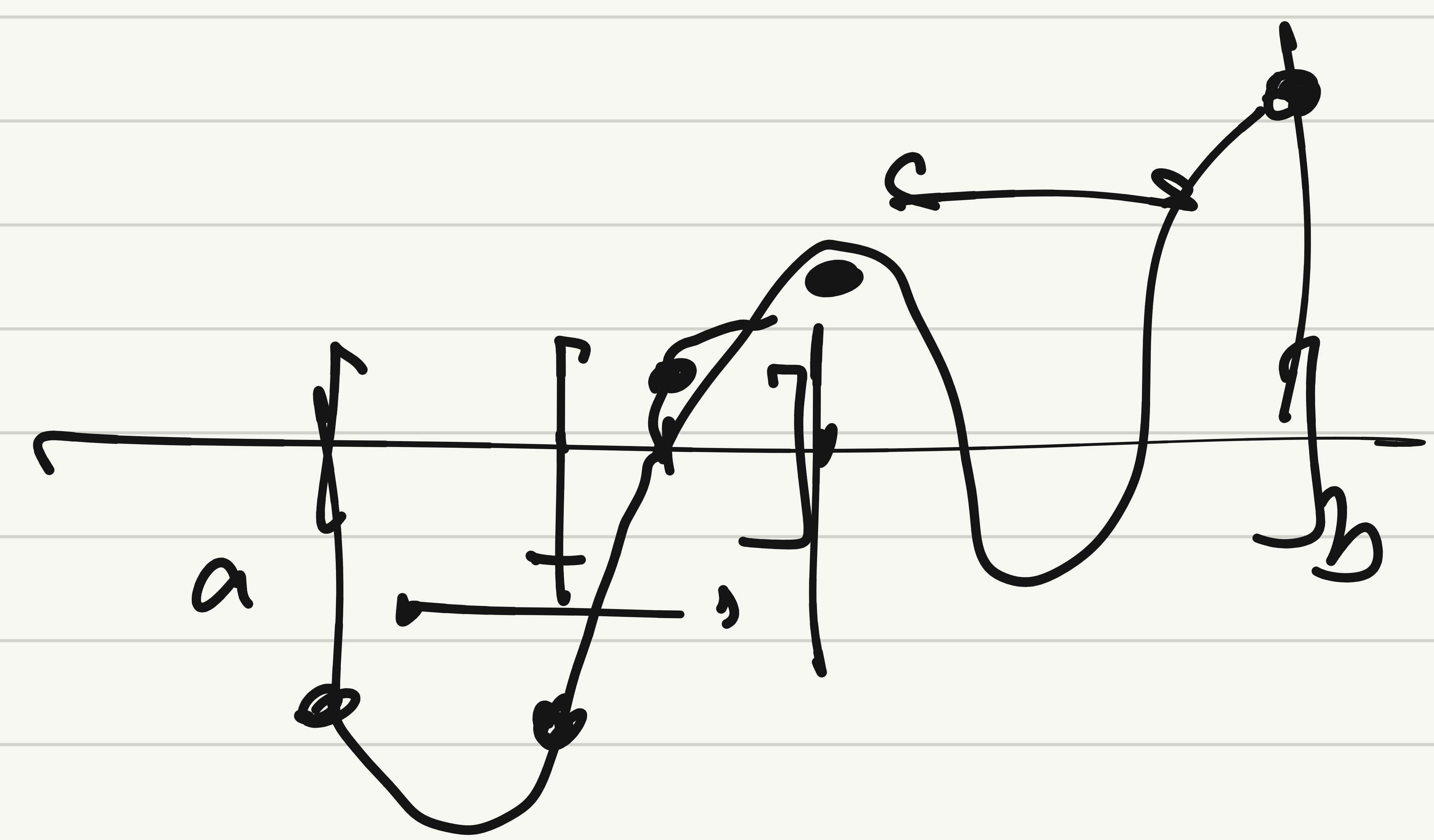
$r=2$: quadratic , $r=3$: cubic

1D algorithms : $f : \mathbb{R} \to \mathbb{R}$

- bisection method
- fixed-point iterations
- Newton's method
- Secant method

Bisection : input $[a, b]$ s.t. $\mathrm{sign}(f(a)) \neq \mathrm{sign}(f(b))$

```
while b-a > tolerance:
    m = a + (b-a)/2
    if sign(f(m)) = sign(f(a))
        a = m
    else
        b = m
```



No x_k, $[a_k, b_k]$ instead

Error $e_k = \underbrace{b_k - a_k}$ $\qquad \dfrac{|e_{k+1}|}{|e_k|} = \dfrac{1}{2}$

<u>fixed point iteration</u>

$f(x) = 0$: find $g : \mathbb{R} \to \mathbb{R}$ s.t. $g(x) = x$ iff $f(x) = 0$

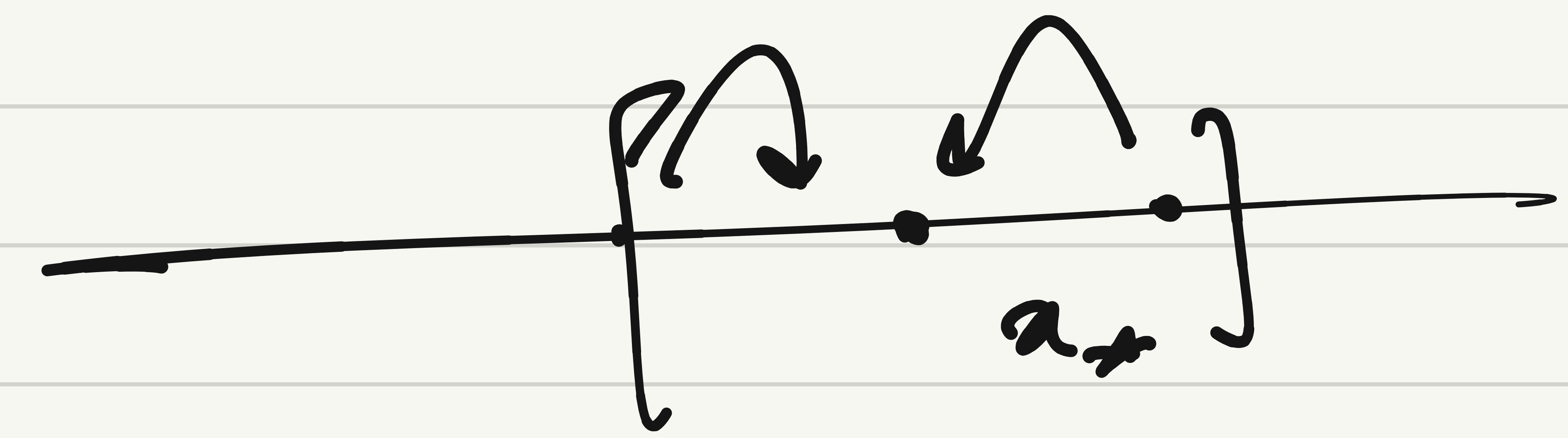
$x_0 \to x_1 = g(x_0) \to x_2 = g(x_1) \to \cdots$

$f(x) = \underbrace{x^2 - x - 2 = 0}$

$\to \quad x = x^2 - 2 = g(x) \qquad \to x - 1 - 2/x = 0 \Rightarrow x = 1 + \frac{2}{x}$

$\to \quad x^2 = x + 2 \Rightarrow x = \sqrt{x+2} = g(x) \qquad \to \quad \ldots$

$$x_0 \to \cdots \to x_k \longrightarrow x_{k+1} = g(x_k) \longrightarrow \cdots$$



When does fixed point iteration converge?

If there is some neighborhood S containing x_*

s.t. g is contraction mapping S to itself,

and $x_0 \in S$ then $|e_k| = |x_k - x_*|$ decreases every iteration

$$\left| \overset{x_{k+1}}{g(x_k)} - \overset{x_*}{g(x_*)} \right| \leq |x_k - x_*|$$