COL865: Special Topics in Computer Applications

Physics-Based Animation

19 — Position-based and projective dynamics

Reading

- Bender et al., *A survey on position based dynamics*, Eurographics 2017 course notes, Ch. 3–5.5
- Bouaziz et al., "Projective dynamics: Fusing constraint projections for fast simulation", SIGGRAPH 2014

Optional:

- Overby et al., "ADMM $\supseteq$ projective dynamics: Fast simulation of hyperelastic models with dynamic constraints", TVCG 2017
- Liu et al., "Quasi-Newton methods for real-time simulation of hyperelastic materials", TOG 2017

Reminder: Backward Euler

$$\ddot{\mathbf{x}} = \mathbf{M}^{-1} \left(\mathbf{f}(\mathbf{x}) + \mathbf{f}_{\text{ext}}\right)$$

Backward Euler:

$$\mathbf{v}^{n+1} = \mathbf{v}^{n} + \Delta t\, \mathbf{M}^{-1} \left(\mathbf{f}(\mathbf{x}^{n+1}) + \mathbf{f}_{\text{ext}}\right)$$
$$\mathbf{x}^{n+1} = \mathbf{x}^{n} + \Delta t\, \mathbf{v}^{n+1}$$

Rearrange:

$$\tilde{\mathbf{x}} = \mathbf{x}^{n} + \Delta t\, \mathbf{v}^{n} + \Delta t^{2}\, \mathbf{M}^{-1}\, \mathbf{f}_{\text{ext}}$$
$$\mathbf{x}^{n+1} = \tilde{\mathbf{x}} + \Delta t^{2}\, \mathbf{M}^{-1}\, \mathbf{f}(\mathbf{x}^{n+1})$$

Constrained dynamics

$$\mathbf{x}^{n+1} = \tilde{\mathbf{x}} + \Delta t^2\, \mathbf{M}^{-1}\, \mathbf{f}(\mathbf{x}^{n+1})$$

Replace forces with constraints $\Rightarrow$ constraint projection:

$$\mathbf{x}^{n+1} = \tilde{\mathbf{x}} + \Delta t^2\, \mathbf{M}^{-1}\, \mathbf{J}(\mathbf{x}^{n+1})^\mathsf{T}\, \boldsymbol{\lambda}$$
$$\mathbf{c}(\mathbf{x}^{n+1}) = \mathbf{0}$$

Fast projection:

$$\mathbf{x} \leftarrow \mathbf{x} + \Delta\mathbf{x}$$
where $\Delta\mathbf{x} = \Delta t^2\, \mathbf{M}^{-1}\, \mathbf{J}(\mathbf{x})^\mathsf{T}\, \Delta\boldsymbol{\lambda}$,
$$\mathbf{c}(\mathbf{x}) + \mathbf{J}(\mathbf{x})\, \Delta\mathbf{x} = \mathbf{0}$$

Position-based dynamics

$$\mathbf{x} \leftarrow \mathbf{x} + \Delta\mathbf{x}$$

$$\text{where } \Delta\mathbf{x} = \Delta t^2\, \mathbf{M}^{-1}\, \mathbf{J}(\mathbf{x})^\mathsf{T}\, \Delta\boldsymbol{\lambda},$$

$$\mathbf{c}(\mathbf{x}) + \mathbf{J}(\mathbf{x})\, \Delta\mathbf{x} = \mathbf{0}$$

PBD [Müller et al. 2007] = fast projection + Gauss-Seidel on constraints:

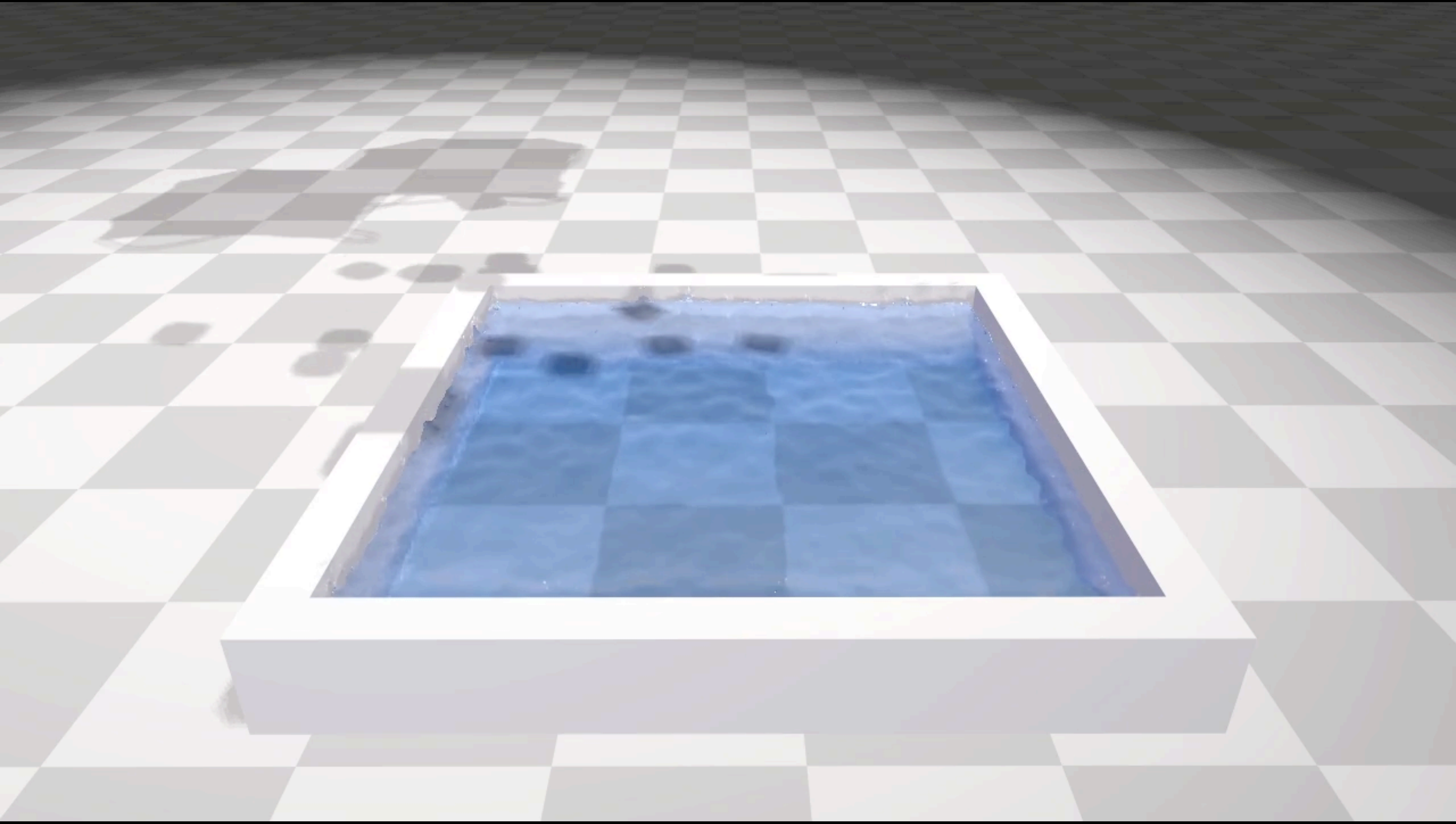
$$\mathbf{x} \leftarrow \mathbf{x} + \Delta\mathbf{x}$$

$$\text{where } \Delta\mathbf{x} = \Delta t^2\, \mathbf{M}^{-1}\, \nabla c_j\, \Delta\lambda_j,$$

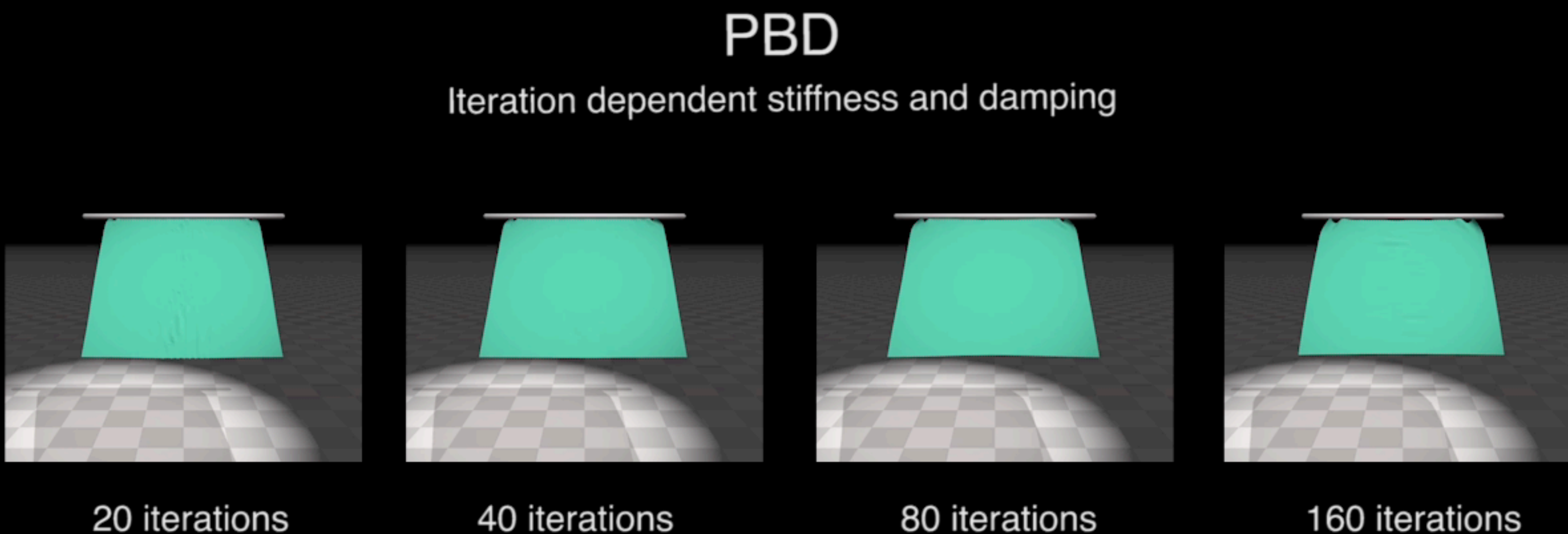
$$c_j(\mathbf{x} + \Delta\mathbf{x}) \approx 0$$

$$\Rightarrow \Delta\lambda_j = -c_j(\mathbf{x})/(\Delta t^2\, \nabla c_j^\mathsf{T}\, \mathbf{M}^{-1}\, \nabla c_j)$$

[Müller et al. 2007]



[Macklin et al. 2014]



[Macklin et al. 2016]

XPBD

Hard constraints: $\mathbf{f}_j = \nabla c_j \lambda_j,\ c_j(\mathbf{x}) = 0$

Soft constraints: $\mathbf{f}_j = \nabla c_j \lambda_j,\ c_j(\mathbf{x}) + \alpha_j \lambda_j = 0$

XPBD [Macklin et al. 2016]:

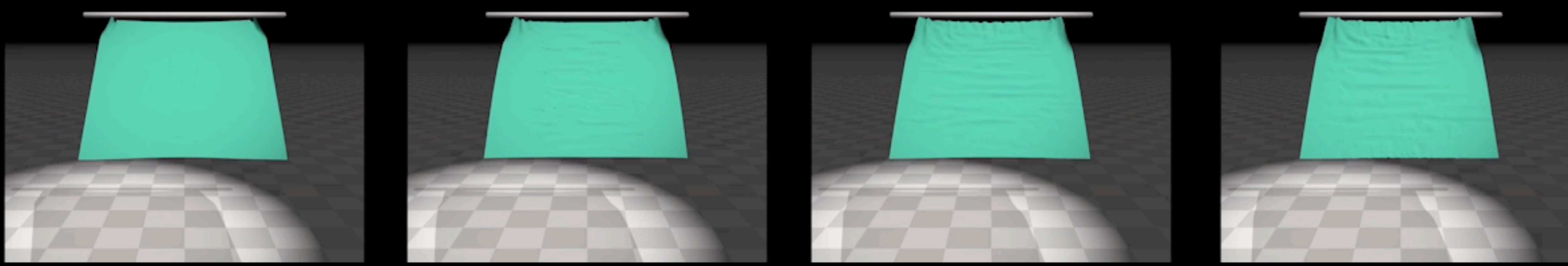
$$\mathbf{x} \leftarrow \mathbf{x} + \Delta\mathbf{x}$$

where $\Delta\mathbf{x} = \Delta t^2\, \mathbf{M}^{-1} \nabla c_i\, \Delta\lambda_i$,

$$c_i(\mathbf{x} + \Delta\mathbf{x}) + \alpha_j\, (\lambda_j + \Delta\lambda_i) \approx 0$$

$$\Rightarrow \Delta\lambda_i = -(c_i(\mathbf{x}) + \alpha_j \lambda_j)/(\Delta t^2\, \nabla c_i^{\mathsf{T}}\, \mathbf{M}^{-1}\, \nabla c_i + \alpha_j)$$

Our Method



20 iterations | 40 iterations | 80 iterations | 160 iterations

[Macklin et al. 2016]

Course schedule

	Mon	Wed	Thu
24, 26, 27 Sep	Lecture	Lecture	Lecture
1, 3, 4 Oct	Friday time table	Minor 2	
8, 10, 11 Oct	Lecture		
15, 17, 18 Oct	Mid-semester break		
22, 24, 25 Oct	Presentations	Presentations	
29 Oct, 1, 2 Nov			Presentations
5, 7, 8 Nov	Presentations	Diwali	No class
12, 14, 15 Nov			Project presentations
	Major		

Optimization-based backward Euler

Backward Euler for conservative forces:

$$\tilde{\mathbf{x}} = \mathbf{x}^n + \Delta t\, \mathbf{v}^n + \Delta t^2\, \mathbf{M}^{-1}\, \mathbf{f}_{\text{ext}}$$
$$\mathbf{x}^{n+1} = \tilde{\mathbf{x}} - \Delta t^2\, \mathbf{M}^{-1}\, \nabla U(\mathbf{x}^{n+1})$$

Optimization form:

$$\min_{\mathbf{x}} \tfrac{1}{2}\, (\mathbf{x} - \tilde{\mathbf{x}})^{\mathsf{T}}\, \mathbf{M}\, (\mathbf{x} - \tilde{\mathbf{x}}) + \Delta t^2\, U(\mathbf{x})$$

Projective dynamics

$$\min_{\mathbf{x}} \tfrac{1}{2} (\mathbf{x} - \tilde{\mathbf{x}})^T \mathbf{M} (\mathbf{x} - \tilde{\mathbf{x}}) + \Delta t^2 U(\mathbf{x})$$

Constraint energy:

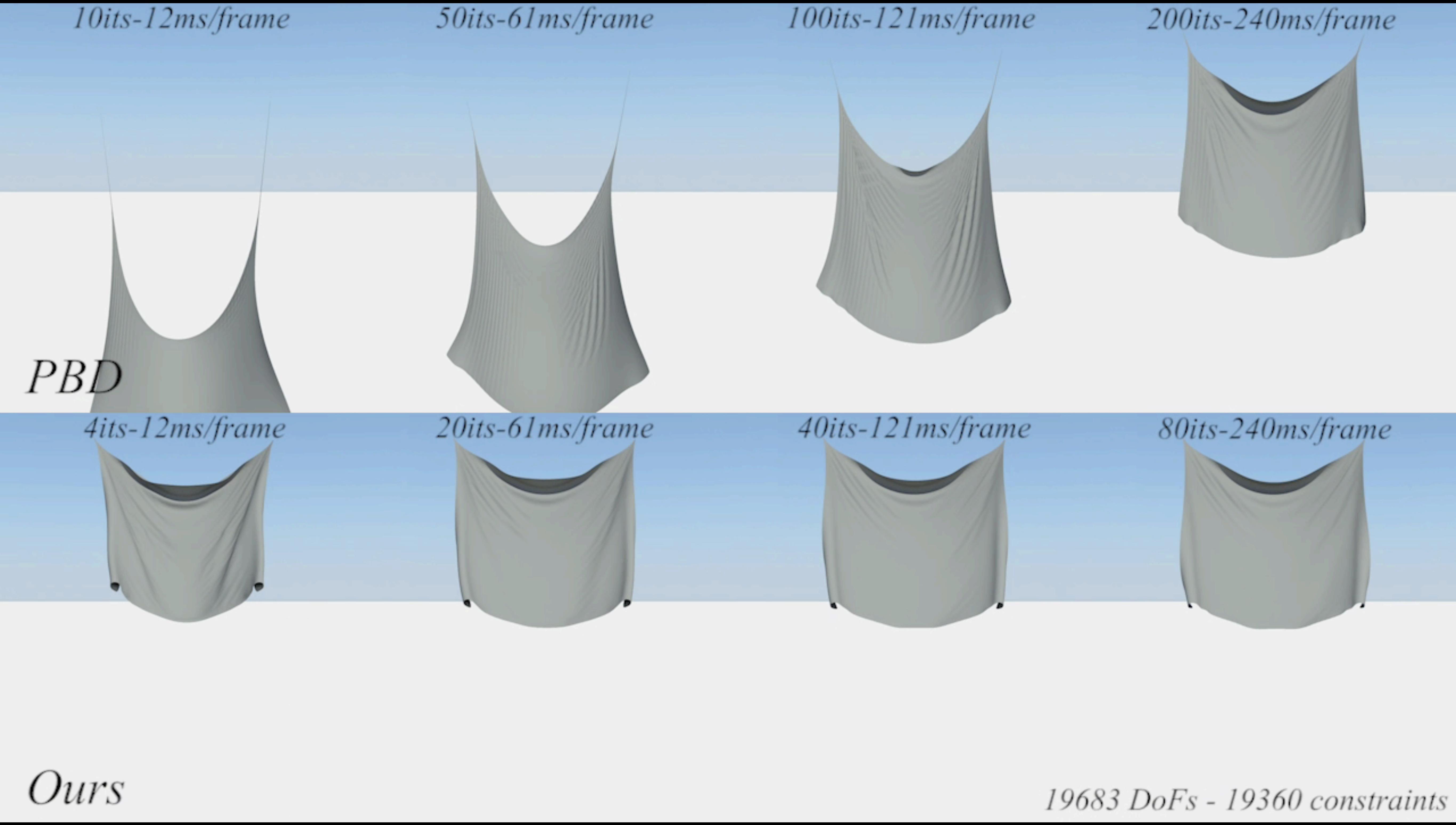
$$U_j(\mathbf{x}) = \tfrac{1}{2} k_j \, d(\mathbf{A}_j \mathbf{x}, S_j)^2$$
$$= \min_{\mathbf{p}_j \in S_j} \tfrac{1}{2} k_j \|\mathbf{A}_j \mathbf{x} - \mathbf{p}_j\|^2$$

Backward Euler (optimization form):

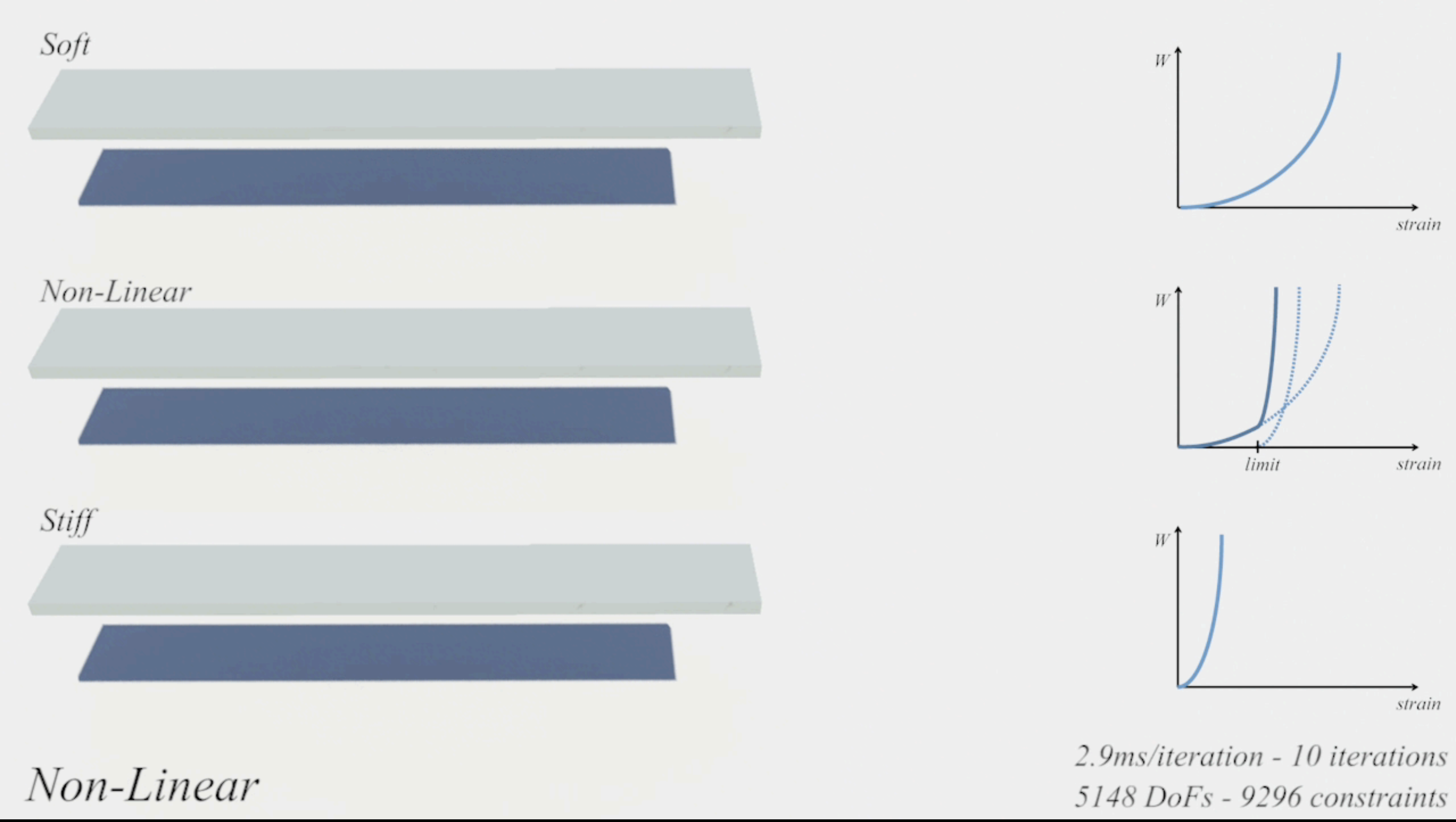
$$\min_{\mathbf{x}, \mathbf{p}_1, \mathbf{p}_2, \dots} \tfrac{1}{2} (\mathbf{x} - \tilde{\mathbf{x}})^T \mathbf{M} (\mathbf{x} - \tilde{\mathbf{x}}) + \tfrac{1}{2} \Delta t^2 \sum k_j \|\mathbf{A}_j \mathbf{x} - \mathbf{p}_j\|^2$$

Projective dynamics [Bouaziz et al. 2014]:

- Update $\mathbf{p}_j$ by projecting $\mathbf{A}_j \mathbf{x}$ to S_j
- Update $\mathbf{x}$ by minimizing quadratic



[Bouaziz et al. 2014]



[Bouaziz et al. 2014]

Nonlinear projective dynamics

ADMM form [Overby et al. 2017]:

$$\min_{\mathbf{x},\mathbf{z}} \tfrac{1}{2} (\mathbf{x} - \tilde{\mathbf{x}})^\top \mathbf{M} (\mathbf{x} - \tilde{\mathbf{x}}) + \tfrac{1}{2} \Delta t^2 \sum k_j \|\mathbf{z}_j - \mathbf{p}_j\|^2$$
$$\text{s.t. } \mathbf{A}_j \mathbf{x} = \mathbf{z}_j$$

Quasi-Newton form [Liu et al. 2017]:

$$f(\mathbf{x}) = \tfrac{1}{2} (\mathbf{x} - \tilde{\mathbf{x}})^\top \mathbf{M} (\mathbf{x} - \tilde{\mathbf{x}}) + \tfrac{1}{2} \Delta t^2 \sum k_j \min_{\mathbf{p}_j} \|\mathbf{A}_j \mathbf{x} - \mathbf{p}_j\|^2$$
$$\nabla f(\mathbf{x}) = \mathbf{M} (\mathbf{x} - \tilde{\mathbf{x}}) + \Delta t^2 \sum k_j \mathbf{A}_j^\top (\mathbf{A}_j \mathbf{x} - \mathbf{p}_j(\mathbf{x}))$$