Q3. $UX + XL = Y$

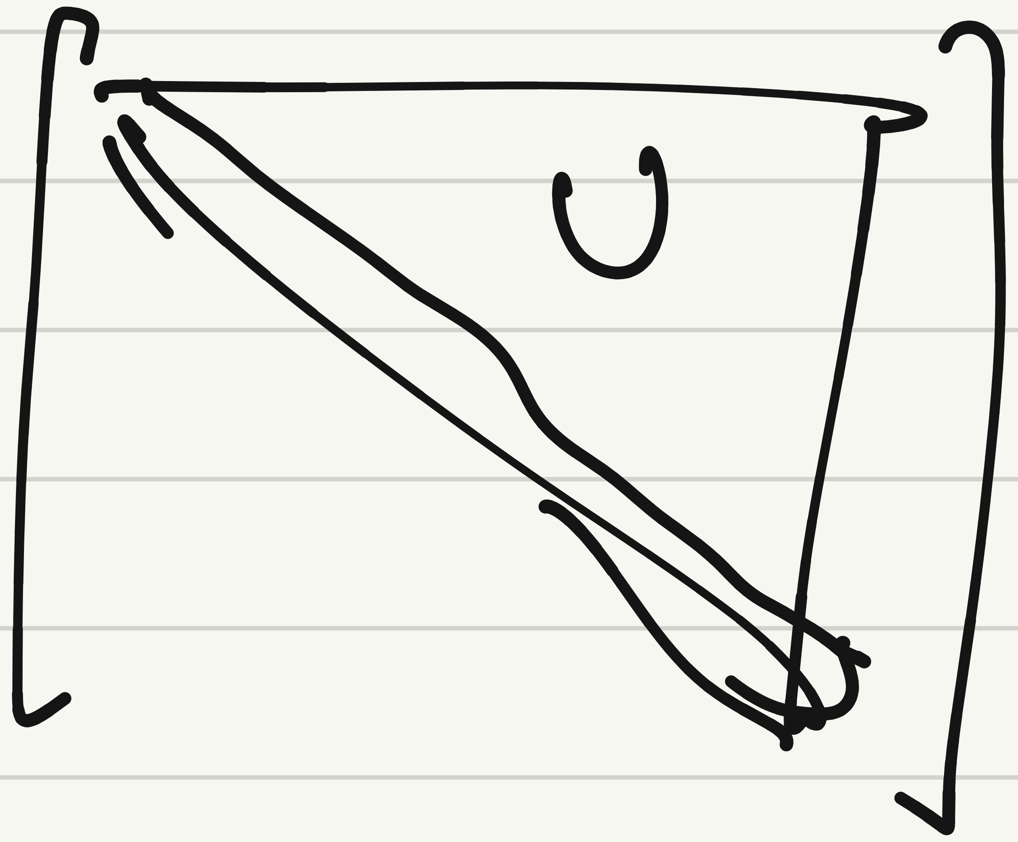
jth col: $U\vec{x}_j + X\vec{l}_j = \vec{y}_j$

$$\vec{l}_j = \begin{bmatrix}0\\0\\\vdots\\l_{jj}\\l_{j+1,j}\\\vdots\\l_{mj}\end{bmatrix}$$

$$= U\vec{x}_j + l_{jj}\vec{x}_j + \cdots + l_{mj}\vec{x}_m = \vec{y}_j$$

$j = m \Rightarrow U\vec{x}_m + l_{mm}\vec{x}_m = \vec{y}_m$

$$\underbrace{(U + l_{mm}I)}\vec{x}_m = \vec{y}_m$$




$\vec{x}_m = \text{backsub}(U + l_{mm}I, \vec{y}_m)$

$A \in \mathbb{C}^{m\times m}$, $\underline{A^* = A}$

$q : \mathbb{C}^m \to \mathbb{R}$

$q(\vec{x}) = \vec{x}^* A \vec{x}$

$q(\vec{x}) = a_{11}x_1^2 + 2a_{12}x_1x_2 + \cdots$

If $A^* = A$
and $\vec{x}^* A \vec{x} > 0$ for all $\vec{x} \neq 0$
then $A$ is $\boxed{\text{Hermitian positive definite}}$

$\vec{x}^* A \vec{x} \geq 0$ : H. p. semidefinite

$\boxed{\text{indefinite}}$ : $\exists\, \vec{x}_1, \vec{x}_2$ : $\vec{x}_1^* A \vec{x}_1 > 0$, $\vec{x}_2^* A \vec{x}_2 < 0$

1. $A+B$ is HPD $\Leftrightarrow$ $x^*(A+B)x = x^*Ax + x^*Bx > 0$

2. $cA$ is HPD if $c > 0$

$\underline{AB}$ is HPD ? Not necessarily!

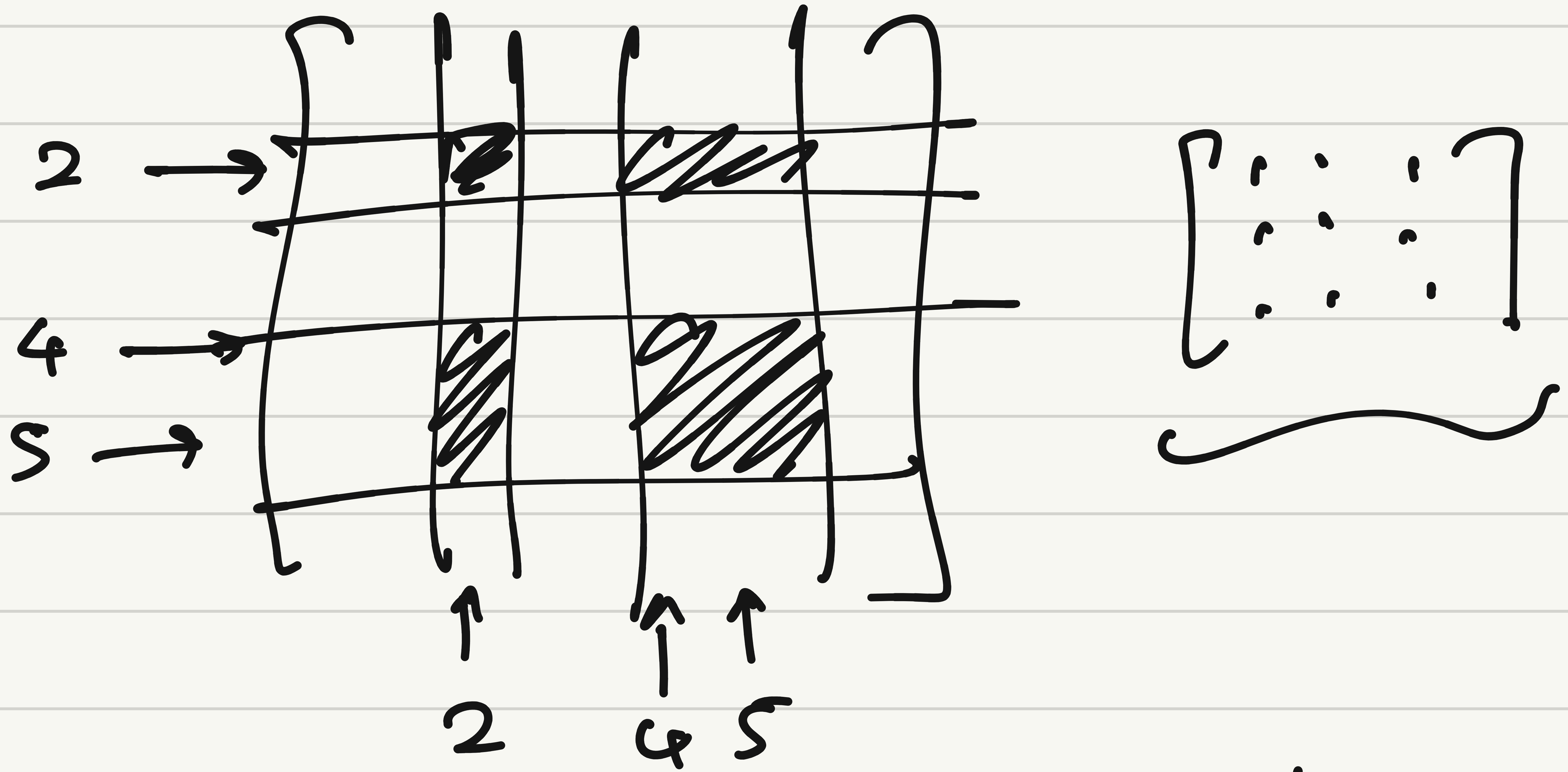
3. $A \in \mathbb{C}^{m\times m}$ HPD, $X \in \mathbb{C}^{m\times n}$, $m \geq n$, full rank

then $\underbrace{X^*AX}_{\in \mathbb{C}^{n\times n}}$ is HPD

$$\underbrace{\vec{x}^*(X^*}_{\vec{y}^*}A\underbrace{X)\vec{x}}_{\vec{y}} > 0$$

(a) Any principal submatrix of $A$ is HPD

$$\vec{y} = X\vec{x}$$
$$\vec{y}^* = (X\vec{x})^* = \vec{x}^* X^*$$



$$\underbrace{\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \\ & & \end{bmatrix}}_{X}$$

(b) $X = \vec{e}_j \Rightarrow \vec{e}_j^{\,*} A \vec{e}_j = a_{jj} > 0$

(c) $A = I \Rightarrow X^* X$ is HPD

(4) Hermitian $\Rightarrow$ eigenvalues are real,

eigenvectors with diff. eigenvalues are orthogonal

HPD $\Rightarrow$ eigenvalues are positive.

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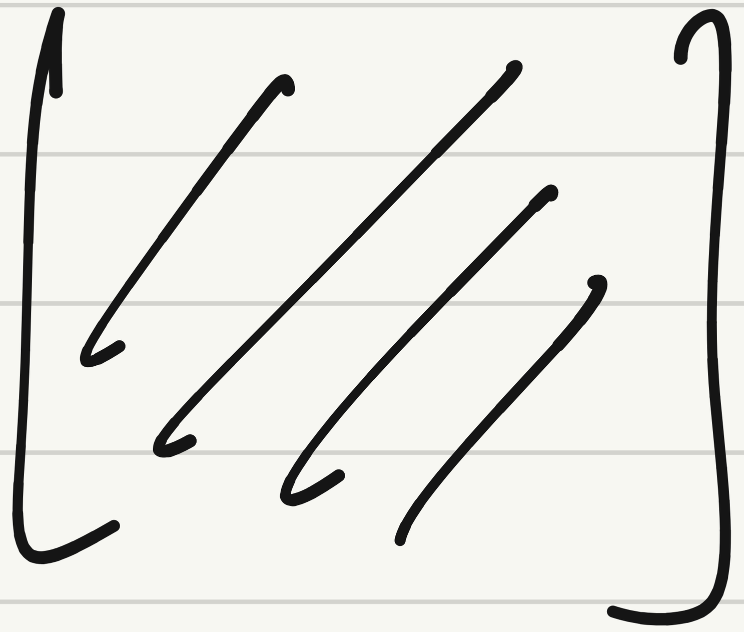
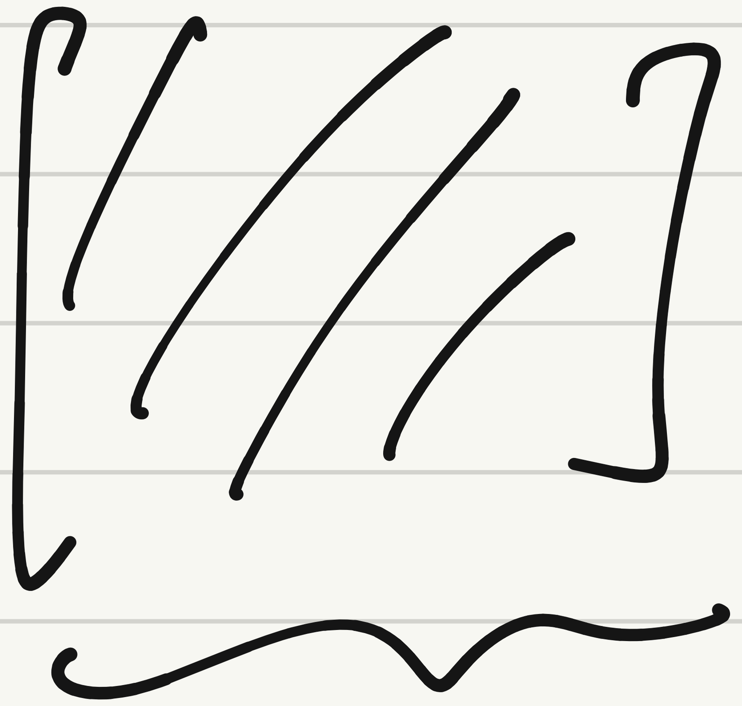
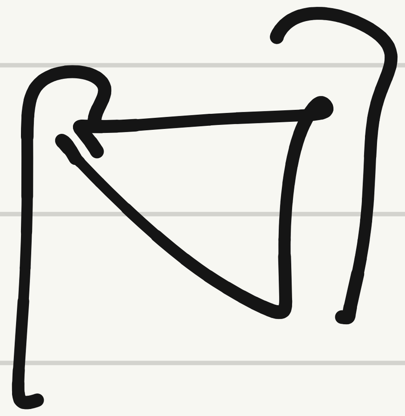
**Cholesky factorization**

$$A \longrightarrow L_1 A = \begin{bmatrix} & \vdots & \\ & \cdot & \cdot \end{bmatrix} \longrightarrow L_2 L_1 A = \begin{bmatrix} & \cdot & & \\ & & \cdot & \cdot \\ \cdot & & \cdot & \end{bmatrix} \longrightarrow \cdots$$

$$\longrightarrow L_m \cdots L_2 L_1 A = \begin{bmatrix} \diagdown \end{bmatrix}$$

$A \to L_1 A =$ 

$A = L_1$  $= L_1 \; L_2$ 

$= \cdots = L_1 L_2 \cdots$ 

$$A = \underbrace{\left[\begin{array}{c|c} 1 & \vec{b}^* \\ \hline \vec{b} & C \end{array}\right]}_{A} = \underbrace{\left[\begin{array}{c|c} 1 & 0 \\ \hline \vec{b} & I \end{array}\right]}_{L_1} \left[\begin{array}{c|c} 1 & \vec{b}^* \\ \hline 0 & C - \vec{b}\vec{b}^* \end{array}\right]$$

$$= \underset{L_1}{\left[\begin{array}{c|c} 1 & 0 \\ \hline \vec{b} & I \end{array}\right]} \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & C - \vec{b}\vec{b}^* \end{array}\right] \underset{L_1^*}{\left[\begin{array}{c|c} 1 & \vec{b}^* \\ \hline 0 & I \end{array}\right]}$$

$$A = \left[\begin{array}{c|c} a & \vec{b}^* \\ \hline \vec{b} & C \end{array}\right] = \underbrace{\left[\begin{array}{c|c} \alpha & 0 \\ \hline \vec{\beta} & I \end{array}\right]}_{L_1} \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & C - \beta\beta^* \end{array}\right] \underbrace{\left[\begin{array}{c|c} \bar{\alpha} & \vec{\beta}^* \\ \hline 0 & I \end{array}\right]}_{L_1^*}$$

$$|\alpha|^2 = a \quad \longrightarrow \quad \text{let } \alpha = \sqrt{a}$$

$$\bar{\alpha}\vec{\beta} = \vec{b} \qquad \beta = \vec{b}/\bar{\alpha}$$

$$C - \frac{\vec{b}\vec{b}^*}{a}$$

$$A = \underbrace{\left[\begin{array}{c|c} \sqrt{a} & 0 \\ \hline \vec{b}/\sqrt{a} & I \end{array}\right]}_{L_1} \underbrace{\left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & C - \frac{\vec{b}\vec{b}^*}{a} \end{array}\right]} \underbrace{\left[\begin{array}{c|c} \sqrt{a} & \vec{b}^*/\sqrt{a} \\ \hline 0 & I \end{array}\right]}_{L_1^*}$$

$$A = L_1 \left[\begin{array}{c|c} 1 & \\ \hline & A_2 \end{array}\right] L_1^* = L_1 L_2 \left[\begin{array}{c|c|c} 1 & & \\ & 1 & \\ \hline & & A_3 \end{array}\right] L_2^* L_1^*$$

$\hookrightarrow \underbrace{c - \vec{b}\vec{b}^*/a}$, HPD?

$$= \cdots = \underbrace{L_1 L_2 \cdots L_m}_{L} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} \underbrace{L_m^* \cdots L_2^* L_1^*}_{L^*}$$

$A = LL^*$

$c$ is HPD,

$$A = \left[\begin{array}{c|c} a & \vec{b}^* \\ \hline \vec{b} & \underset{\uparrow \text{ HPD}}{c} \end{array}\right] = L_1 \overbrace{\left[\begin{array}{c|c} 1 & \\ \hline & \underset{\uparrow \text{ HPD? } \checkmark}{c - \vec{b}\vec{b}^*/a} \end{array}\right]}^{= \underbrace{L_1^{-1}} A \underbrace{L_1^{-*}}} L_1^*$$

$A = LL^* = R^*R$

↑

lower tri

<u>Thm</u>: Every HPD matrix has a <u>unique</u> Cholesky factorization with $l_{kk} > 0$.

op count $\sim \frac{1}{3} m^3$ flops

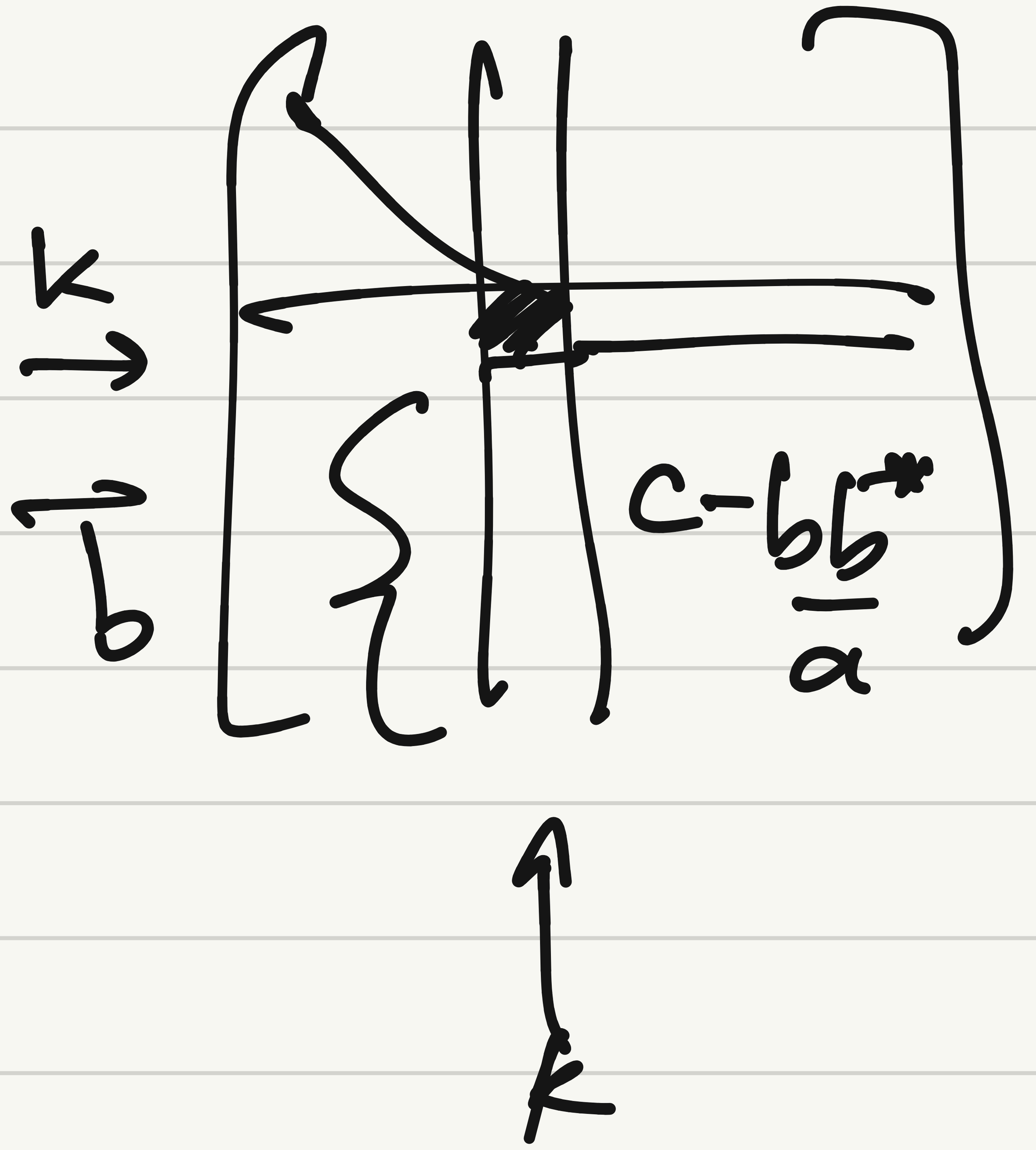
$$\frac{\|\delta A\|}{\underbrace{\|L\|\,\|L^*\|}_{\to \|A\|}} = O(\varepsilon_m)$$

Alg.: (only work on lower triangle of L)

$L = A$

for cols $k = 1, .., m$:

- let $\vec{b} = L_{k+1:m,\,k}$
- $L_{k+1:m,\,k+1:m} \mathrel{-}= \vec{b}\vec{b}^* / l_{kk}$
- $l_k \mathrel{/}= \sqrt{l_{kk}}$




$$\underbrace{\|L\|_2 = \|L^*\|_2 = \sqrt{\|A\|_2}}_{\text{prove using SVD of } L}$$

Thm: for sufficiently small $\varepsilon_m$,

1. no $\ell_{kk} < 0$
2. computed $\tilde{L}\tilde{L}^* = A + \delta A$ with $\frac{\|\delta A\|}{\|A\|} = O(\varepsilon_m)$

for HPD matrix, use Cholesky to solve $A\vec{x} = \vec{b}$

$\Rightarrow \underline{L}L^*\vec{x} = \underline{\vec{b}}$

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If $X$ is full rank, $X^*X$ is HPD

If $A$ is HPD, $\exists X$ s.t. $X^*X = A$ ?

Cholesky $\longrightarrow$ $X = \underline{\underline{L^*}}$

$X = \underline{P}L^*\underline{Q}$

Next class:

<u>Iterative methods</u>

$A\vec{x} = \vec{b} \sim O(m^3)$

$\underbrace{x_1 \rightarrow x_2 \rightarrow x_3 \rightarrow \cdots}$ $\vec{x}_*$