COL865: Special Topics in Computer Applications

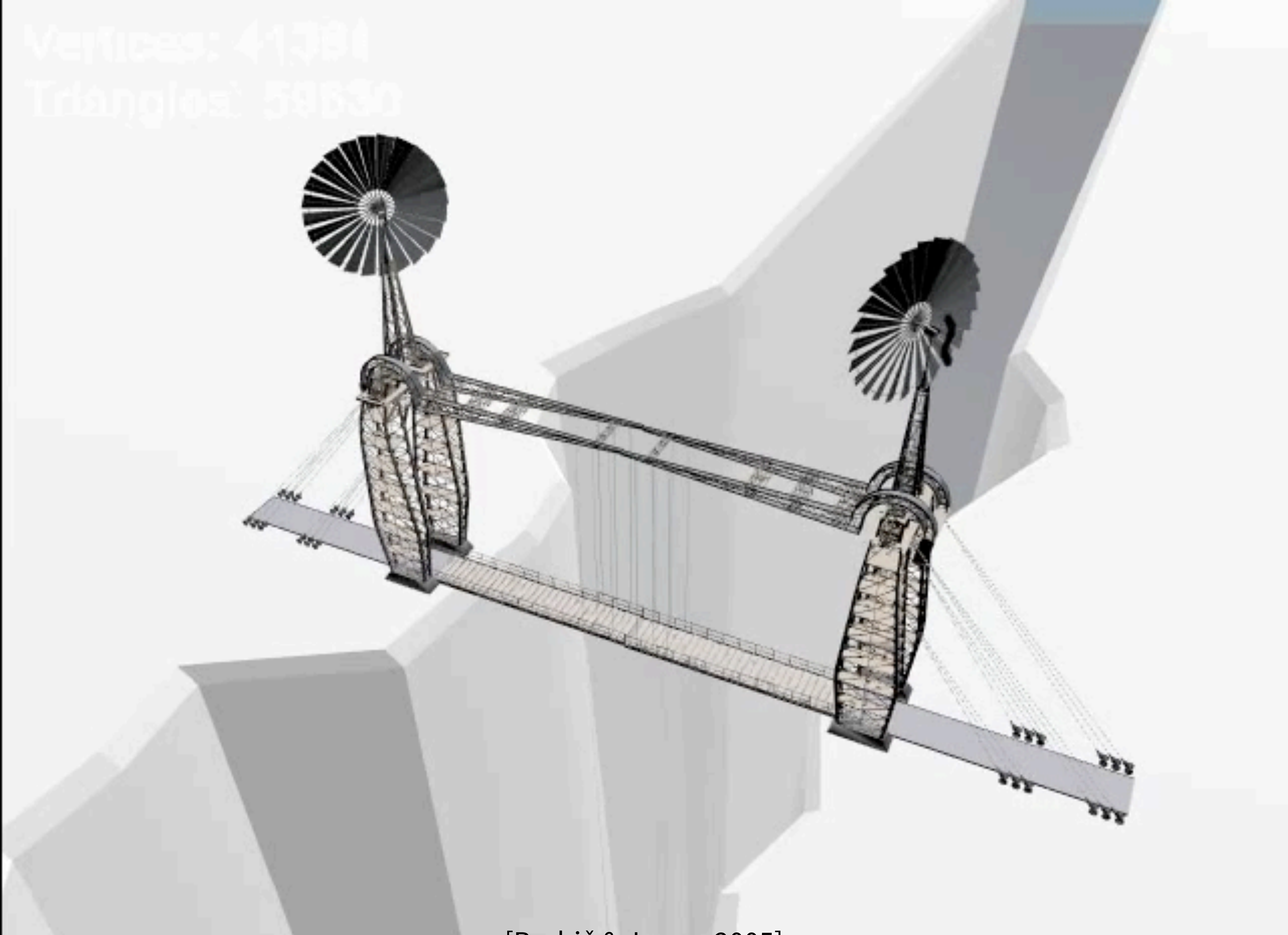
Physics-Based Animation

---

20 — Model reduction

Announcements

- ***Project proposal*** due this Friday, 12 October
    - Talk to me during office hours
- Increased ***office hours***: Mon–Wed, noon–1pm
- Pick ***topics for 2nd paper presentation*** by Sunday, 21 October



[Barbič & James 2005]

Reading

- Sifakis & Barbič, *FEM Simulation of 3D Deformable Solids*, Part 2: “Model reduction”

Model reduction

$$\mathbf{M}\,\ddot{\mathbf{u}} = \mathbf{f}(\mathbf{u}, \dot{\mathbf{u}}) + \mathbf{f}_{\text{ext}}$$

Choose a low-dimensional basis $\mathbf{u} = \mathbf{U}\,\mathbf{q} = q_1\,\mathbf{u}_1 + \cdots + q_r\,\mathbf{u}_r$

Equations of motion via Galerkin projection:

$$\mathbf{U}^{\mathsf{T}}\,\mathbf{M}\,\mathbf{U}\,\ddot{\mathbf{q}} = \mathbf{U}^{\mathsf{T}}\,\mathbf{f}(\mathbf{U}\,\mathbf{q}, \mathbf{U}\,\dot{\mathbf{q}}) + \mathbf{U}^{\mathsf{T}}\,\mathbf{f}_{\text{ext}}$$

1. How to choose a good basis?
2. How to efficiently compute $\mathbf{U}^{\mathsf{T}}\,\mathbf{f}$?

Linear modal analysis

$$\mathbf{M}\,\ddot{\mathbf{u}} = -\mathbf{K}\,\mathbf{u}$$

Generalized eigenvectors:

$$\mathbf{K}\,\boldsymbol{\psi}_i = \lambda_i\,\mathbf{M}\,\boldsymbol{\psi}_i$$

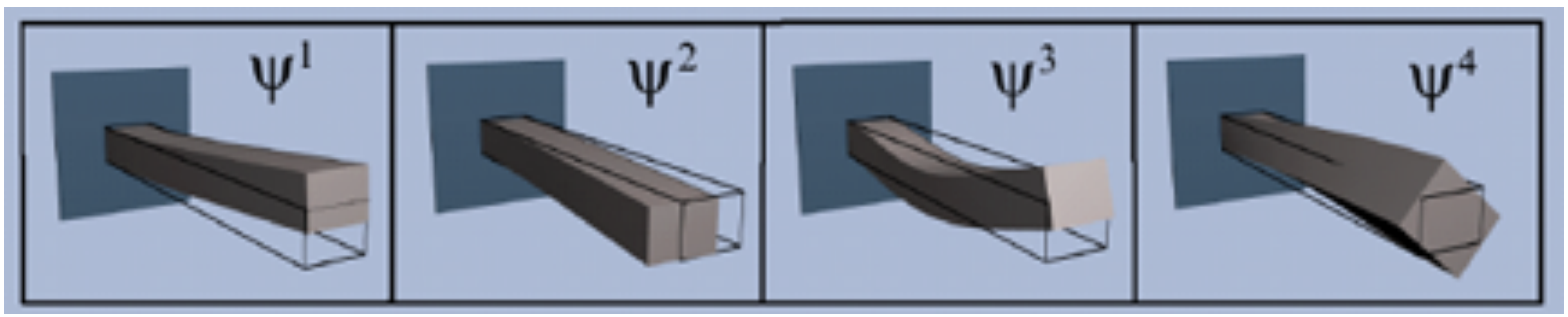
$\mathbf{M}$, $\mathbf{K}$ symmetric, $\mathbf{M}$ nonsingular
$\Rightarrow$ mass-orthogonality $\boldsymbol{\psi}_i^{\mathsf{T}}\,\mathbf{M}\,\boldsymbol{\psi}_j = 0$

Normalize $\boldsymbol{\psi}_i$ so that $\boldsymbol{\psi}_i^{\mathsf{T}}\,\mathbf{M}\,\boldsymbol{\psi}_i = 1$

Then let $\mathbf{u} = q_1\,\boldsymbol{\psi}_1 + \cdots + q_r\,\boldsymbol{\psi}_r$

$$\ddot{q}_i = -\lambda_i\,q_i$$

Linear modes



[Sifakis & Barbic]

Linear modal analysis

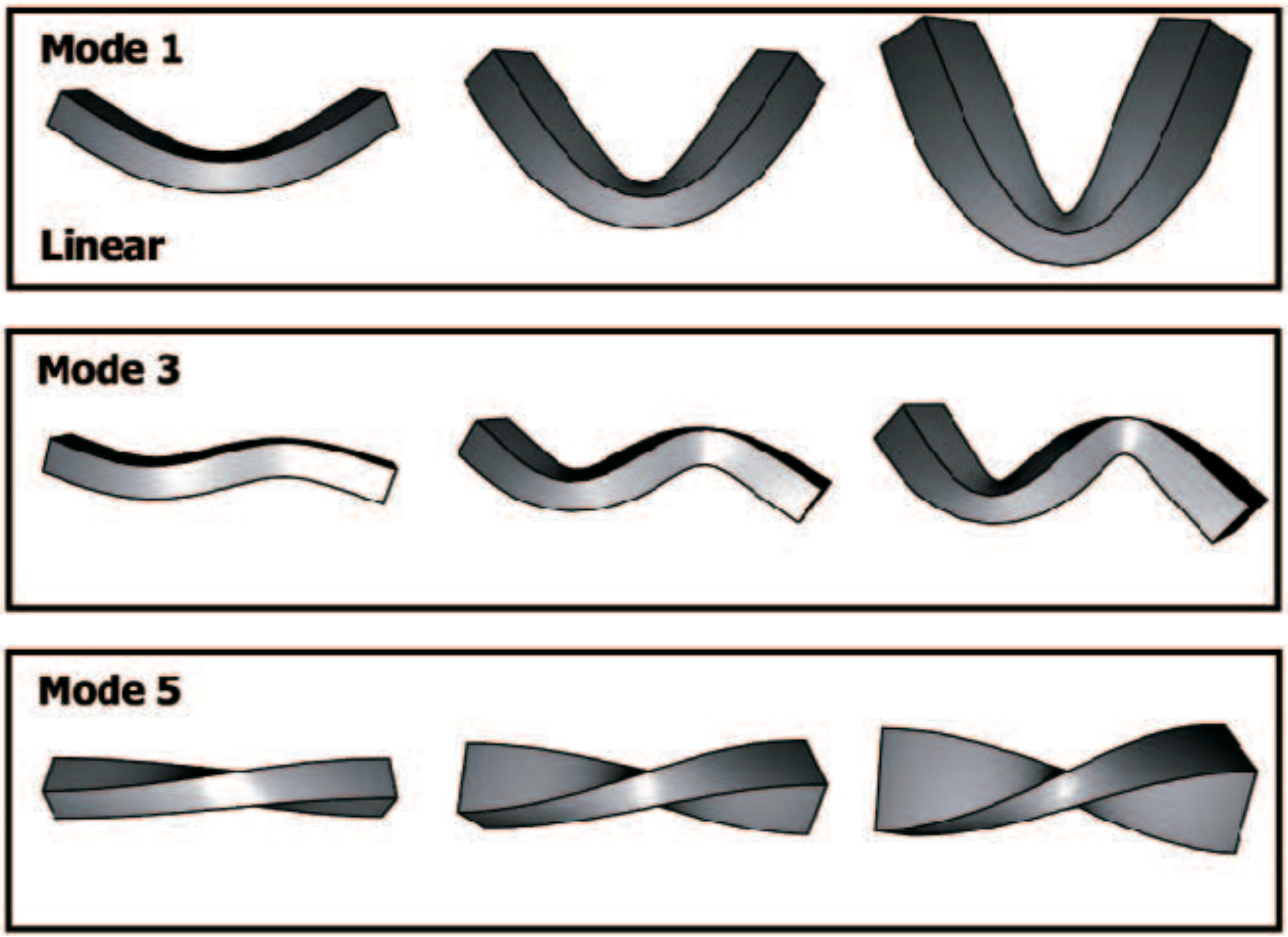
With damping and external forces:

$$\ddot{\mathbf{q}} + \mathbf{\Lambda}\, \mathbf{q} + \mathbf{U}^\mathsf{T}\, \mathbf{D}\, \mathbf{U}\, \dot{\mathbf{q}} = \mathbf{U}^\mathsf{T}\, \mathbf{f}_{\mathrm{ext}}$$

Rayleigh damping: $\mathbf{D} = \alpha\, \mathbf{M} + \beta\, \mathbf{K}$

$$\ddot{q}_i + \lambda_i\, q_i + (\alpha + \beta\, \lambda_i)\, \dot{q}_i = \boldsymbol{\psi}_i^\mathsf{T}\, \mathbf{f}_{\mathrm{ext}}$$

Dealing with nonlinearity



[Choi & Ko 2005]

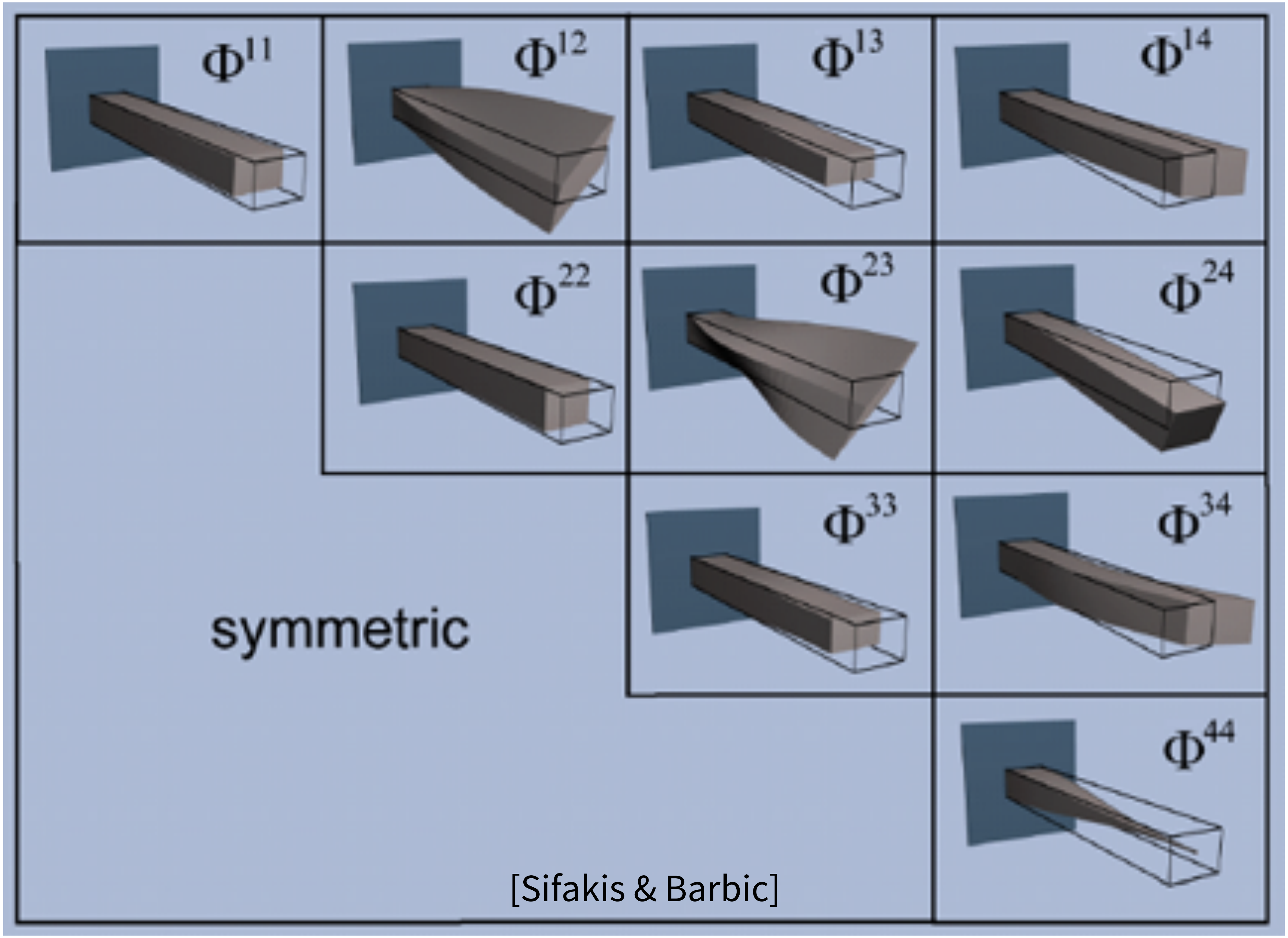
Choice of basis

- Linear modes using $\mathbf{K} = d\mathbf{f}/d\mathbf{u}$ at undeformed pose
- PCA basis
- Modal derivatives [Barbič & James 2005]

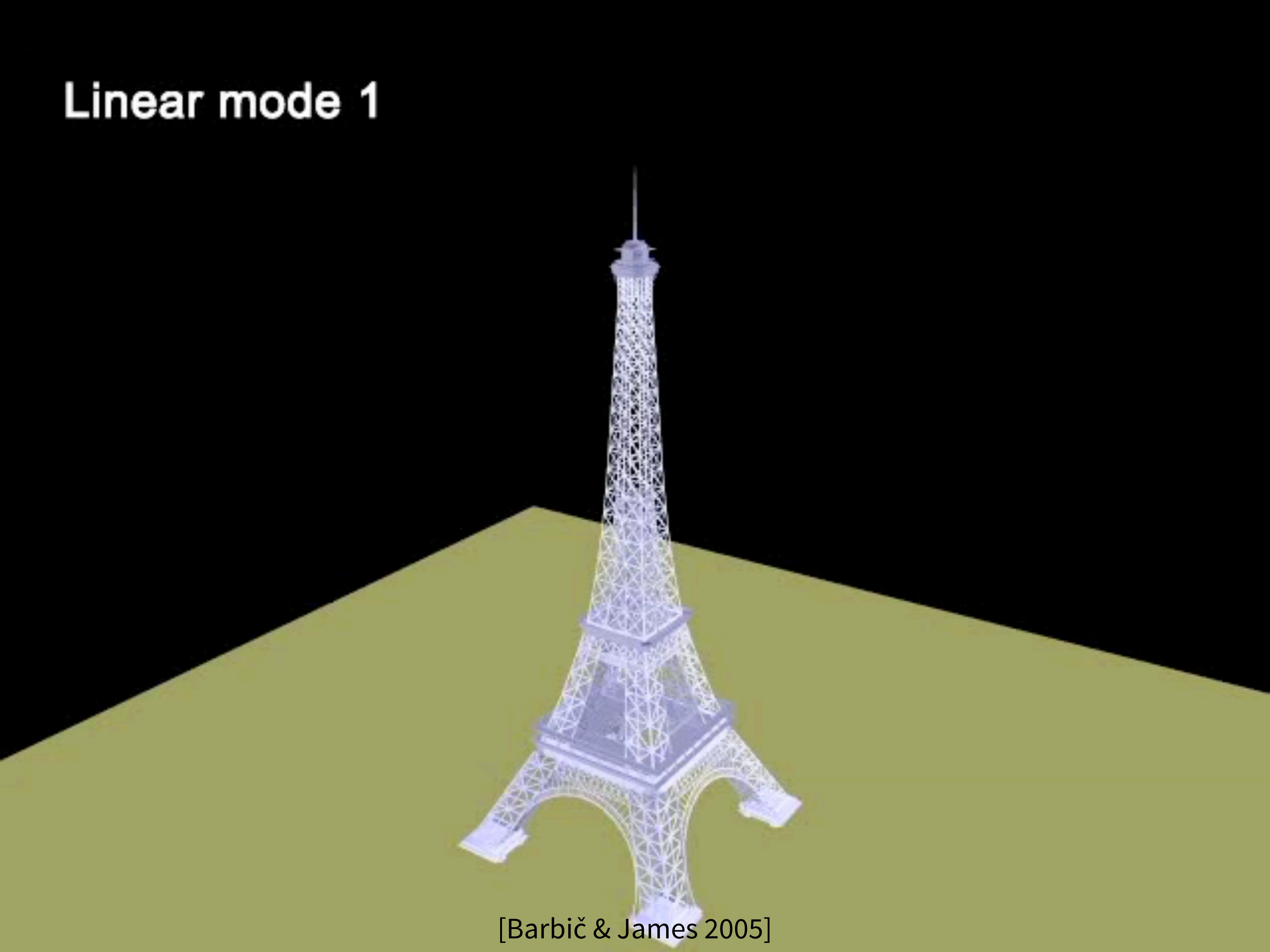
$$\boldsymbol{\psi}_i(\mathbf{u}) = \text{eigenvector}_i(\mathbf{M}, \mathbf{K}(\mathbf{u}))$$

$$\boldsymbol{\psi}_i(\mathbf{u} + h\,\boldsymbol{\psi}_j) = \boldsymbol{\psi}_i(\mathbf{u}) + h\,\boldsymbol{\varphi}_{ij}(\mathbf{u}) + O(h^2)$$

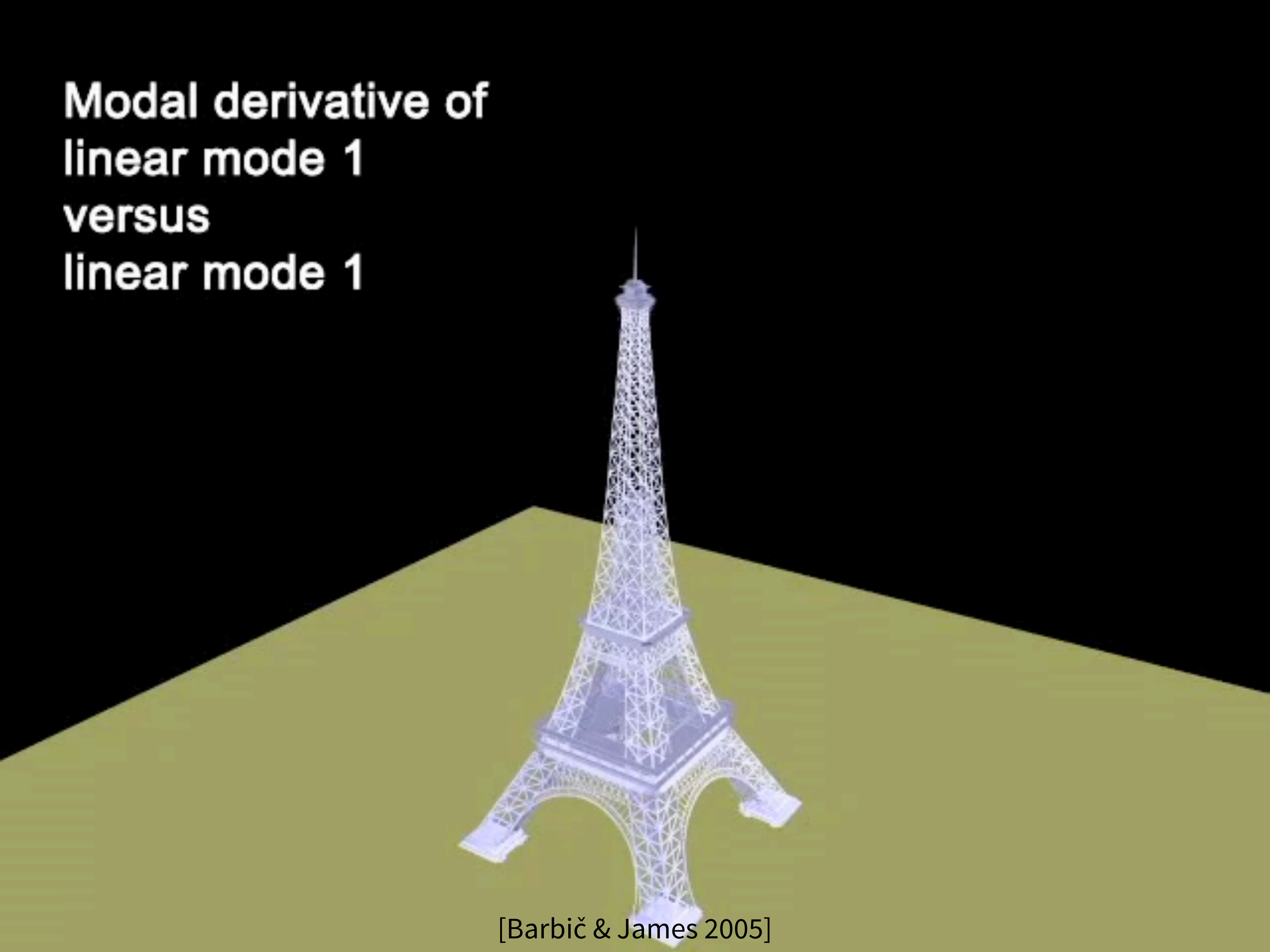
Modal derivatives



[Sifakis & Barbic]



[Barbič & James 2005]



[Barbič & James 2005]

[Barbič & James 2005]

Modal warping

Keep original basis, change reconstruction: $\mathbf{u} \neq \mathbf{U}\,\mathbf{q}$
[Choi & Ko 2005, Huang et al. 2011]

1. For each element, $\mathbf{I} + \mathbf{G} = \mathrm{d}(\mathbf{U}\,\mathbf{q})/\mathrm{d}\mathbf{X}$
2. $\mathbf{G} = \boldsymbol{\varepsilon} + \boldsymbol{\omega}$
3. $\mathbf{S} = \mathbf{I} + \boldsymbol{\varepsilon}$, $\mathbf{R} = \exp(\boldsymbol{\omega})$, $\mathbf{F} = \mathbf{R}\,\mathbf{S}$
4. Reconstruct **u** from **F** via least-squares

Modal Analysis Extension

Linear Mode

Our Non-linear Geometric Extension

[Huang et al. 2011]

Evaluation of reduced forces

$$\bar{\mathbf{f}}(\mathbf{q}) = \mathbf{U}^T \mathbf{f}(\mathbf{U}\,\mathbf{q})$$
$$\bar{\mathbf{K}}(\mathbf{q}) = \mathbf{U}^T \mathbf{K}(\mathbf{U}\,\mathbf{q})\,\mathbf{U}$$

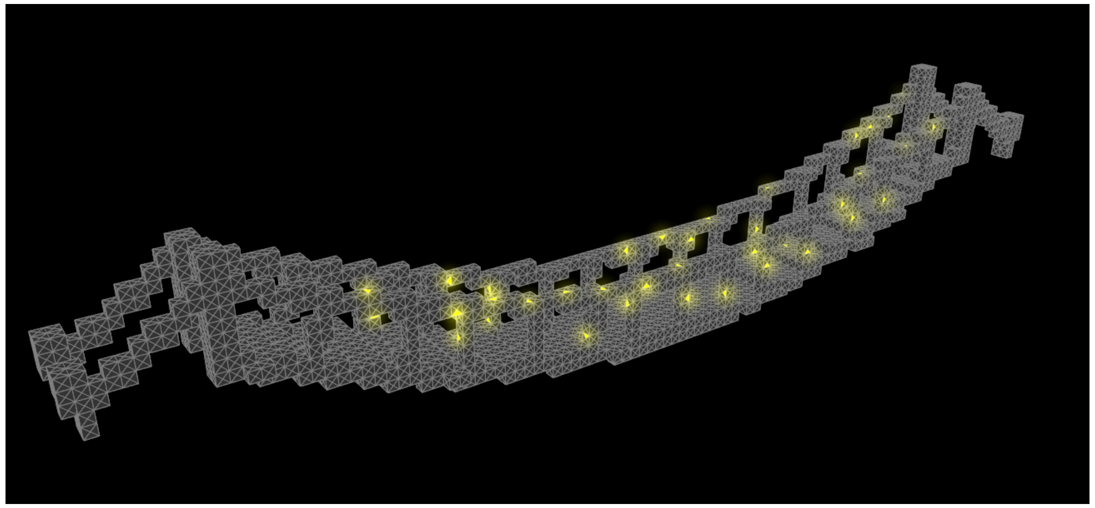
Naive approach: $O(n\,r)$ cost

- If $\mathbf{f}$ is polynomial in $\mathbf{x}$ $\Rightarrow$ $\bar{\mathbf{f}}$ is polynomial in $\mathbf{q}$

  Precompute all r^{p+1} coefficients

- Optimized cubature
  [An et al. 2008]

$$\bar{f}_i(\mathbf{q}) = \iiint \boldsymbol{\psi}_i(\mathbf{X})^T\,\mathbf{f}(\mathbf{X};\,\mathbf{U}\,\mathbf{q})\,\mathrm{d}V$$
$$\approx \sum w_j\,\boldsymbol{\psi}_i(\mathbf{X}_j)^T\,\mathbf{f}(\mathbf{X}_j;\,\mathbf{U}\,\mathbf{q})$$



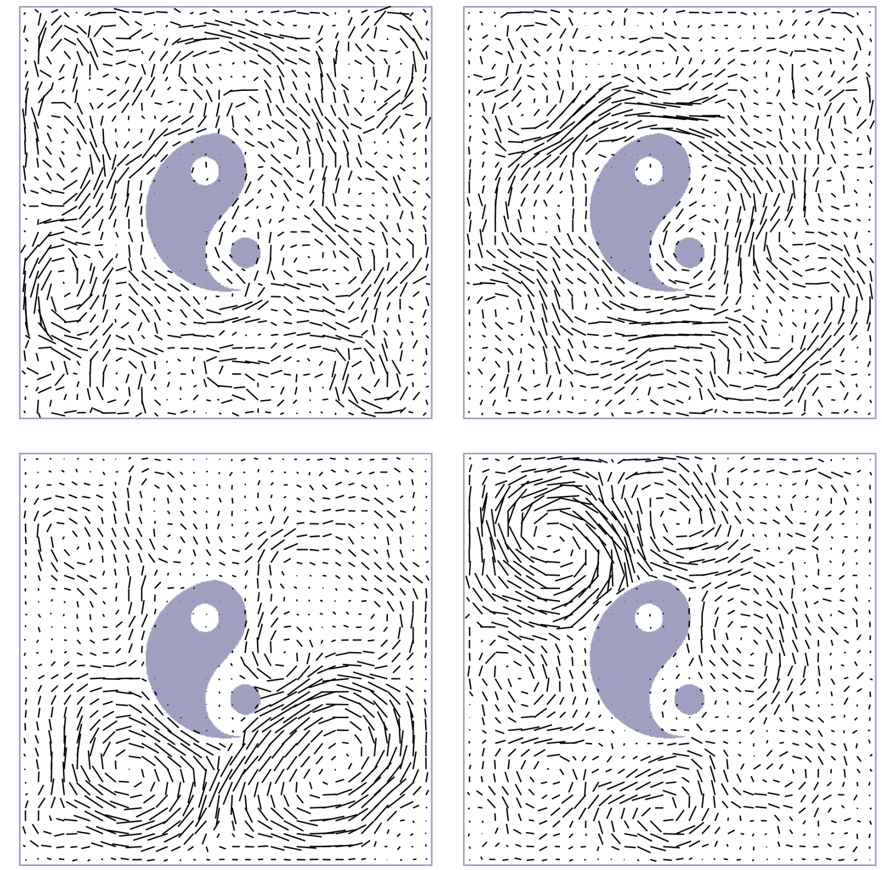
[An et al. 2008]

Model reduction for fluids

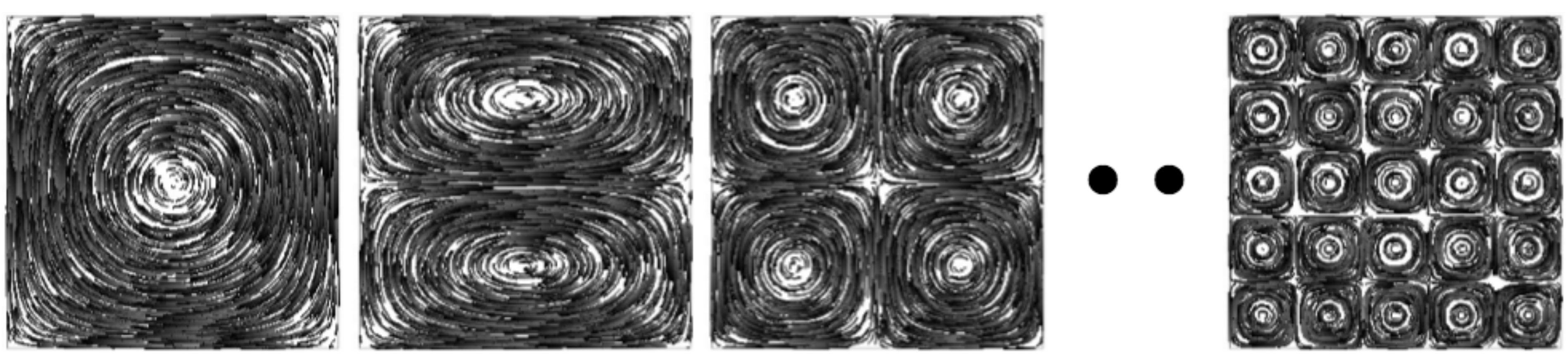
$$\mathbf{u} = w_1\, \boldsymbol{\psi}_1 + \cdots + w_r\, \boldsymbol{\psi}_r$$

Choice of basis:

- PCA basis [Treuille et al. 2006]
- Laplacian eigenfunctions [de Witt et al. 2011]



[Treuille et al. 2006]



[de Witt et al. 2011]

Model reduction for fluids

$$\mathbf{u} = w_1\,\boldsymbol{\psi}_1 + \cdots + w_r\,\boldsymbol{\psi}_r$$

Reduced forces:

- No pressure step needed
- Advection term $(\mathbf{u} \cdot \nabla)\,\mathbf{u}$ is quadratic in u:
  store r^3 coefficients $C_{k,i,j}$

$$(\boldsymbol{\psi}_i \,.\, \nabla)\,\boldsymbol{\psi}_j \approx C_{1,i,j}\,\boldsymbol{\psi}_1 + \cdots + C_{r,i,j}\,\boldsymbol{\psi}_r$$

$$\begin{aligned}\dot{w}_k &= \textstyle\sum_{i,j} C_{k,i,j}\,w_i\,w_j \\ &= \mathbf{w}^\mathrm{T}\,\mathbf{C}_k\,\mathbf{w}\end{aligned}$$

[Treuille et al. 2006]