COL 726 Minor

Tuesday, 16 March, 2021

4:15–5:15pm

- There are five questions. Each question is worth 5 marks.
- You are permitted to refer to the textbook, the lecture notes, and online resources during the exam. You are not permitted to collaborate or discuss with other people, other than asking for clarifications on Piazza.
- You are given 15 minutes for scanning and uploading. Answer sheets must be uploaded to Gradescope by 5:30pm.

---

1. (a) Find the relative condition number of the inner product $f(\mathbf{u}, \mathbf{v}) = \mathbf{u}^*\mathbf{v}$ with respect to $\mathbf{u}$.

   (b) Show that computing the inner product of two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{C}^m$ in floating-point arithmetic using the obvious algorithm is backward stable.

2. Prove that $\|\mathbf{ABC}\|_F \le \|\mathbf{A}\|_2 \|\mathbf{B}\|_F \|\mathbf{C}\|_2$ for any three square matrices $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathbb{C}^{m\times m}$.

   Note: There are two different norms being used on the right-hand side!

3. Consider two matrices $\mathbf{X} = \begin{bmatrix}\mathbf{x}_1 & \cdots & \mathbf{x}_r\end{bmatrix} \in \mathbb{C}^{m\times r}$ and $\mathbf{Y} = \begin{bmatrix}\mathbf{y}_1 & \cdots & \mathbf{y}_r\end{bmatrix} \in \mathbb{C}^{n\times r}$, both with orthogonal (but not orthonormal) columns, and $r \ll n < m$. Let $\mathbf{A} = \mathbf{X}\mathbf{Y}^*$.

   (a) State a singular value decomposition of $\mathbf{A}$.

   (b) Given a vector $\mathbf{b} \in \mathbb{C}^m$, give an efficient algorithm for finding the vector $\mathbf{y} \in \operatorname{range}(\mathbf{A})$ closest to $\mathbf{b}$. What are the condition numbers of $\mathbf{y}$ with respect to $\mathbf{b}$, $\mathbf{X}$, and $\mathbf{Y}$?

4. (a) Show that for any vector $\mathbf{x} \in \mathbb{C}^m$, there exist Householder reflectors $\mathbf{F}_2, \mathbf{F}_3, \ldots, \mathbf{F}_m$ such that $\mathbf{F}_2\mathbf{F}_3 \cdots \mathbf{F}_m\mathbf{x} = \|\mathbf{x}\|_2 \mathbf{e}_1$, and each reflector acts on only two entries of $\mathbf{x}$.

   (b) Show how such reflectors can be used to find the QR decomposition of a square matrix $\mathbf{A} \in \mathbb{C}^{m\times m}$ of the form

$$\left[\begin{array}{c|cccc} \times & \times & \times & \cdots & \times \\ \times & & \times & \cdots & \times \\ \vdots & & & \ddots & \vdots \\ \times & & & & \times \\ \hline \times & & & & \end{array}\right]$$

in $O(m^2)$ time.

Note: In part (a) there are many choices of which two rows each reflector should act on, but only one choice will work well in part (b).

5. Suppose you attempt to perform Cholesky factorization of a Hermitian matrix $\mathbf{A} \in \mathbb{C}^{m \times m}$ which you are not sure is positive definite. Partway through, you reach a factorization

$$\mathbf{A} = \begin{bmatrix} \mathbf{L} & \mathbf{0} \\ \mathbf{M} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{C} \end{bmatrix} \begin{bmatrix} \mathbf{L}^* & \mathbf{M}^* \\ \mathbf{0} & \mathbf{I} \end{bmatrix}$$

where $c_{11} < 0$, and you are unable to proceed. Show how to construct a vector $\mathbf{x}$ for which $\mathbf{x}^* \mathbf{A} \mathbf{x} < 0$, thus demonstrating that $\mathbf{A}$ is not even positive semidefinite.