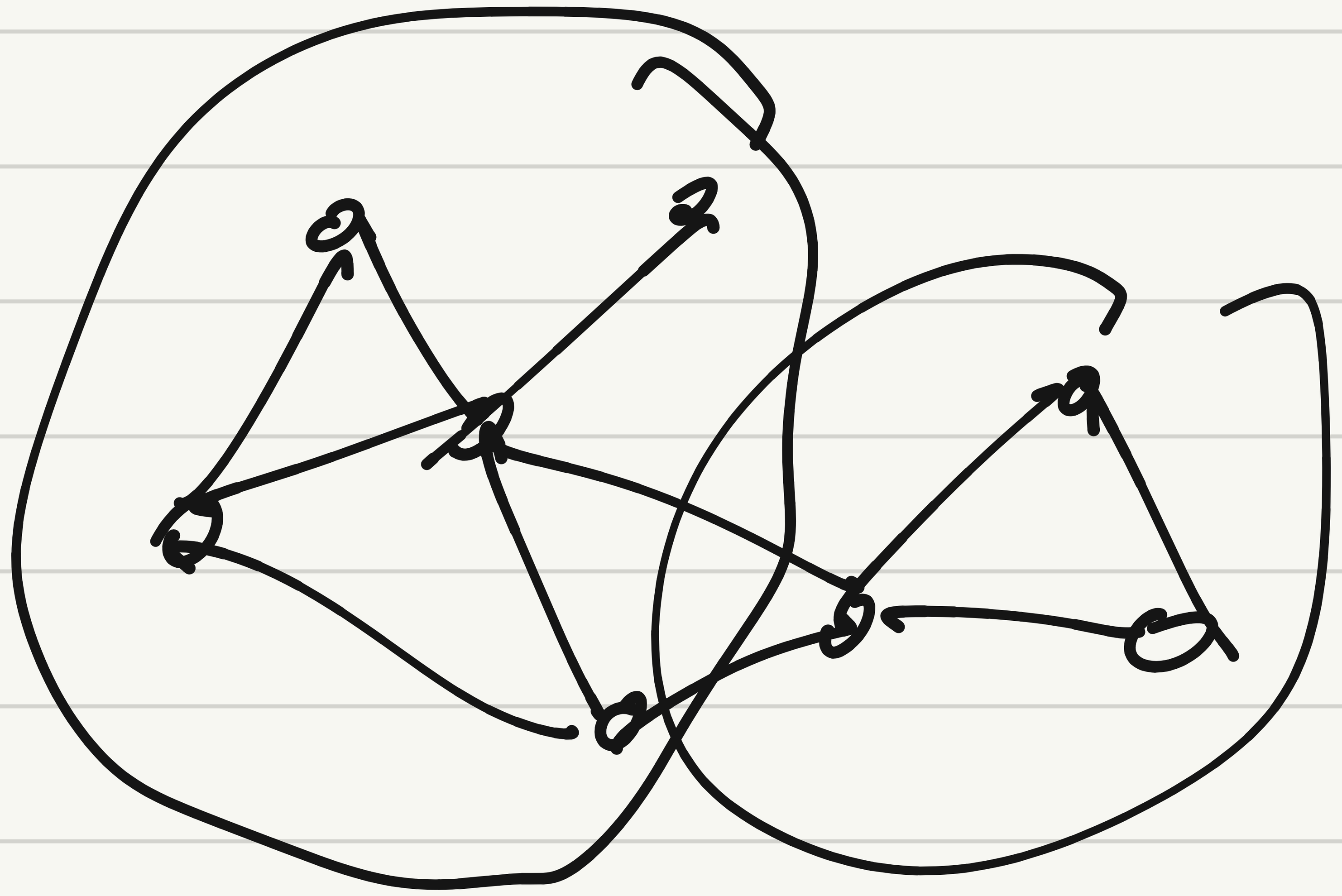
COL726: Eigenvalues

$A \in \mathbb{C}^{m \times m}$

$A\vec{v} = \lambda \vec{v}$ $\qquad \lambda_1, \lambda_2, \ldots, \lambda_m$

$\vec{v} \neq 0$ $\qquad \lambda \in \mathbb{C}$

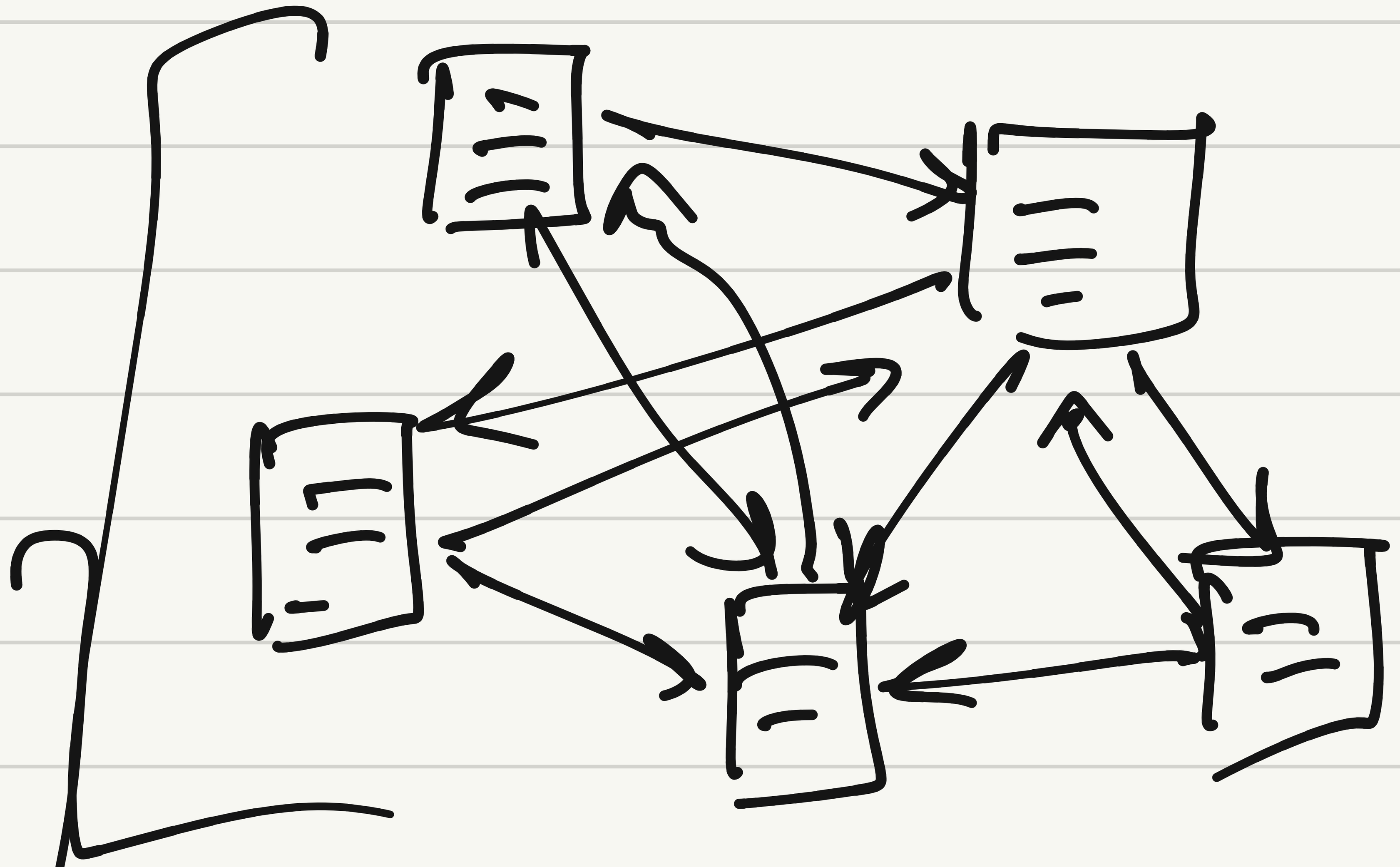
$\vec{v} \in \mathbb{C}^m$



graph Laplacian

$$\lambda_1 + \lambda_2 + \cdots + \lambda_m = \mathrm{tr}(A) = \sum_{i=1}^{m} a_{ii}$$

$$\lambda_1 \lambda_2 \cdots \lambda_m = \det(A)$$



$\boxed{\text{characteristic polynomial}}$

$$p_A(z) = \det(zI - A)$$

$$\begin{bmatrix} z-a_{11} & -a_{12} & \cdots \\ -a_{21} & z-a_{22} & \cdots \\ \vdots & \vdots & \end{bmatrix}$$

deg. m polynomial.

coeff of z^m is 1 : $\boxed{\text{monic}}$ poly.

$A \in \mathbb{C}^{m \times m}$ $\qquad p_A(z) = \det(zI - A)$

roots of p_A are eigenvalues of A.

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$p_A(z) = z^2 + 1$$

$$z^m + a_{m-1} z^{m-1} + \cdots + a_1 x^1 + a_0 = 0$$

$(a_0, a_1, \ldots, a_{m-1}) \mapsto (\lambda_1, \ldots, \lambda_m)$ badly conditioned!

If A has m lin. indep. eigenvectors, $\vec{v}_1, \ldots, \vec{v}_m$

then A is $\boxed{\text{diagonalizable}}$ $\qquad AV = V\Lambda$

$$\Lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_m \end{bmatrix}$$

$$\Rightarrow A = V\Lambda V^{-1} \iff V^{-1} A V = \Lambda$$

Eigenvalues = roots of p_A (deg. n)

$\Rightarrow$ n roots (counted with multiplicity)

$\Rightarrow$ A has n eigenvalues counted with $\boxed{\text{algebraic multiplicity}}$

$$A = \begin{bmatrix} 2 & & \\ & 3 & \\ & & 3 \end{bmatrix} \rightarrow p_A(z) = (z-2)(z-3)^2$$

a.m. = 1   a.m. = 2

$$E_3 = \left\langle \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\rangle = \underline{\left\{ \begin{bmatrix} 0 \\ y \\ z \end{bmatrix} \right\}}$$

$$A\vec{v} = \lambda \vec{v} \iff (A - \lambda I)\vec{v} = \vec{0} \iff \vec{v} \in \underline{\text{null}(A - \lambda I)} = E_\lambda : \boxed{\text{eigenspace}} \text{ of } \lambda$$

$\dim E_\lambda$: $\boxed{\text{geometric multiplicity}}$

$$E_2 = \left\langle \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\rangle = \left\{ \begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix} \right\}$$

alg. mult. $\geq$ geom. mult. ≥ 1

$$\begin{bmatrix} a_1 & & & & \\ & \underbrace{a_2 \;\; a_2 \;\; \ddots}_{k} & & & \\ & & & & a_m \end{bmatrix}$$

$$\left.\begin{array}{l} \text{- a.m.} = \text{g.m.} = 1 \\ \text{- a.m.} = \text{g.m.} > 1 \end{array}\right\}$$ can find m lin. indep. eigenvectors

- a.m. $>$ g.m. ≥ 1

$$\begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix} \qquad p_A(z) = (z-3)^2 \qquad A - 3I = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$E_\lambda = \left\langle \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\rangle$$

$\boxed{\text{defective}}$ matrix or $\boxed{\text{nondiagonalizable}}$ matrix

$A = QR = LU = LL^* = LDL^*$

$\boxed{\text{similarity transformation}}$

A, invertible X, define $B = \overbrace{X^{-1}AX}$: B is $\boxed{\text{similar}}$ to A

B has same char. poly., same eigenvalues with same a.m., g.m.

$\vec{v}$ is eigenvector of $B \iff X\vec{v}$ is eigenvector of A

$A = V\Lambda V^{-1} \iff \Lambda = V^{-1}AV$

$\boxed{\text{Unitary diagonalization}}$ Suppose $\exists$ unitary Q s.t. $A = Q\Lambda Q^*$

A is $\boxed{\text{unitarily diagonalizable}}$

If this is possible, cols of Q are eigenvectors of A, $\Rightarrow$ eigenspaces of A are orthogonal

$$A = A^*$$

- A is Hermitian $\Longrightarrow$ A is unitarily diag., eigenvalues are real.
- skew-Hermitian: $A = -A^*$
- unitary
- circulant

$$\begin{bmatrix} a & b & c & & & \\ d & a & b & c & \cdot & \\ e & d & a & b & \cdot & \cdot \\ & e & \ddots & \ddots & \ddots & b \\ & \cdot & e & \ddots & \ddots & a \\ & & \cdot & \ddots & d & \end{bmatrix}$$

$\boxed{\text{Spectral theorem}}$: A is unit. diag. iff $\underbrace{A^*A = AA^*}_{\boxed{\text{normal matrix}}}$

$$\frac{\|\vec{r}_n\|}{\|\vec{b}\|} \leq \cancel{\kappa(V)} \cdot \inf \|p_n\|_{\Lambda(A)}$$

if A is normal then $\kappa(V) = 1$

$A \in \mathbb{C}^{m\times m}$: find X s.t. $A = XBX^{-1}$, eigenvalues of B are easy to find

- $A = V\Lambda V^{-1}$: diagonalization, only if A is nondefective
- $A = \underline{\underline{Q}}\Lambda \underline{\underline{Q}}^*$: unitary diag., only if A is normal
- $A = QTQ^*$: unitary triangularization, always exists!

triangular

$\boxed{\text{Schur factorization}}$

$$\begin{bmatrix} t_{11} & t_{12} & \cdots & * \\ & t_{22} & \cdots & \vdots \\ & & \ddots & \vdots \\ & & & t_{mm} \end{bmatrix}$$

eigenvalues $= t_{11}, t_{22}, \ldots, t_{mm}$

$p_T(z) = (z - t_{11}) \cdot \ldots \cdot (z - t_{mm})$

If A is Hermitian

$\Rightarrow T = Q^*AQ$ also Hermitian

$\Rightarrow T$ diagonal!

Proof of existence of Schur factorization

Q: Given $\vec{v}$, find W using Householder
$\vec{e}_1 \mapsto \vec{v}$

A, pick $(\lambda, \vec{v})$: $\boxed{\text{eigenpair}}$. $\vec{v}$ unit

$$U = \begin{bmatrix} \vec{v} & W \end{bmatrix} \qquad U^*AU = U^* \begin{bmatrix} \lambda\vec{v} & AW \end{bmatrix}$$

$W^* \vec{v} = 0$

$$= \begin{bmatrix} \vec{v}^* \\ W^* \end{bmatrix} \begin{bmatrix} \lambda\vec{v} & AW \end{bmatrix}$$

$$\overbrace{U_m^* \cdots U_2^* U_1^*}^{Q^*} A \overbrace{U_1 U_2 \cdots U_m}^{Q} = T \qquad = \begin{bmatrix} \lambda & \vec{v}^* AW \\ \vec{0} & C \end{bmatrix}$$

$A = QTQ^*$

There is <u>no</u> direct alg. for finding $A = QTQ^*$ when $m \geq 5$.

<u>Abel-Ruffini Thm:</u> for any $m \geq 5$, there exist degree-m polynomials whose roots <u>cannot</u> be expressed with finite no. of

$+, -, \times, \div, \sqrt[n]{\ }$

$$p(z) = z^m + a_{m-1} z^{m-1} + \dots + a_1 z^1 + a_0$$

$$\begin{bmatrix} 0 & & & & & -a_0 \\ 1 & 0 & & & & -a_1 \\ & 1 & 0 & & & -a_2 \\ & & 1 & \ddots & & \vdots \\ & & & \ddots & 0 & -a_{m-2} \\ & & & & 1 & -a_{m-1} \end{bmatrix}$$

$\boxed{\text{Companion matrix}}$

$\longrightarrow$ Any eigenvalue alg. <u>must</u> be iterative

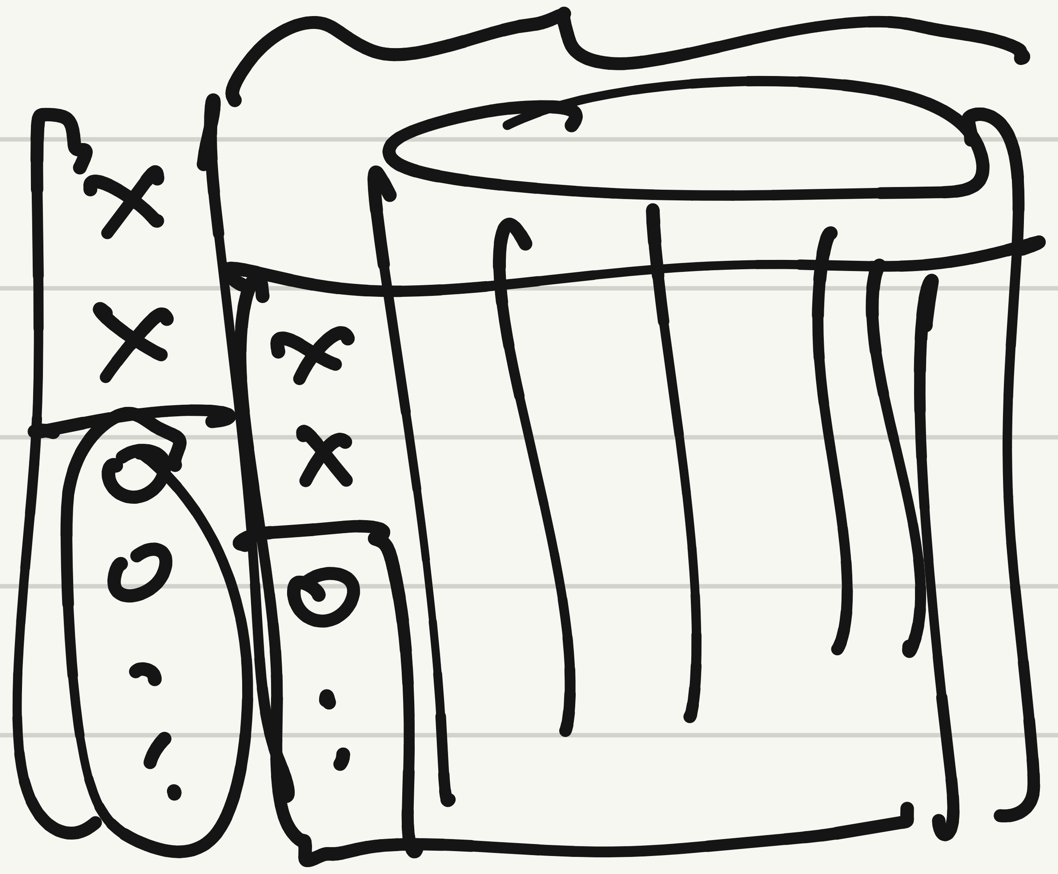
$$A \to Q_1^* A Q_1 \to Q_2^* A Q_2 \to \cdots \to T$$

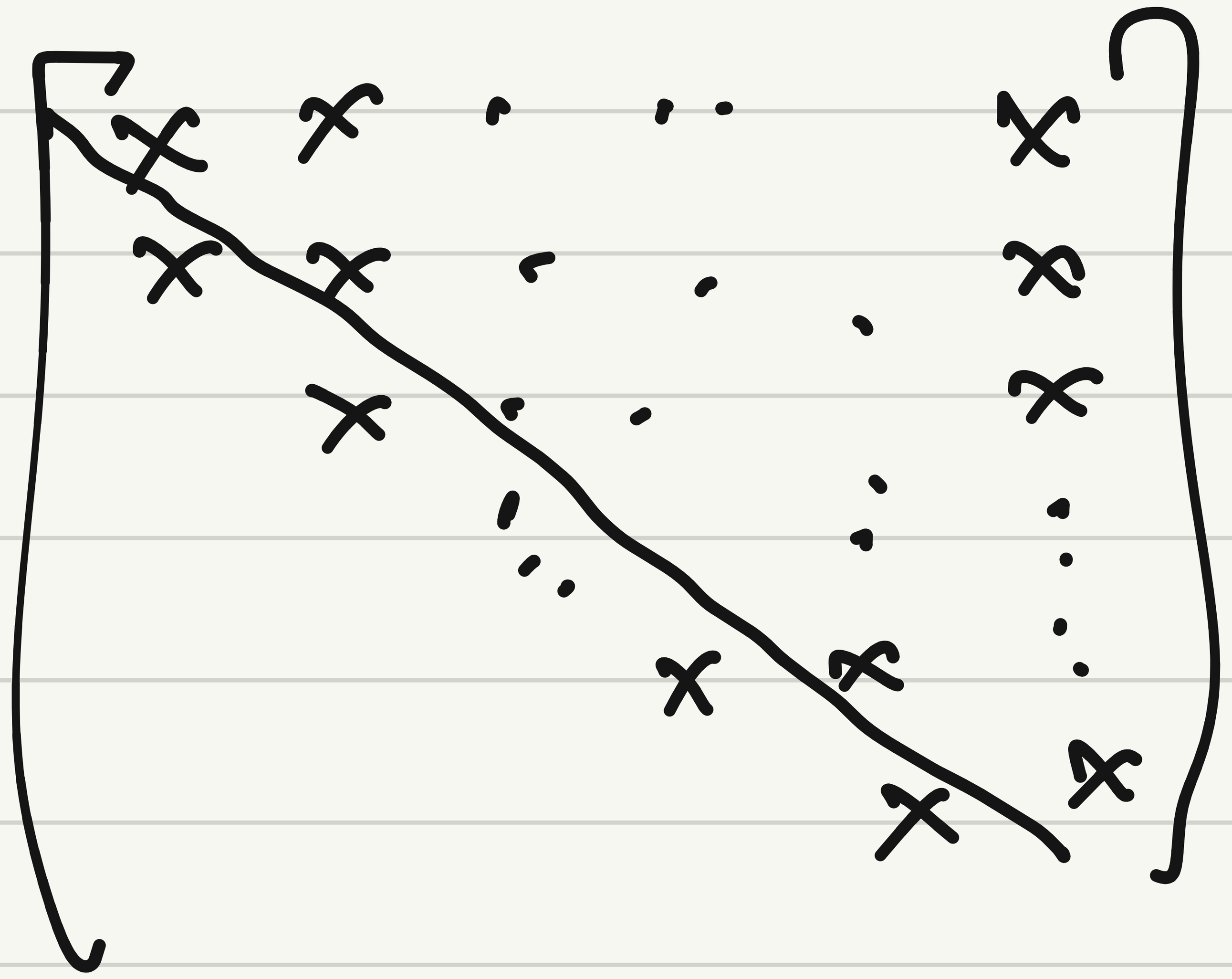
$$A = \underbrace{Q\,T}\,Q^* \iff T = \underbrace{Q^* A}\,Q$$

$$\begin{bmatrix} & & \\ & Q^* & \\ & & \end{bmatrix} \underset{A}{\begin{bmatrix} x & x & \cdot & - \\ x & x & \cdot & \cdot \\ x & x & \cdot & \cdot \\ \vdots & \vdots & & \end{bmatrix}} = \underset{Q^*A}{\begin{bmatrix} x & x & & \\ 0 & x & \cdots & \\ 0 & x & & \\ \vdots & \vdots & & \end{bmatrix}}$$

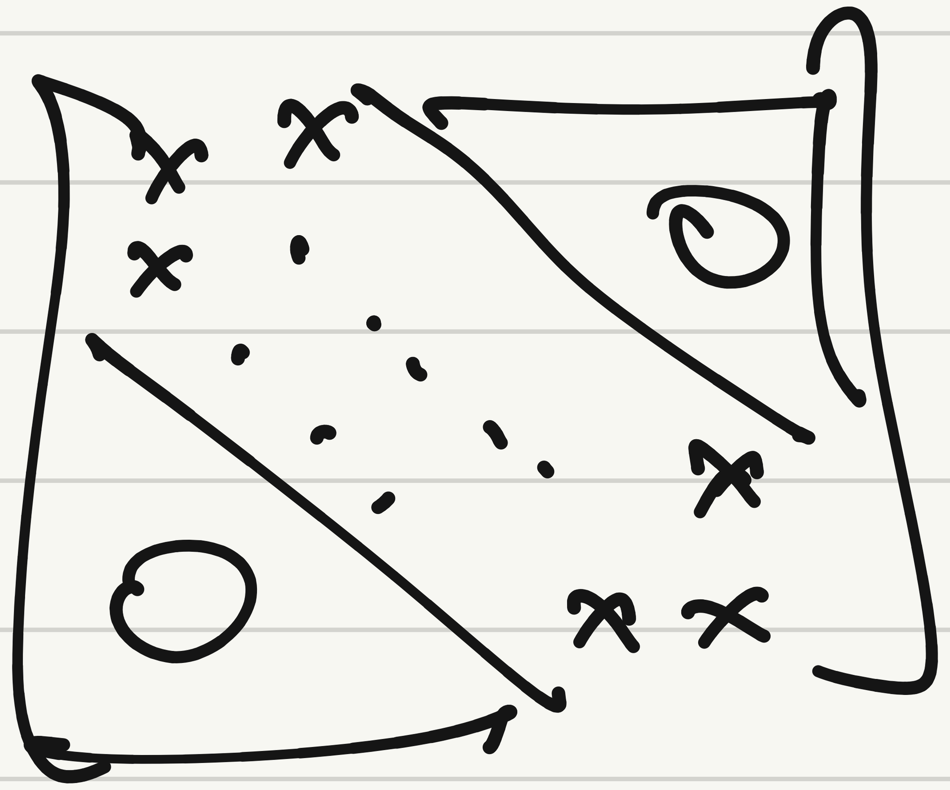
$$Q^*AQ = \underset{Q^*A}{\begin{bmatrix} x & x & x & \\ 0 & x & x & \\ 0 & x & x & \\ \vdots & \vdots & \vdots & \end{bmatrix}} \begin{bmatrix} & & \\ & Q & \\ & & \end{bmatrix} = \begin{bmatrix} x & x & x & \cdots \\ x & x & x & \cdots \\ x & x & x & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix}$$

$$Q_1^* A = \begin{bmatrix} \times & \times & \times & \cdots \\ \times & \times & \times & \cdots \\ 0 & \times & \times & \\ 0 & \times & \times & \\ \vdots & \vdots & \vdots & \end{bmatrix} \qquad Q^* = \begin{bmatrix} 1 & 0 \\ 0 & F \end{bmatrix}$$

$Q_1^* A Q_1 =$ 

$\longrightarrow \cdots$ $Q^* A Q =$ 

$$Q^* A Q = H \iff A = Q H Q^*$$

If A is Hermitian $\Rightarrow$ $H =$  : $\boxed{\text{tridiagonal}}$