COL 726 Assignment 1

15 February – 1 March, 2020

Note: Answers should be accompanied by a rigorous justification wherever applicable.

Updated text is highlighted in blue.

1. Let $f : X \to Y$ and $g : Y \to Z$ be continuous functions between arbitrary vector spaces, and let $h = g \circ f$. Suppose $y = f(\mathbf{x})$ and $z = g(\mathbf{y}) = h(\mathbf{x})$.

   (a) Find the tightest possible upper bound on $\kappa_h(\mathbf{x})$ in terms of $\kappa_f(\mathbf{x})$ and $\kappa_g(\mathbf{y})$, where $\kappa_f, \kappa_g, \kappa_h$ are the condition numbers of f, g, h respectively.

   (b) The actual condition number can be far smaller than the bound derived above. Give a concrete example where $\kappa_f(\mathbf{x}), \kappa_g(\mathbf{y}) \geq 10$ but $\kappa_h(\mathbf{x}) \leq 1$.

2. Suppose you want to compute the function $f(x) = x(1 - x)$ for some number $x \in \mathbb{R}$, so you proceed by rounding to a floating-point number $\hat{x} = \mathrm{fl}(x) \in \mathbb{F}$ and evaluating $\hat{x} \otimes (1 \ominus \hat{x})$.

   (a) Derive an upper bound on the forward error of the result.

   (b) For what value(s) of x does the result fail to be accurate? For what value(s) does it fail to be backward stable?

3. Consider the equation $\mathbf{UX} + \mathbf{XL} = \mathbf{Y}$, where $\mathbf{U}, \mathbf{L}, \mathbf{Y} \in \mathbb{C}^{m \times m}$ are known matrices, $\mathbf{U}$ is upper triangular, and $\mathbf{L}$ is lower triangular. Give an algorithm to find the unknown matrix $\mathbf{X} \in \mathbb{C}^{m \times m}$ in $O(m^3)$ time using backsubstitution.

   (Backsubstitution is the $O(m^2)$-time algorithm for solving $\mathbf{Tx} = \mathbf{y}$ when $\mathbf{T}$ is upper triangular, by solving for x_m using the last row, then solving for x_{m-1} using the second-last row, and so on.)

   Hint: Try finding the columns of $\mathbf{X}$ one at a time.

4. When does a rectangular matrix $\mathbf{A} \in \mathbb{C}^{m \times n}$ have the property that $\|\mathbf{Ax}\|_2 = \|\mathbf{x}\|_2$ for all $\mathbf{x} \in \mathbb{C}^n$? For both cases ($m > n$ and $m < n$), explain whether it is possible, and if so, give necessary and sufficient conditions.

5. Say I am interested in matrices of the form $\mathbf{M}(\lambda) = \mathbf{I} + \lambda\mathbf{A}$ for $\lambda \in \mathbb{C}$ and $\mathbf{A} \in \mathbb{C}^{m \times m}$. I know the value of $\|\mathbf{A}\|$ in some arbitrary induced norm. Find a positive real number δ such that $\mathbf{M}(\lambda)$ is nonsingular whenever $|\lambda| < \delta$.

   Hint: Consider the action of $\mathbf{M}(\lambda)$ on an arbitrary vector $\mathbf{x}$.

6. Consider the sequence defined by the difference equation $x_{k+1} = 2.25x_k - 0.5x_{k-1}$, with starting values $x_0 = 1/3$ and $x_1 = 1/12$.

(a) Write a function `computeSequence`(n) which computes the first n terms of this sequence, and returns them in a Numpy array $[x_0, x_1, \ldots, x_{n-1}]$.

(Better yet, write a function `computeSequence`($[a_0, \ldots, a_m], b, [x_0, \ldots, x_m], n$) which solves the general difference equation $x_{k+1} = a_0 x_k + a_1 x_{k-1} + \cdots + a_m x_{k-m} + b$ with initial terms $x_0, \ldots, x_m$.)

(b) Make a semilog plot of the computed terms as a function of k, up to $k = 80$. The exact solution to the equation is $x_k = 4^{-k}/3$; show this on the same plot.

(c) The exact solution decreases monotonically as k increases. Does your computed solution have the same behaviour? Can you explain your results?

Collaboration policy: Refer to the policy on the course webpage.

If you collaborated with others to solve any question(s) of this assignment, give their names in your submission. If you found part of a solution using some online resource, give its URL.

Submission: Submit a PDF of your answers for all questions to Gradescope. Submit the code for Question 6 to Moodle. Both submissions must be uploaded before the assignment deadline.

Code submissions should contain a single .py file which contains all the functions. Functions are permitted but not required to produce any side-effects like printing out values or drawing plots. Any results you are asked to show should go in the PDF.